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MULTIPLICATIVITY OF THE UNIFORM NORM AND INDEPENDENT FUNCTIONS

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It has long been known that there is a close connection between stochastic independence of continuous functions on an interval and space-filling curves [**9**]. In fact any two nonconstant continuous functions on $[0, 1]$ which are independent relative to Lebesgue measure are the coordinate functions of a space filling curve. (The results of Steinhaus [**9**] have apparently been overlooked in more recent work in this area [**3**, **5**, **6**].)

The purpose of this note is to establish a connection with a multiplicative property for the uniform norm on function spaces. Only the uniform norm will be considered on $C(\Omega)$, the space of continuous real valued functions on a compact Hausdorff space Ω. If $f \in C(\Omega)$ then we define $Q(f)$ to be the linear span of $\{1, f, f^2\}$, that is the space of quadratic polynomials in f. $\Omega(f)$ is known to play a role in the isometric theory of $C(\Omega)$. For example, if f separates points of Ω then $Q(f)$ is a *Korovkin set*; so if φ is a contraction on $C(\Omega)$ which fixes $Q(f)$ then φ is the identity map [**1**, **7**, **8**].

PROPOSITION 1. *Let Ω be a compact Hausdorff space, let $f_j \in C(\Omega)$ $(j = 1, 2)$ and define $\gamma\colon \Omega \to \mathbb{R}^2$ by $\gamma(t) = (f_1(t), f_2(t))$. The following statements are equivalent:*

(a) $\gamma(\Omega) = f_1(\Omega) \times f_2(\Omega)$,
(b) $\|g_1 g_2\| = \|g_1\|\,\|g_2\|$ *for all* $g_j \in Q(f_j)\,(j = 1, 2)$.

PROOF: (b) $\Rightarrow$ (a). Let $(x_1, x_2) \in f_1(\Omega) \times f_2(\Omega)$. There exist quadratic polynomials p_j such that $p_j(x_j) = 1$ and $|p_j(x)| < 1$ for $x \in f_j(\Omega) \setminus \{x_j\}$. Let $g_j = p_j \circ f_j$, so that $g_j \in Q(f_j)$ $(j = 1, 2)$.

By hypothesis, $1 = \|g_1\|\,\|g_2\| = \|g_1 g_2\|$. It follows that there is a point $t \in \Omega$ such that $|g_1(t) g_2(t)| = 1$, whence $f_1(t) = x_1$ and $f_2(t) = x_2$. This proves (a).

(a) $\Rightarrow$ (b). Any quadratic polynomial p_j attains its maximum absolute value on $f_j(\Omega)$ at some point x_j. Our assumption implies that there is a point $t \in \Omega$ with $f_j(t) = x_j$ $(j = 1, 2)$. Writing $g_j = p_j \circ f_j$ we see that $|g_1(t) g_2(t)|$ takes the value $\|g_1\|\,\|g_2\|$, and the result follows. ∎

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REMARK. Simple examples show that the condition $\|f_1 f_2\| = \|f_1\| \, \|f_2\|$ is not sufficient to imply (a).

We now show how independence of functions gives rise to the multiplicative property of the norm. A Borel measure μ on Ω is said to be *faithful* if $\mu(G) > 0$ for each nonempty open subset G of Ω. We refer to [**2**] for details on the notion of independence.

PROPOSITION 2. *Let Ω be a compact Hausdorff space and μ a faithful Borel probability measure on Ω. Let f_1 and f_2 be functions on Ω that are μ-independent and such that $|f_1|$ and $|f_2|$ are lower semicontinuous. Then $\|f_1 f_2\| = \|f_1\| \, \|f_2\|$.*

PROOF: We may suppose that f_1 and f_2 are both nonzero.

Let $\varepsilon > 0$ and let $V_j = \{t \in \Omega\colon |f_j(t)| > \|f_j\| - \varepsilon\}$. Then $\mu(V_j) > 0$, since V_j is a nonempty open set $(j = 1, 2)$. By independence of $|f_1|$ and $|f_2|$, we have

$$\mu(V_1 \cap V_2) = \mu(V_1)\mu(V_2) > 0.$$

In particular $V_1 \cap V_2 \neq \emptyset$. It follows that there exists $t \in \Omega$ such that $|f_j(t)| > \|f_j\| - \varepsilon$ $(j = 1, 2)$. Therefore $\|f_1 f_2\| > (\|f_1\| - \varepsilon)(\|f_2\| - \varepsilon)$. Since $\varepsilon > 0$ was arbitrary this proves the result. ∎

Now if we suppose that f_1, $f_2 \in C[0, 1]$ are independent relative to Lebesgue measure then the same is true for quadratic polynomials in f_1 and f_2, so Proposition 2 shows that condition (b) of Proposition 1 is satisfied. Thus if f_1 and f_2 are nonconstant then $\gamma[0, 1] = f_1[0, 1] \times f_2[0, 1]$ is a rectangle in $\mathbb{R}^2$. (Note that the result of [**3**] is a simple consequence of this fact.) Peano's original space-filling curve arises in this way [**9, 6**]. In fact [**6**] gives a detailed proof that the n-dimensional version of Peano's curve is measure preserving and hence has independent coordinate functions.

The step from independent coordinate functions to measure preserving mappings is often a small one.

PROPOSITION 3. *Let $f, g \in C[0, 1]$ be independent and have continuous distribution functions F, G respectively. Then the function $\varphi(t) = (F(f(t)), G(g(t)))$ defines a continuous measure preserving transformation of $[0, 1]$ onto $[0, 1] \times [0, 1]$.*

PROOF: $F \circ f$ and $G \circ g$ are uniformly distributed over $[0, 1]$, by [**2**, p.169 Exercise 14.4], and they are also independent since f and g are. The result follows easily. ∎

In conclusion we recall that space filling curves arise naturally in functional analysis whenever we have a Hilbert space embedded isometrically in $C[0, 1]$, [**4, 10**]. It is therefore interesting to note that the multiplicative norm condition fails in this case.

PROPOSITION 4. *Let $H \subset C[0, 1]$ be a real Hilbert space, and let $f, g \in H$ be linearly independent. Then $\|fg\| < \|f\| \, \|g\|$.*

PROOF: The proof of [4] shows that if there is a point $t_0 \in [0, 1]$ such that $|f(t_0)| = \|f\|$ and $|g(t_0)| = \|g\|$ then f and g must be linearly dependent. ∎

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