ON SOME CLASSES OF WEIGHTED COMPOSITION OPERATORS

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Introduction. Let (X, Σ, μ) denote a complete σ-finite measure space and $T: X \rightarrow X$ a measurable ($T^{-1}A \in \Sigma$ for each $A \in \Sigma$) point transformation from X into itself with the property that the measure $\mu \circ T^{-1}$ is absolutely continuous with respect to μ. Given any measurable, complex-valued function $w(x)$ on X, and a function f in $L^2(\mu)$, define $W_T f(x)$ via the equation

$$W_T f(x) = w(x) f(Tx). \tag{1}$$

Known regularity conditions (given below) are both necessary and sufficient for the linear transformation $f \rightarrow W_T f$ to define a bounded operator on $L^2(\mu)$; such operators are called *weighted composition operators*. In case $w(x) = 1$ the operator C_T defined via composition with T is simply a *composition operator*. In this paper we characterize the normal, quasi-normal and hermitian weighted composition operators in terms of w, T, and $d\mu \circ T^{-1}/d\mu$. Some known results on seminormal composition operators and weighted composition operators are obtained as corollaries. We given an example of quasinormal W_T with non-constant weight which is not normal. Campbell and Dibrell [**2**] give sufficient conditions for a composition operator C_T to be *power hyponormal*; that is, for $(C_T)^n$ to be hyponormal for all natural numbers n. We give a sufficient condition for W_T to be power hyponormal, generalizing in a natural way the corresponding result for weighted shifts on the integers.

Preliminaries. To avoid semantic complexities we take $T^{-1}\Sigma$ as the relative completion of the σ-algebra generated by $\{T^{-1}A : A \in \Sigma\}$. If $f \in L^p$ $(p \geq 1)$ or f is non-negative and measurable, there exists a unique (a.e.) $T^{-1}\Sigma$ measurable function F whose integral over $T^{-1}\Sigma$ measurable sets agrees with the integral of f over the same sets, whenever the integral of f over such a set converges. Following Lambert ([**4**]) we refer to F as the conditional expectation of f with respect to $T^{-1}\Sigma$, and write $F = E(f \mid T^{-1}\Sigma)$, or simply $E(f)$. Let $h = d\mu \circ T^{-1}/d\mu$; we always assume that h is finite-valued a.e. An arbitrary function f is $T^{-1}\Sigma$ measurable if and only if there is some Σ-measurable function g so that $g \circ T = f$ a.e. We would like to point out, however, that this function g need not be unique (a.e.) even if T is surjective. Indeed, if H denotes the support of h, and k is any measurable function, then $k \circ T = 0$ a.e. if and only if k is supported on $X - H$. Hence the equation $g \circ T = f$ a.e. has a unique a.e. solution g for each $T^{-1}\Sigma$ measurable function if if and only if $h > 0$ a.e. on X. In particular, *property* ($E4$) *of* [**4**, p. 396] *is false*. In fact, we have the following example.

Example. Let $X = [0, 1]$ equipped with Lebesgue measure m on the Lebesgue measurable sets, and let C denote the Cantor set in X. If Q is any measurable transformation mapping C bijectively onto $[1/2, 1]$ and S is any measurable transformation mapping the complement of C onto $[0, 1/2)$ so that $m \circ S^{-1}$ is absolutely continuous with respect to m, then define $T: X \rightarrow X$ by $Tx = Qx$, $x \in C$, and $Tx = Sx$, $x \notin C$. This T is measurable, surjective, satisfies $m \circ T^{-1} \ll m$, and $h = 0$ on $[1/2, 1]$.

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Such examples are important both here and in **[4]** because of possible interplay between the weight and the Radon–Nikodym derivative in the weighted case. However we note that if we require a solution g of $g \circ T = f$ to be supported in H, then g is (a.e.) uniquely determined. For $T^{-1}\Sigma$-measurable functions f, we define $f \circ T^{-1}$ as the unique (a.e.) Σ-measurable function g, supported in H, which satisfies $g \circ T = f$ a.e. (One should not interpret this as implying the invertibility of T as a transformation.) With this convention the following change of variables formula holds for measurable functions $p(x)$:

$$\int_X p \, d\mu = \int_X hE(p) \circ T^{-1} \, d\mu,$$

in the sense that if one of the integrals exists then so does the other, and they have the same value.

The transformation W_T is a bounded operator on L^2 if and only if $\|hE(w^2) \circ T^{-1}\|_\infty$ is finite, and in this case the operator norm of W_T is related to this function norm by $\|W_T\|^2 = \|hE(w^2) \circ T^{-1}\|_\infty$. In case $w(x) = 1$, C_T induces a bounded operator precisely when $h \in L^\infty$, and then $\|C_T\| = \|h\|_\infty^{1/2}$. When this occurs, the closure of the range of C_T consists of those L^2 functions which are $T^{-1}\Sigma$ measurable, and $E: L^2 \to L^2$ is the projection whose range is the closure of the range of C_T. One of the reasons weighted composition operators are interesting is that many natural and apparently innocuous measurable transformations T do not induce bounded composition operators (for example $x \to x^2$ on $[0, 1]$), but it us usually easy to weight them to make the resultant weighted operator bounded. If f and g are arbitrary $T^{-1}\Sigma$-measurable functions then $(fg) \circ T^{-1} = (f \circ T^{-1})(g \circ T^{-1})$. If f and g are Σ-measurable functions for which $E(f)$ and $E(fg \circ T)$ are defined we have $E(fg \circ T) = g \circ TE(f)$. If $B \in \Sigma$, we define Σ_B as $\{C \cap B : C \in \Sigma\}$, and $L^2(B)$ as those Σ-measurable functions supported on B whose modulus squared has a finite integral over B.

Results.

THEOREM 1. *Let A be the support of $w(x)h(Tx)$. Then W_T is normal if and only if*
(i) *A = support of w,*
(ii) *$T^{-1}A = A$ and $T^{-1}\Sigma_A \cap \Sigma_A = \Sigma_A$, and*
(iii) *w is $T^{-1}\Sigma$ measurable and $h \circ T\,|w|^2 = h\,|w|^2 \circ T^{-1}$ a.e.*

The statement that w is $T^{-1}\Sigma$ measurable in part (iii) actually follows from (i) and (ii), but we include it for emphasis.

For ease during calculation, we assume that $w(x) \geq 0$. The general complex case is easily deduced from our calculations. The proof in the case when A is all of X is enlightening so we will present it first. It follows from the following lemma.

LEMMA 1. *W_T has dense range if and only if $\mu\{w = 0\} = 0$ and $T^{-1}\Sigma = \Sigma$.*

Proof. Suppose that W_T has dense range. If B is a measurable set of finite measure on which w is zero, then $L^2(B)$ is orthogonal to the range of W_T and hence B must have zero measure. Since X is σ-finite we have $\mu\{w = 0\} = 0$. Let $B \in \Sigma$ have finite measure.

Since W_T has dense range we may find a sequence $\{f_n\}$ of L^2 functions so that $\lim_{n\to\infty} W_T f_n = \chi_B$, both in L^2 and a.e. Since $\mu\{w=0\}=0$, we have $f_n \circ T \to w^{-1}\chi_B$ a.e. Thus on any set B of finite measure $w^{-1}\chi_B$ is $T^{-1}\Sigma$ measurable; hence w^{-1} is $T^{-1}\Sigma$ measurable and therefore so is w. Since $\chi_B = \lim W_T f_n$ a.e., B is also $T^{-1}\Sigma$ measurable so that $T^{-1}\Sigma = \Sigma$.

Suppose that $\mu\{w=0\}=0$ and $T^{-1}\Sigma=\Sigma$. Let $\sigma_m = \{w > 1/m\} \in \Sigma$. Let $B \in \Sigma$ have finite measure, set $B_m = B \cap \sigma_m$ and $F_m = w^{-1}\chi_{B_m}$. We claim there exist sequences $\{f_{m,n}\}$ in L^2 so that $\lim_{n\to\infty} f_{m,n} \circ T = F_m$ in L^2. (This would be immediate if C_T were bounded.) Define $p_m(x) = [(\chi_{B_m}) \circ T^{-1}](x)$, and let P_m denote the support of p_m, so that (i) $P_m \subseteq H$ (the support of h) and (ii) $T^{-1}P_m = B_m$. We have $0 = \mu\{\chi_{B_m} w < 1/m\} = \mu T^{-1}\{p_m < 1/m\}$; since $h > 0$ on P_m, this implies $\mu\{p_m < 1/m\} = 0$. Let $P_{m,n}$ be an increasing sequence of measurable sets, each of finite measure, whose union is P_m. Then $f_{m,n} = \chi_{P_{m,n}}/p_m$, is a sequence of L^2-functions which satisfies

$$\|f_{m,n} \circ T - F_m\|_2^2 = \int_X \left|\frac{1}{w}\right|^2 (\chi_{P_m} - \chi_{P_{m,n}})^2 \circ T \, d\mu$$

$$\leq m^2 \mu(T^{-1}P_m - T^{-1}P_{m,n}),$$

which converges to 0 as $n \to \infty$. Then $W_T f_{m,n} \to \chi_{B_m}$ in L^2, so that χ_{B_m} and hence χ_B are in the closure of the range of W_T and W_T has dense range.

COROLLARY 1. *Suppose that W_T has dense range. Then W_T is normal if and only if $h \circ T w^2 = h w^2 \circ T^{-1}$ a.e.*

Proof. By calculating $W_T W_T^* f$ and $W_T W_T f$ we see that W_T is normal if and only if $wh \circ TE(wf) = hE(w^2) \circ T^{-1} f$ a.e., for each $f \in L^2$. Since W_T has dense range, Lemma 1 implies that E is the identity map on measurable, conditionable functions, and the result follows by allowing f to vary through the characteristic functions of sets of finite measure.

The idea behind the proof of Theorem 1 is to show that the action of W_T may be localized to $L^2(A)$, and then apply Lemma 1 and Corollary 1.

Proof of Theorem 1. Suppose first that W_T is normal. For each $f \in L^2$ write $f = \chi_A f + \chi_{X-A} f$ so that

$$W_T^* W_T f = hE(w^2) \circ T^{-1}[\chi_A f + \chi_{X-A} f], \tag{2}$$

$$W_T W_T^* f = wh \circ T[E(w\chi_A f) + E(w\chi_{X-A} f)]. \tag{3}$$

The right-hand side of (3) is 0 a.e. on $X - A$ and by normality so is the r.h.s. of (2). Because we may choose $f \in L^2$ which is positive on $X - A$ we must have $hE(w^2) \circ T^{-1} = 0$ a.e. on $X - A$, i.e. $\operatorname{supp} hE(w^2) \circ T^{-1} \subseteq A$. On the other hand it is elementary to calculate that $\operatorname{supp} w \subseteq \operatorname{supp} hE(w^2) \circ T^{-1}$. Since $A \subseteq \operatorname{supp} w$ all these inclusions must be equalities. This proves (i).

Now we may rewrite (2) and (3) in their equivalent forms

$$W_T^*W_Tf = hE(w^2)\circ T^{-1}[\chi_A f], \tag{2$'$}$$

$$W_TW_T^*f = wh\circ T[E(w\chi_A f)]. \tag{3$'$}$$

Observe that $L^2(A)$ is reducing for W_T; we claim W_T has dense range in $L^2(A)$. This may be seen by setting $J(x) = [hE(w^2)\circ T^{-1}](x)$ and $A_n = \{J > 1/n\}$. Then $A = \bigcap_1^\infty A_n$, for each $f \in L^2(A)$, $\chi_{A_n}f \to f$ in L^2, and $g_n = \chi_{A_n}f/J$ is in $L^2(A)$ for every n. But $W_T^*W_Tg_n = W_TW_T^*g_n = \chi_{A_n}f$, so that f is in the closure of the range of W_T, and W_T has dense range in $L^2(A)$; by applying Lemma 1 we see that $T^{-1}\Sigma = \Sigma_A$, so that (ii) holds. It follows that for each Σ-measurable function g supported on A for which $E(g)$ may be defined we have $E(g) = g$; and in particular w is $T^{-1}\Sigma$ measurable. Thus for each $f \in L^2$

$$W_T^*W_Tf = hw^2\circ T^{-1}f, \tag{2$''$}$$

$$W_TW_T^*f = w^2h\circ Tf. \tag{3$''$}$$

(iii) follows by letting f vary through characteristic functions of measurable subsets of finite measure. Now suppose that (i) (ii) and (iii) hold. (ii) implies that if a measurable, conditionable function f is supported in either $X - A$ or A then so is $E(f)$; and if f is supported in A, $E(f) = f$. Thus for each $f \in L^2$,

$$W_T^*W_Tf = hw^2\circ T^{-1}f = hw^2\circ T^{-1}\chi_A f,$$

and

$$W_TW_T^*f = wh\circ TE(w\chi_A f) + E(w\chi_{X-A}f) = w^2h\circ T\chi_A f.$$

Normality follows immediately from (iii).

Special Cases: In case $w = 1$ we are considering a composition operator C_T and Theorem 1 specializes to the following result.

COROLLARY 2. *C_T is normal if and only if*
(i) *$T^{-1}\Sigma = \Sigma$, and*
(ii) *$h = h\circ T > 0$ a.e.*

This is the content of Lemma 2 of Whitley [**6**], although he does not include the positivity of h in his statement (it follows from the proof).

COROLLARY 3 (BASTIAN [**1**]). *Suppose that T is measure preserving.*

(a) *If T is ergodic and non-invertible, then W_T is not normal for any (non-zero) choice of w.*

(b) *If $\mu(X) < \infty$ and T is invertible, then W_T is hyponormal if and only if W_T is normal.*

Proof. (a) By Theorem 1, W_T is normal iff w is $T^{-1}\Sigma$ measurable and (apply C_T to both sides of the equation appearing in condition (iii)) $|w|^2$ is invariant under C_T. Since T is ergodic, $|w|^2$ must be constant. Since w is nonzero, A must be all of X. But T is non-invertible and measure-preserving;hence $T^{-1}\Sigma$ cannot be all of Σ, so that condition (ii) of Theorem 1 cannot hold.

(b) From Corollary 2 of (Lambert [**4**]) we see that when T is invertible, hyponormality is equivalent to $|w|^2 \le |w|^2\circ T^{-1}$ a.e. Because C_T is order preserving and invertible,

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this is equivalent to $|w|^2 \circ T \leq |w|^2$. By the boundedness of W_T we have that $|w|^2$ (and hence $|w|^2 \circ T$) is in L^∞. Hence $|w|^2 - |w|^2 \circ T$ is a bounded non-negative function. But T is measure preserving and therefore $\int_X |w|^2 - |w|^2 \circ T \, d\mu = 0$, so that $|w|^2 = |w|^2 \circ T$ a.e.

Remarks. Part (a) of Corollary 3 is stated in **[1]** only in the case of finite measure. Part (b) is not true in the infinite measure case (consider weighted shifts on $\mathbb{Z}$).

Corollary 4. *W_T is hermitian if and only if*

(a) *T is periodic on A with period at most 2, and*

(b) *$h\bar{w} \circ T = w$ a.e.*

Proof. We may suppose that W_T is normal; since $L^2(X - A) = \ker W_T$ and $L^2(A)$ is reducing for W_T we may assume that $A = X$.

Suppose that W_T is hermitian. For each measurable set B of finite measure we have

$$W_T(\chi_B) = w\chi_{T^{-1}B} = W_T^*(\chi_B) = h\bar{w} \circ T^{-1}\chi_B \circ T^{-1}.$$

We claim that $g \to g \circ T^{-1}$ is the composition operator $g \to g \circ T$, i.e., that $T^2 = I$. Because w is non-zero a.e. we may divide both sides of the equality above by w and obtain

$$\chi_{T^{-1}B} = U\chi_B \circ T^{-1} \text{ a.e.},$$

where $U(x) = (h\bar{w} \circ T^{-1}/w)(x)$. Composing with T we have

$$\chi_B \circ T^2 = (U \circ T)\chi_B \text{ a.e.}$$

$U(x)$ is positive a.e. and hence so is $U \circ T$. Since B is arbitrary we must have $T^2 = I$ and $U \circ T = 1$ a.e., so that $U = 1$ a.e. This shows that (a) and (b) must hold. The converse is immediate.

Example. (Deborah Hart) It is not true that $T^{-1}\Sigma = \Sigma$ implies that T is invertible as a transformation or that $g \to g \circ T^{-1}$ is a composition operator. Consider $X = \{0, 1\}$, $\Sigma = \{X, \varnothing\}$ and $T(0) = T(1) = 0$. T is measurable and $T^{-1}\Sigma = \Sigma$ but T does not take the measurable set X to a measurable set.

Example. Suppose T has period 2; then $1 = d\mu \circ T^{-2}/d\mu = h \circ Th$. If $w = \sqrt{h} = \bar{w}$ then a direct calculation shows that this choice of a weight always gives a hermitian W_T.

Example. w need not be real for W_T to be hermitian. Let $X = [0, 1)$ equipped with Lebesgue measure on the Borel sets, $T : x \to (1 - x)$, and $w(x) = (2x - 1)i$.

Quasinormality. The characterization of quasinormality may be approached in many ways; the following we found most interesting. Via change of variables we have

$$\|W_T f\|^2 = \int w^2 |f|^2 \circ T^2 \, d\mu = \int hE(w^2) \circ T^{-1} |f|^2 \, d\mu. \tag{4}$$

Let $B =$ support of J, where J is the function $hE(w^2) \circ T^{-1}$. Then $\ker W_T = L^2(X - B) = L^2(B)^\perp$. For each f in l^2 write

$$f = \chi_B f + \chi_{X-B} f, \tag{5}$$

so that $W_T f = W_T \chi_B f$. We may define a partial isometry V with initial space $(\ker W_T)^\perp = L^2(B)$ and final space Ran W_T by

$$V_g = w\left(\frac{\chi_B g}{\sqrt{J}}\right) \circ T, \qquad g \in L^2. \tag{6}$$

If we write multiplication by ϕ as M_ϕ, then the (unique, canonical) polar form for W_T is given by

$$W_T = V(M_{\sqrt{J}}). \tag{7}$$

Since an operator is quasinormal if and only if the factors in its canonical polar from commute, direct calculation yields the following result.

THEOREM 2. *W_T is quasinormal if and only if $hE(w^2)\circ T^{-1} = h\circ TE(w^2)$.*

It is known that quasinormality is strictly weaker than normality in the non-weighted case ([**6**]). An example illustrating this phenomenon in the weighted case, with a non-constant weight, is given by following the composition $x \to 2x \pmod 1$ on $L^2(0, 1)$ (Lebesgue measure) with the weight $w = \chi_{(0,\,1/2)}$.

Power Hyponormality. We now state and prove the sufficiency of a condition for the power hynormality of W_T. This theorem says that the general σ-finite ease generalizes the case of a weighted shift on the integers in the natural way.

THEOREM 3. *Suppose that W_T is hyponormal and $T^{-1}\Sigma = \Sigma$. Then W_T is power hyponormal.*

Proof. In order to prove Theorem 3 we need the following lemma and its corollary.

LEMMA 2. *If f, $g \in L^\infty$ satisfy $f\circ T \geq g\circ T$ a.e., and $h = d\mu\circ T^{-1}/d\mu$, then $f \geq g$ a.e. on the support of h.*

Proof of Lemma 2. Whenever $B \subseteq \operatorname{supp} h$ has finite measure we have

$$0 \leq \int_{T^{-1}B} f\circ T - g\circ T\, d\mu = \int_B (f-g)h\, d\mu,$$

and the result follows since $h > 0$ a.e. on every such B.

COROLLARY. *Suppose that $T^{-1}\Sigma = \Sigma$, and $J = hw^2\circ T^{-1}$. Then $J \geq J\circ T$ a.e. implies that $J\circ T^{-1} \geq J$ a.e.*

Proof. The desired inequality is true a.e. on the support of h by Lemma 2. Off the support of h the right-hand side is 0 a.e. Since the left-hand side is ≥ 0 a.e., the desired inequality holds a.e.

The hyponormality of W_T, coupled with the hypothesis that $T^{-1}\Sigma = \Sigma$ implies that $h\circ Tw^2 \leq hw^2\circ T^{-1}$ a.e. (Lambert [**4**]). We will use this fact to complete the proof of Theorem 3 by inductively establishing the following inequality: for each $f \in L^2$ we have

$$\|(W_T^*)^n f\|_2^2 \leq \int h^n\circ Tw^{2n}\,|f|^2\, d\mu \leq \|(W_T)^n f\|_2^2, \qquad n \in \mathbb{N}. \tag{8}$$

Fix f. For $n = 1$ we have

$$\|W_T^* f\|_2^2 = \int h^2 w^2 \circ T^{-1} |f|^2 \circ T^{-1} \, d\mu$$

$$= \int h \circ T w^2 |f|^2 \, d\mu$$

$$\leq \int h w^2 \circ T^{-1} |f|^2 \, d\mu \text{ (by hyponormality and } T^{-1}\Sigma = \Sigma)$$

$$= \int w^2 |f|^2 \circ T \, d\mu = \|W_T f\|_2^2.$$

Suppose (8) holds for $n = 0, 1, \ldots, k-1$. Then for $n = k$ we have

$$\|(W_T^*)^k f\|_2^2 = \|(W_T^*)^{k-1} W_T^* f\|_2^2 \leq \int (h \circ T)^{k-1} w^{2k-2} |W_T^* f|^2 \, d\mu$$

$$= \int (h \circ T)^{k-1} w^{2k-2} h^2 w^2 \circ T^{-1} |f|^2 \circ T^{-1} \, d\mu$$

$$= \int (h \circ T^2)^{k-1} w^{2k-2} \circ T h \circ T w^2 |f|^2 \, d\mu$$

$$= \int (h \circ T)^{k-1} w^{2k-2} h \circ T w^2 |f|^2 \, d\mu \text{ (by Corollary 3)}$$

$$= \int h^k \circ T w^{2k} |f|^2 \, d\mu.$$

On the other hand we have

$$\|(W_T)^k f\|_2^2 = \|(W_T)^{k-1} W_T f\|_2^2 \geq \int (h \circ T)^{k-1} w^{2k-2} w^2 |f|^2 \circ T \, d\mu$$

$$= \int h^{k-1} w^{2k-2} \circ T^{-1} w^2 \circ T^{-1} h \, |f|^2 \, d\mu$$

$$= \int h^k w^{2k} \circ T^{-1} |f|^2 \, d\mu$$

$$\geq \int h^k \circ T w^{2k} |f|^2 \, d\mu.$$

This completes the induction and also the proof of Theorem 3.

Remark. In the unweighted case this is easily proved using Corollary 11 of (Harrington and Whitley [3]) and Corollary 3 in (Campbell and Dibrell [2]), as follows. Corollary 11 says that is h is $T^{-1}\Sigma$ measurable then C_T is hyponormal if and only if $h \circ T \leq h$ a.e. Thus C_T hyponormal and $T^{-1}\Sigma = \Sigma$ implies $h \circ T \leq h$, and the aforementioned Corollary 3 states that this is sufficient for power hyponormality.

We note here that the converse to Corollary 3 in Campbell and Dibrell is not true; Lambert ([**5**]) has constructed an example of a shift on $l^2(\mathbb{N}, m)$ (m a non-constant weight sequence) for which $h \circ T \leq h$ (so that C_T is power hyponormal) but $(h_2 \circ T)(n) > h_2(n)$ for some n. (Here, $h_2 = d\mu \circ T^{-2}/d\mu$). We also remark that the power hyponormal class (unweighted) is larger than the subnormal class (see Example 14 in Harrington and Whitley [**3**])). It would be interesting to characterize the power hyponormal class.

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