CYCLES AND ENDOMORPHISMS OF ABELIAN VARIETIES

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In the present paper we shall remark that to each class of algebraically equivalent cycles on a Jacobian variety we can attach a symmetric element of the ring of endomorphisms of the variety and shall prove some formulae concerning attached symmetric elements.

§ 1. Endomorphism $\delta(\mathbf{X}, \mathbf{Y})$

We shall denote by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$ cycles on an abelian variety $\mathbf{A}$. Throughout in this section we shall fix this abelian variety $\mathbf{A}$ of dimension n. When $\mathbf{X}$ and $\mathbf{Y}$ have complemental dimensions and $\mathbf{X}\cdot\mathbf{Y}$ is defined, $S[\mathbf{X}\cdot\mathbf{Y}]$ means $e_1x_1+\ldots+e_fx_f$, where $x_1, x_2, \ldots, x_f$ are points contained in $\mathbf{X}\cdot\mathbf{Y}$ with multiplicities $e_1, e_2, \ldots, e_f$ respectively. When $\mathbf{A}, \mathbf{X}, \mathbf{Y}$ are defined over a field k, $S[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]$ defines a function on $\mathbf{A}$ within values in $\mathbf{A}$ and defined over k. Since $S[\mathbf{X}\cdot(\mathbf{Y}-\mathbf{Y})]=0$, this function is an endomorphism of $\mathbf{A}$. We denote it by $\delta(\mathbf{X}, \mathbf{Y})$. Even when $\mathbf{X}\cdot\mathbf{Y}_{t_0}$ is not defined, we define $S[\mathbf{X}\cdot(\mathbf{Y}_{t_0}-\mathbf{Y})]$ by $\delta(\mathbf{X}, \mathbf{Y})t_0$.

We say that two cycles $\mathbf{X}, \mathbf{Y}$ are immediately algebraically equivalent if and only if there exists a complete curve Γ, a cycle $\mathbf{Z}$ on $\mathbf{A}\times\Gamma$ and two points $\mathbf{M}, \mathbf{M}'$ on Γ such that $\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{P})$ is defined for any point $\mathbf{P}$ on Γ and $\mathbf{X}\times\mathbf{M}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M})$, $\mathbf{Y}\times\mathbf{M}'=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M}')$.

We say that two cycles $\mathbf{X}, \mathbf{Y}$ are algebraically equivalent——$\mathbf{X}\equiv\mathbf{Y}$ in notation——if and only if there exist a finite number of cycles $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_f$ such that $\mathbf{X}_1=\mathbf{X}$, $\mathbf{X}_f=\mathbf{Y}$ and $\mathbf{X}_i$ is immediately algebraically equivalent to $\mathbf{X}_{i+1}$ for each $i=1, 2, \ldots, f-1$.

PROPOSITION 1. *If* $\mathbf{X}\equiv\mathbf{X}'$, *then* $\delta(\mathbf{X}, \mathbf{Y})=\delta(\mathbf{X}', \mathbf{Y})$.

Proof. It is sufficient to prove this for immediately algebraically equiva-

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lent $\mathbf{X}$, $\mathbf{X}'$. Let Γ be a complete curve and let $\mathbf{Z}$ be a cycle on $\mathbf{A}\times\Gamma$ such that $\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{P})$ is defined for any point $\mathbf{P}$ on Γ and $\mathbf{X}\times\mathbf{M}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M})$, $\mathbf{X}\times\mathbf{M}'=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M}')$ with points $\mathbf{M}$, $\mathbf{M}'$ on Γ. Let Γ' be a non-singular model of Γ and let J be the Jacobian variety of Γ' over a definition field of Γ'. We can replace Γ by Γ'. $S[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]$ defines a function on $\Gamma'\times\mathbf{A}$, where $\mathbf{X}(\mathbf{N})\times\mathbf{N}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{N})$. This function can be extended to a function on $\mathbf{J}\times\mathbf{A}$. Since $S[(\mathbf{X}-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]=0$, it is a homomorphism from $\mathbf{J}\times\mathbf{A}$ into $\mathbf{A}$. A homomorphism from a product of abelian varieties is a sum of homomorphisms from each component. Hence $S[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]=\alpha_1(\varphi(\mathbf{N}))+\alpha_2 t$, where φ is the canonical function from Γ' into $\mathbf{J}$. Since $S[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}-\mathbf{Y})]=0$, $\alpha_1(\varphi(\mathbf{N}))=0$ for all $\mathbf{N}$ on Γ'. This shows that $S[\mathbf{X}(\mathbf{N})\cdot(\mathbf{Y}_t-\mathbf{Y})]=S[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]$, namely $\delta(\mathbf{X}(\mathbf{N}),\mathbf{Y})=\delta(\mathbf{X},\mathbf{Y})$.

Similarly we get

PROPOSITION 2. *If* $\mathbf{X}\equiv\mathbf{X}'$, *then* $\delta(\mathbf{X}\mathbf{Y},\mathbf{Z})=\delta(\mathbf{X}'\mathbf{Y},\mathbf{Z})$.

By repeated application of this proposition we get

PROPOSITION 3. *If* $\mathbf{X}_1\equiv\mathbf{X}'_1$, $\mathbf{X}_2\equiv\mathbf{X}'_2,\ldots,\mathbf{X}_r\equiv\mathbf{X}'_r$, *then* $\delta(\mathbf{X}_1\cdot\mathbf{X}_2\cdot\mathbf{X}_3\cdot\ldots\cdot\mathbf{X}_r,\mathbf{Y})=\delta(\mathbf{X}'_1\cdot\mathbf{X}'_2\cdot\ldots\cdot\mathbf{X}'_r,\mathbf{Y})$.

Since any two points t, s are connected by a finite number of curves on $\mathbf{A}$, $\mathbf{X}_t$ and $\mathbf{X}_s$ are algebraically equivalent.

PROPOSITION 4. $\delta(\mathbf{X},\mathbf{Y})+\delta(\mathbf{Y},\mathbf{X})=deg(\mathbf{X}\cdot\mathbf{Y})\,\delta$[1].

Proof.

$$\begin{aligned}S[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]&=S[\mathbf{X}\cdot\mathbf{Y}_t]-S[\mathbf{X}\cdot\mathbf{Y}]\\&=S[(\mathbf{X}\cdot\mathbf{Y}_t)_{-t}]-(deg(\mathbf{X}\cdot\mathbf{Y}))(-t)-S[\mathbf{X}\cdot\mathbf{Y}]\\&=S[\mathbf{X}_{-t}\cdot\mathbf{Y}]-S[\mathbf{X}\cdot\mathbf{Y}]+(deg(\mathbf{X}\cdot\mathbf{Y}))t\\&=(deg(\mathbf{X}\cdot\mathbf{Y}))t-\delta(\mathbf{Y}\cdot\mathbf{X})t.\end{aligned}$$

From Proposition 3 and Proposition 4 we get:

PROPOSITION 5. *If* $\mathbf{X}_1\equiv\mathbf{X}'_1,\ldots,\mathbf{X}_r\equiv\mathbf{X}'_r$, $\mathbf{Y}_1\equiv\mathbf{Y}'_1,\ldots,\mathbf{Y}_e\equiv\mathbf{Y}'_e$, *then* $\delta(\mathbf{X}_1\cdot\mathbf{X}_2\cdot\ldots\cdot\mathbf{X}_r,\mathbf{Y}_1\cdot\mathbf{Y}_2\cdot\ldots\cdot\mathbf{Y}_e)=\delta(\mathbf{X}'_1\cdot\mathbf{X}'_2\ldots\mathbf{X}'_r,\mathbf{Y}'_1\cdot\mathbf{Y}_2\ldots\mathbf{Y}'_e)$.

Thus $\delta(\mathbf{X},\mathbf{Y})$ depends only on the algebraic equivalence of $\mathbf{X}$ and $\mathbf{Y}$.

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1) δ is the identity endomorphism of $\mathbf{A}$.

PROPOSITION 6. $\delta(X, Y_1 \cdot Y_2 \ldots Y_r) = \delta(X \cdot Y_2 \ldots Y_r, Y_1)$
$+ \ldots + \delta(X \cdot Y_1 \ldots Y_{i-1} \cdot Y_{i+1} \ldots Y_r, Y_i)$
$+ \ldots + \delta(X \cdot Y_1 \ldots Y_{r-1}, Y_r).$

Proof. $\delta(X, Y_1 \cdot Y_2 \ldots Y_r)t = S[X((Y_1 \cdot Y_2 \ldots Y_r)_t - (Y_1 \cdot Y_2 \ldots Y_r))]$

$$= S[X \cdot (Y_{1t} \cdot Y_{2t} \ldots Y_{rt} - Y_{1t} \cdot Y_{2t} \ldots Y_{r-1\,t} \cdot Y_r + Y_{1t} \cdot Y_{2t} \ldots Y_{r-1\,t} \cdot Y_r$$
$$- Y_{1t} \cdot Y_{2t} \ldots Y_{r-2\,t} \cdot Y_{r-1} \cdot Y_r + \ldots + Y_{1t} \cdot Y_2 \ldots Y_r - Y_1 \cdot Y_2 \ldots Y_r)]$$
$$= S[X \cdot Y_{1t} \cdot Y_{2t} \ldots Y_{r-1\,t} \cdot (Y_{rt} - Y_r)]$$
$$+ \ldots + S[X \cdot Y_{1t} \ldots Y_{i-1\,t} \cdot Y_{i+1} \ldots Y_r (Y_{it} - Y_i)]$$
$$+ \ldots + S[X \cdot Y_2 \ldots Y_r \cdot (Y_{1t} - Y_1)]$$
$$= \delta(X \cdot Y_1 \ldots Y_{r-1}, Y_r)t + \ldots + \delta(X \cdot Y_1 \ldots Y_{i-1} \cdot Y_{i+1} \ldots Y_r, Y_i)t$$
$$+ \ldots + \delta(X \cdot Y_2 \ldots Y_r, Y_1)t.$$

PROPOSITION 7. $\delta(X_1, X_2 \cdot X_3 \ldots X_r) + \ldots + \delta(X_i, X_1 \ldots X_{i-1} \cdot X_{i+1} \ldots X_r)$
$+ \ldots + \delta(X_r, X_1 \cdot X_2 \ldots X_{r-1}) = (r-1)\, deg\,(X_1 \ldots X_r)\,\delta.$

Proof. We shall prove this by induction on r. We assume that this is true for $r-1$ and we put $Y = X_{r-1} \cdot X_r$. Then

$$\delta(X_1, X_2 \ldots X_{r-2} \cdot Y) + \ldots + \delta(X_i, X_1 \ldots X_{i-1} \cdot X_{i+1} \ldots X_{r-2} \cdot Y)$$
$$+ \ldots + \delta(X_{r-2}, X_1 \ldots X_{r-3} \cdot Y) + \delta(Y, X_1 \ldots X_{r-2})$$
$$= (r-2)\, deg\,(X_1 \ldots X_{r-2} \cdot Y)\,\delta = (r-2)\, deg\,(X_1 \ldots X_r)\,\delta.$$

On the other hand, by virtue of Proposition 4 and Proposition 5,

$$\delta(Y, X_1 \ldots X_{r-2}) = \delta(X_{r-1} \cdot X_r, X_1 \ldots X_{r-2})$$
$$= deg\,(X_1 \ldots X_r)\,\delta - \delta(X_1 \ldots X_{r-2}, X_{r-1} \cdot X_r)$$
$$= deg\,(X_1 \ldots X_r)\,\delta - \delta(X_1 \ldots X_{r-2} \cdot X_{r-1}, X_r) - \delta(X_1 \ldots X_{r-2} \cdot X_r, X_{r-1})$$
$$= deg\,(X_1 \ldots X_r)\,\delta - (deg\,(X_1 \ldots X_r)\,\delta - \delta(X_r, X_1 \ldots X_{r-1}))$$
$$- (deg\,(X_1 \ldots X_r)\,\delta - \delta(X_{r-1}, X_1 \ldots X_{r-2} \cdot X_r))$$
$$= \delta(X_{r-1}, X_1 \ldots X_{r-2} \cdot X_r) + \delta(X_r, X_1 \ldots X_{r-1}) - deg\,(X_1 \ldots X_r)\,\delta.$$

Hence we have

$$\delta(X_1, X_2 \ldots X_r) + \ldots + \delta(X_i, X_1 \ldots X_{i-1} \cdot X_{i+1} \ldots X_r)$$
$$+ \ldots + \delta(X_r, X_1 \ldots X_{r-1}) = (r-1)\, deg\,(X_1 \cdot X_2 \ldots X_r)\,\delta.$$

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The case $r=1$ is Proposition 4 itself.

COROLLARY 1. *If* $\mathbf{X}$ *is a non-degenerate divisor, then*

$$\delta(\mathbf{X}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-1})^{2)} = (n-1)/n \, deg(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{n})\delta.$$

COROLLARY 2. *If* $\mathbf{X}$ *is a non-degenerate divisor, then*

$$\delta(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{r}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-r}) = (n-r)/n \, deg(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{n})\,\delta.$$

§ 2. Case of Jacobian variety

Let Γ be a non-singular curve of genus n defined over k, let $\mathbf{J}$ be the Jacobian variety of Γ and let φ be the canonical function from Γ into $\mathbf{J}$. We assume that $\mathbf{J}$ and φ are also defined over k. We denote by $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \ldots, \mathbf{W}^{(n-1)}$ the locuses of $\varphi(\mathbf{P}_1)$, $\varphi(\mathbf{P}_1)+\varphi(\mathbf{P}_2), \ldots, \varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_{n-1})$, respectively over k, where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{n-1}$ are independent generic points over k of Γ. We denote by $\mathbf{W}'^{(1)}, \ldots, \mathbf{W}'^{(n-1)}$ the locuses of $-\varphi(\mathbf{P}_1)$, $-\varphi(\mathbf{P}_1)-\varphi(\mathbf{P}_2), \ldots, -\varphi(P_1)-\ldots-\varphi(\mathbf{P}_{n-1})$, respectively, over k.

PROPOSITION 8. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}'^{(r)}) = \binom{n-1}{r}\delta.$

Proof. Let $x=\varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_n)$ where $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are independent generic points of Γ over k. Then by virtue of Proposition 17, N° 40, § V, [1], $\mathbf{W}^{(n+r)}\mathbf{W}'^{(r)}_x = \sum_{(i)}(w_{i_1 i_2\ldots i_{n-r}})$, where $w_{i_1 i_2\ldots i_{n-1}} = \varphi(\mathbf{P}_{i_1})+\varphi(\mathbf{P}_{i_2})+\ldots+\varphi(\mathbf{P}_{i_{n-r}})$ and the summation which means the summation as cycles of dimension zero runs over all $\binom{n}{n-r}$ combinations of indices $1, 2, \ldots, n$. Hence

$$\begin{aligned} S[\mathbf{W}^{(n-r)}(\mathbf{W}'^{(r)}_x-\mathbf{W}^{(r)})] &= \sum_{i_1<i_2<\ldots<i_{n-r}}(\varphi(\mathbf{P}_{i_1})+\ldots+\varphi(\mathbf{P}_{i_{n-r}})) \\ &= \frac{n-r}{n}\binom{n}{n-r}[\varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_n)] = \binom{n-1}{r}x. \end{aligned}$$

LEMMA. $(n-r)\mathbf{W}^{(r)} = \mathbf{W}^{(r+1)}\cdot\theta_a{}^{3)}$ *with suitable* a.

This is proved in a similar manner as in the proof of Proposition 17, N° 40, § V, [1].

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2) $\delta(\mathbf{X}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-1})$ means $\delta(\mathbf{X}, \mathbf{X}_{a_1}\cdot\mathbf{X}_{a_2}\ldots\mathbf{X}_{a_{n-1}})$ with suitable $a_1, a_2, \ldots, a_{n-1}$ which is independent of the choice of $a_1, \ldots, a_{n-1}$.

3) θ means $\mathbf{W}^{(n-1)}$.

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Consequently we get:

PROPOSITION 9. $(n-r)\mathbf{W}^{(r)}=\theta\cdot\theta_{a_1}\ldots\theta_{a_{n-r-1}}$ *with suitable* $a_1,\ldots,a_{n-r-1}$.

PROPOSITION 10. $\delta(\mathbf{W}^{(n-r)},\mathbf{W}^{(r)})=\delta(\mathbf{W}^{(n-r)},\mathbf{W}'^{(r)})=\binom{n-1}{r}\delta.$

Proof. $\delta(\mathbf{W}^{(n-r)},\mathbf{W}^{(r)})=\frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-1)},\overbrace{\theta\ldots\theta}^{n-r})$

$$=\frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-r)},\overbrace{\theta_{-c}\ldots\theta_{-c}}^{n-r})=\frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-r)},\overbrace{\theta'\ldots\theta'}^{n-r})$$

$$=\delta(\mathbf{W}^{(n-r)},\mathbf{W}'^{(r)})=\binom{n-1}{r}\delta,$$

where $c=\varphi(\mathfrak{K})$ with a canonical divisor $\mathfrak{K}$.

PROPOSITION 11. $\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)},\mathbf{W}^{(2n-r-s)})$

$$=\frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)},\mathbf{W}^{(2n-r-s)}).$$

Proof. $\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)},\mathbf{W}^{(2n-r-s)})=\frac{1}{(n-r)!\,(n-s)!}(\overbrace{\theta\ldots\theta}^{2n-s-r},\mathbf{W}^{(2n-s-r)})$

$$=\frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)},\mathbf{W}^{(2n-s-r)}).$$

Similarly we get

PROPOSITION 12. $\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)}\mathbf{X},\mathbf{Y})=\frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)}\mathbf{X},\mathbf{Y}).$

PROPOSITION 13. *Let* $\mathbf{C}$ *be a cycle of dimension one. Then* $\delta(\mathbf{C},\theta)$, $\delta(\theta,\mathbf{C})$ *are symmetric.*

Proof. It is sufficient to show this for simple irreducible curves. Let Γ_1 be a non-singular model of $\mathbf{C}$ and let f be the birational correspondence from Γ_1 to $\mathbf{C}$ regular at each simple point of $\mathbf{C}$. Let $\mathbf{J}_1$ be the Jacobian variety of Γ_1 and φ be the canonical function of Γ_1 into $\mathbf{J}_1$. We denote by λ the extension of f onto a homomorphism from $\mathbf{J}_1$ into $\mathbf{J}$. Then $\mathbf{S}[\mathbf{C}\cdot(\theta_t-\theta)]$ $=\lambda\mathbf{S}[\varphi_1(\mathbf{P}_{r\Gamma_1}\wedge_f\Gamma_1\times(\theta_t-\theta))]=\lambda\lambda_0' t$, where $\wedge_1$ is the graph of f in $\Gamma_1\times\mathbf{J}$. By virtue of formulae in N° 77, § XI, [1],

$$\mathbf{M}_l((\lambda\lambda_0')')=\mathbf{E}_l(\theta)^{-1\,t}\mathbf{M}_l(\lambda\lambda_0')\,\mathbf{E}_l(\theta)=\mathbf{E}_l(\theta)^{-1\,t}\mathbf{M}_l(\lambda_0')\,{}^t\mathbf{M}_l(\lambda)\,\mathbf{E}_l(\theta)$$
$$=\mathbf{E}_l(\theta)^{-1\,t}(\mathbf{E}_l(\theta)^{-1\,t}\mathbf{M}_l(\lambda)\,\mathbf{E}_l(\theta))\,{}^t\mathbf{M}_l(\lambda)\,\mathbf{E}_l(\theta)$$
$$=\mathbf{E}_l(\theta)^{-1\,t}\mathbf{E}_l(\theta)\,\mathbf{M}_l(\lambda)\,{}^t\mathbf{E}_l(\theta)^{-1\,t}\mathbf{M}_l(\lambda)\,\mathbf{E}_l(\theta)$$

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$$= \mathbf{E}_l(\theta)^{-1}(-\mathbf{E}_l(\theta))\mathbf{M}_l(\lambda)(-\mathbf{E}_l(\theta_1))^{-1\,t}\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)$$
$$= \mathbf{M}_l(\lambda)\mathbf{E}_l(\theta_1)^{-1\,t}\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta) = \mathbf{M}_l(\lambda\lambda'_\theta)$$

where θ_1 is the theta divisor of $\mathbf{J}_1$. This shows that $\delta(\mathbf{C}, \theta)$ is symmetric. Since $\delta(\theta, \mathbf{C}) = (deg\,\mathbf{C}\cdot\theta)\delta - \delta(\mathbf{C}, \theta)$, $\delta(\theta, \mathbf{C})$ is also symmetric.

PROPOSITION 14. *Let* $\mathbf{X}$ *be a cycle of dimension* r. *Then*

$$\delta(\mathbf{X}, \mathbf{W}^{(n-r)}) = \delta(\mathbf{X}\cdot\mathbf{W}^{(n-r+1)}, \theta).$$

Proof. $\delta(\mathbf{X}, \mathbf{W}^{(n-r)}) = \delta\Big(\mathbf{X}, \frac{1}{r!}\overbrace{\theta\ldots\theta}^{r}\Big) = \frac{1}{r!}\delta(\mathbf{X}, \overbrace{\theta\ldots\theta}^{r})$

$$= \frac{1}{r!}r\delta(\mathbf{X}, \overbrace{\theta\ldots\theta}^{r-1}) = \frac{1}{(r-1)!}\delta(\mathbf{X}\overbrace{\theta\theta\ldots\theta}^{r-1}, \theta) = \delta(\mathbf{X}\mathbf{W}^{(n-r+1)}, \theta).$$

From this proposition and Proposition 13 we get

THEOREM 1. $\delta(\mathbf{X}, \mathbf{W}^{(n-r)})$, $\delta(\mathbf{W}^{(n-r)}, \mathbf{X})$ *are symmetric.*

PROPOSITION 15. $\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\mathbf{W}^{(2)})\delta - \lambda'\lambda$

$$= \frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda.$$

Proof.

$$(\delta(\lambda^{-1}(\theta))\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\mathbf{W}^{(2)}\theta) - \delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \delta(\theta\cdot\mathbf{W}^{(2)}, \lambda^{-1}(\theta)) - \delta(\theta\cdot\lambda^{-1}(\theta), \mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - (n-1)\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) - \delta(\theta\cdot\lambda^{-1}(\theta), \mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - (n-1)\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) - (n-2)\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta).$$

Hence

$$\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta))$$
$$= \frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda^{4)}.$$

PROPOSITION 16. $\delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)}) = \dfrac{(n-2)}{2}\sigma(\lambda'\lambda)\delta + \lambda'\lambda.$

Proof. $\delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)}) = (deg\,\theta\lambda^{-1}(\theta)\mathbf{W}^{(2)}) - \delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta)$

$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \Big(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\Big)$$
$$= \frac{(n-1)}{2}\sigma(\lambda'\lambda)\delta - \frac{1}{2}\sigma(\lambda'\lambda)\delta + \lambda'\lambda$$

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4) $\sigma(\alpha) = \text{Spur}\, M_l(\alpha)$. See N° 49, § VI, [1].

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$$= \lambda'\lambda + \frac{(n-2)}{2}\sigma(\lambda'\lambda)\delta.$$

PROPOSITION 17. $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)})$

$$= \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right).$$

Proof. $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)}) = (deg\,\mathbf{W}^{(r+1)}\mathbf{W}^{(n-r)}\lambda^{-1}(\theta))\delta$

$$- \delta(\mathbf{W}^{(n-r)}\lambda^{-1}(\theta), \mathbf{W}^{(r+1)}) - \delta(\mathbf{W}^{(n-r)}\mathbf{W}^{(r+1)}\lambda^{-1}(\theta))$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}(deg\,\mathbf{W}^{(1)}\lambda^{-1}(\theta))\delta - \delta(\mathbf{W}^{(n-r)}\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \theta)$$

$$- \frac{(n-1)!}{r!\,(n-r-1)!}\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) = \frac{(n-1)!}{2r!\,(n-r-1)!}\sigma(\lambda'\lambda)\delta$$

$$- \frac{1}{r}\delta(\mathbf{W}^{(n-r+1)}\theta\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \theta) - \frac{(n-1)!}{r!\,(n-r-1)!}\lambda'\lambda$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right) - \frac{1}{r}\delta(\theta\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \mathbf{W}^{(n-r)})$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$$

$$- \frac{(n-r-1)}{r}\delta(\lambda^{-1}(\theta)\mathbf{W}^{(r+1)}, \mathbf{W}^{(n-r)}).$$

Hence $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)}) = \dfrac{(n-2)!}{r!\,(n-r-1)!}\left(\dfrac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$

$$= \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right).$$

PROPOSITION 18. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r+1)}\lambda^{-1}(\theta)) = \binom{n-2}{r-1}\lambda'\lambda + \binom{n-2}{r}\dfrac{\sigma(\lambda'\lambda)}{2}.$

Proof. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r+1)}\lambda^{-1}(\theta))$

$$= (deg\,\mathbf{W}^{(n-r)}\mathbf{W}^{(r+1)}\lambda^{-1}(\theta))\delta - \delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)})$$

$$= \frac{(n-1)!}{2r!\,(n-r-1)!}\sigma(\lambda'\lambda)\delta - \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$$

$$= \binom{n-2}{r-1}\lambda'\lambda + \binom{n-2}{r}\frac{\sigma(\lambda'\lambda)}{2}.$$

PROPOSITION 19. $\delta(\lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \mathbf{W}^{(r)})$

$$= a_0\delta - \sum_i a_i\lambda_i'\lambda_i + \sum_{i_1, i_2, i_1 \neq i_2} a_{i_1 i_2}\lambda_{i_2}'\lambda_{i_1}'\lambda_{i_1}\lambda_{i_2}$$

$$- \ldots \pm \sum_{i_1, i_2, \ldots, i_r, i_j \neq i_k} a_{i_1 i_2, \ldots, i_r}\lambda_{i_r}'\ldots\lambda_{i_1}'\lambda_{i_1}\ldots\lambda_{i_r}$$

where $a_{i_1 i_2 \ldots i_h} = deg\,\lambda_{j_1}^{-1}(\theta)\ldots\lambda_{j_{r-h}}^{-1}(\theta)\cdot\mathbf{W}^{(r-h)}\,i_1, i_2, \ldots, i_h \neq j_1, \ldots, j_h.$

Proof. $\delta(\lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \mathbf{W}^{(r)})$

$$= a_0\delta - \delta(\mathbf{W}^{(r)}, \lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta))$$

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$$= a_0\delta - \sum_i \delta(\mathbf{W}^{(r)} \cdot \lambda_1^{-1}(\theta)\ldots\lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \lambda_i^{-1}(\theta))$$
$$= a_0\delta - \sum_i \lambda_i'\delta(\mathbf{W}^{(r)}\lambda_1^{-1}(\theta)\ldots\lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \theta)$$
$$= a_0\delta - \sum_i \lambda_i'\delta(\lambda_1^{-1}(\theta)\ldots\lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \mathbf{W}^{(r-1)}).$$

If this proposition is true for $r-1$, then by virtue of the above calculations it is also true for r. It is clearly true for $r=1$. Hence our proof is finished.

Consequently we have

PROPOSITION 20. $\delta(\mathbf{W}^{(r)}, \lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta))$
$$= \sum_i a_i\lambda_i'\lambda_i - \sum_{i_1, i_2, i_1 \neq i_2} a_{i_1 i_2}\lambda_{i_2}'\lambda_{i_1}'\lambda_{i_1}\lambda_{i_2} - \cdots \pm \sum_{i_1, i_2, \ldots, i_r\, i_j \neq i_k} a_{i_1 i_2, \ldots, i_r}\lambda_{i_r}'\ldots\lambda_{i_1}'\lambda_{i_1}\ldots\lambda_{i_r},$$

where $a_{i_1 i_2 \ldots i_h} = deg\,\lambda_{j_1}^{-1}(\theta)\ldots\lambda_{j_{r-h}}^{-1}(\theta)\mathbf{W}^{(r-h)}\; i_1, i_2, \ldots, i_h \neq j_1, \ldots, j_h.$

From these propositions, we get

THEOREM 2. *Let r be an arbitrary integer with $1 \leqq r < n$. Then there exists an integer c such that for all symmetric element α there exists a cycle* $\mathbf{X}$ *of dimension r satisfying*

$$c\alpha = \delta(\mathbf{W}^{(n-r)}, \mathbf{X}).$$

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