1

Introduction

In this chapter we introduce basic notions needed in what follows. We also discuss nearest neighbour Markov chains and diffusion processes, which represent the two classes of Markov processes whose invariant measure, in the case of positive recurrence, or Green function, in the case of transience, is available in closed form. A closed form makes possible the direct analysis of such Markov processes: their classification and the tail asymptotics of the invariant probabilities or Green function. This discussion sheds some light on what we may expect for general Markov chains.

1.1 Countable Markov Chains

Let us start with a simple process, a *countable time-homogeneous Markov chain* $X = \{X_n, n \geq 0\}$, which is a stochastic process with a countable *state space*, which can always be reduced to $S = \mathbb{Z}^+$. The process is determined by an initial distribution X_0 and a collection of *transition probabilities* $p_{xy} \geq 0$, $x, y \in S$, such that $\sum_{y \in S} p_{xy} = 1$ for all $x \in S$ and

$$\begin{aligned} &\mathbb{P}\{X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\} \\ &\quad = \mathbb{P}\{X_{n+1} = x_{n+1} \mid X_n = x_n\} \ = \ p_{x_n x_{n+1}}, \end{aligned} \tag{1.1}$$

for all time epochs n and all sequences of states $x_{n+1}, x_n, \ldots, x_0$ in S. In words, the probability of moving from one state to another does not depend on the trajectory that determined how X arrived in its current state. This memoryless property can be equivalently defined as independence of the future and the past given the current state, that is,

$$\mathbb{P}\{BA \mid X_n = x_n\} = \mathbb{P}\{B \mid X_n = x_n\}\mathbb{P}\{A \mid X_n = x_n\}$$

for any $n \geq 1$ and events $B \in \sigma(X_{n+1}, X_{n+2}, \ldots)$ and $A \in \sigma(X_0, \ldots, X_{n-1})$.

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Definition 1.1 A random variable T taking non-negative integer values which is possibly improper, is called the *stopping time* if, for all $n \in \mathbb{Z}^+$, the event $\{T \le n\}$ belongs to the σ-algebra $\sigma(X_0, X_1, \ldots, X_n)$.

The Markov property (1.1) can be extended to stopping times as follows. If T is a stopping time then the process $\{X_{T+n}\}_{n\ge 0}$ is also a Markov chain with initial distribution X_T. Moreover, for any $x \in S$, this chain is independent of $X_0, X_1, \ldots, X_{T-1}$ given $X_T = x$. This property is called the *strong Markov property*.

For any state $x \in S$, denote by τ_x the first hitting time of x,

$$\tau_x := \inf\{n \ge 1 : X_n = x\},$$

with standard convention $\inf \emptyset = \infty$. For all x, τ_x is a stopping time.

Definition 1.2 A state x is called *positive recurrent* if $\mathbb{E}_x \tau_x < \infty$.

Definition 1.3 A state x is called *non-positive* if it is not positive recurrent, more precisely, if either $\mathbb{P}_x\{\tau_x = \infty\} > 0$, or $\mathbb{P}_x\{\tau_x < \infty\} = 1$ and $\mathbb{E}_x \tau_x = \infty$.

Definition 1.4 A state x is called *recurrent* (*persistent*) if $\mathbb{P}_x\{\tau_x < \infty\} = 1$.

Definition 1.5 A state x is called *null recurrent* if $\mathbb{P}_x\{\tau_x < \infty\} = 1$ while $\mathbb{E}_x \tau_x = \infty$.

Definition 1.6 A state x is called *transient* if $\mathbb{P}_x\{\tau_x < \infty\} < 1$.

By the strong Markov property, the time lengths between consecutive visits of the chain to a fixed state x are independent and identically distributed. Therefore, a state x is transient if and only if $\mathbb{P}_x\{\tau_x < \infty\} < 1$, which is equivalent to the convergence of the following series (*a Green function*):

$$\sum_{n=0}^{\infty} \mathbb{P}_x\{X_n = x\} \;=\; \mathbb{E}_x \sum_{n=0}^{\infty} \mathbb{I}\{X_n = x\} \;<\; \infty.$$

Definition 1.7 The *period* of state x is defined as

$$d_x := \gcd\{n \ge 1 : \mathbb{P}_x\{X_n = x\}\}.$$

A state x is called *aperiodic* if $d_x = 1$.

Definition 1.8 A Markov chain X_n is called *irreducible* if, for all x and y, $\mathbb{P}_x\{X_n = y\} > 0$ for some n.

Notice that, for an irreducible countable Markov chain, the following solidarity property holds true: the positive recurrence, non-positivity, recurrence, null recurrence, transience, or aperiodicity of any state implies the same property for all other states.

Definition 1.9 A measure $\{\pi_x\}_{x\in S}$ is called *invariant* (or *stationary*) for a countable Markov chain $\{X_n\}$ if

$$\pi(y) \;=\; \sum_{j\in S} \pi(x) p_{xy} \quad \text{for all } y \in S.$$

Definition 1.10 A probability distribution $\{\pi_x\}_{x\in S}$ is called *asymptotic* (or *limiting*) for a countable Markov chain $\{X_n\}$ if

$$\mathbb{P}_x\{X_n = y\} \;\to\; \pi(y) \quad \text{as } n \to \infty \quad \text{for any } x \in S.$$

An asymptotic distribution – if exists – is necessarily an invariant probability measure; however, this does not hold vice versa. For Markov chains with finitely many states, the following ergodic theorem is a major result.

Theorem 1.11 *Any finite irreducible aperiodic Markov chain possesses an asymptotic distribution.*

For a Markov chain with infinitely many states the last result may fail in general. For example, a simple random walk with transition probabilities $p_{x,x+1} = p > 1/2$ and $p_{x,x-1} = 1 - p < 1/2$ is irreducible; however, there is no convergence to an asymptotic distribution. This Markov chain is transient which is only possible due to the infinite number of states.

Theorem 1.12 *Let $\{X_n\}$ be a countable irreducible Markov chain. Fix some $x \in S$. If $\{X_n\}$ is recurrent then a measure π defined by*

$$\mu(y) := \mathbb{E}_x \sum_{n=1}^{\tau_x} \mathbb{I}\{X_n = y\} \;=\; \sum_{n=1}^{\infty} \mathbb{P}_x\{X_n = y, n \le \tau_x\}, \quad y \in S, \quad (1.2)$$

is a σ-finite invariant measure for $\{X_n\}$.

Proof Let us first check that $\mu(y) < \infty$ for all $y \in S$. By its definition, $\mu(x) = 1$. Since $\{X_n\}$ is irreducible, there exists a state y such that $p_{yx} > 0$. Then the random variable

$$\mathbb{E}_x \sum_{n=1}^{\tau_x} \mathbb{I}\{X_n = y\}$$

is stochastically bounded by a geometric distribution with success probability $p_{yx} > 0$, hence $\mu(y) < \infty$. Then, by the solidarity property, $\mu(z) < \infty$ for all $z \neq x$.

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Now let us show that μ is invariant. Indeed, for $z = x$,

$$\begin{aligned}\sum_{y\in S}\mu(y)p_{yx} &= p_{xx} + \sum_{y\neq x}\sum_{n=1}^{\infty}\mathbb{P}_x\{X_n = y, n \le \tau_x\}p_{yx}\\ &= p_{xx} + \sum_{n=1}^{\infty}\sum_{y\neq x}\mathbb{P}_x\{X_n = y, n < \tau_x\}p_{yx}\\ &= p_{xx} + \sum_{n=1}^{\infty}\mathbb{P}_x\{\tau_x = n+1\}\\ &= \mathbb{P}_x\{\tau_x < \infty\} \;=\; 1 \;=\; \mu(x),\end{aligned}$$

because $\{X_n\}$ is recurrent. For any $z \neq x$,

$$\begin{aligned}\sum_{y\in S}\mu(y)p_{yz} &= p_{xz} + \sum_{n=1}^{\infty}\sum_{y\neq x}\mathbb{P}_x\{X_n = y, n < \tau_x\}p_{yz}\\ &= p_{xz} + \sum_{n=1}^{\infty}\mathbb{P}_x\{X_{n+1} = z, n+1 < \tau_x\}\\ &= \sum_{n=1}^{\infty}\mathbb{P}_x\{X_n = z, n < \tau_x\}\\ &= \mu(z),\end{aligned}$$

by the definition of $\mu(z)$ for $z \neq x$. □

So, any irreducible recurrent Markov chain possesses a σ-finite invariant distribution. However, the existence of a σ-finite invariant distribution does not guarantee recurrence, as the following example demonstrates. For a simple random walk on $\mathbb{Z}$, the Haar measure assigning $\mu(x) = 1$ for all $x \in \mathbb{Z}$ is invariant whatever the success probability p.

For positive recurrence there is a criterion in terms of an invariant measure, as follows.

Theorem 1.13 *For a countable irreducible Markov chain $\{X_n\}$, the following are equivalent:*

(i) *some state is positive recurrent;*
(ii) *all states are positive recurrent;*
(iii) *the measure μ defined in* (1.2) *is finite;*
(iv) *there exists a probability invariant measure π.*

Then

$$\pi(y) \;=\; \frac{1}{\mathbb{E}_y\tau_y} \quad \textit{for all } y \in S.$$

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Proof The equivalence of (i) or (ii) to (iii) is immediate from the definition (1.2), because

$$\begin{aligned}\sum_{y\in S}\pi(y) &= \sum_{y\in S}\mathbb{E}_x\sum_{n=1}^{\tau_x}\mathbb{I}\{X_n=y\}\\ &= \mathbb{E}_x\sum_{n=1}^{\tau_x}\sum_{y\in S}\mathbb{I}\{X_n=y\} \;=\; \mathbb{E}_x\tau_x,\end{aligned}$$

which is only finite if $\{X_n\}$ is positive recurrent.

The most difficult implication is (iv)→(iii). It follows from the observation that any invariant measure π satisfies the equalities

$$\pi(y) := \pi(x)\mathbb{E}_x\sum_{n=1}^{\tau_x}\mathbb{I}\{X_n=y\} \;=\; \pi(x)\sum_{n=1}^{\infty}\mathbb{P}_x\{X_n=y,n\le\tau_x\},\quad y\in S.$$

For a proof, see e.g. Meyn and Tweedie [126, Theorem 10.4.9]. □

1.2 Real-Valued Markov Chains

Now let us proceed with a *time-homogeneous Markov chain* $X=\{X_n,n\ge 0\}$, whose state space is a Borel subset S of $\mathbb{R}$, that is, for all $x\in S$ and Borel sets $B_0,\dots,B_{n-1},B_{n+1}\in\mathcal{B}(S)$,

$$\begin{aligned}&\mathbb{P}\{X_{n+1}\in B_{n+1}\mid X_0\in B_0,\dots,X_{n-1}\in B_{n-1},X_n=x\}\\ &\quad=\mathbb{P}\{X_{n+1}\in B_{n+1}\mid X_n=x\}.\end{aligned}$$

We usually simply say that X_n takes values in $\mathbb{R}$, keeping in mind that the corresponding transition probabilities may be defined only on some subset S of the real line.

Denote by $P(\cdot,\cdot): S\times\mathcal{B}(S)\to[0,1]$ the *transition probabilities* of $\{X_n\}$:

$$P(x,B)=\mathbb{P}\{X_{n+1}\in B\mid X_n=x\};$$

this function is measurable in x for each fixed B and is a probability measure for each fixed x, that is, it is a stochastic transition kernel. Then, for all n and B,

$$\mathbb{P}\{X_{n+1}\in B\}=\int_S P(y,B)\mathbb{P}\{X_n\in dy\}.$$

Let $\mathbb{P}_x\{\cdot\}=\mathbb{P}\{\cdot\mid X_0=x\}$ and $\mathbb{E}_x\{\cdot\}=\mathbb{E}\{\cdot\mid X_0=x\}$.

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Denote by $\xi(x)$, $x \in S$, a random variable corresponding to the *jump* of the chain at point $x \in S$, that is, a random variable with distribution

$$\begin{aligned}\mathbb{P}\{\xi(x) \in B\} &= \mathbb{P}\{X_{n+1} - X_n \in B \mid X_n = x\} \\ &= \mathbb{P}_x\{X_1 \in x + B\}, \quad B \in \mathcal{B}(\mathbb{R}).\end{aligned}$$

In the sequel we will always assume that S is a right unbounded set. Furthermore, for ease of notation, we assume that $P(x, B)$ is defined for all $x \in \mathbb{R}$.

Denote the kth moment of the jump at point x by

$$m_k(x) \ := \ \mathbb{E}\xi^k(x).$$

Definition 1.14 We say that a Markov chain $\{X_n\}$ has an *asymptotically zero drift* if $m_1(x) = \mathbb{E}\xi(x) \to 0$ as $x \to \infty$.

The study of processes with asymptotically zero drift was initiated by Lamperti in a series of papers [111, 112, 113].

The first topic of basic importance is the classification of Markov chains, which is discussed in detail in Chapter 2. For any Borel set $B \subset \mathbb{R}$ denote by τ_B the time of the first entry of $\{X_n\}$ to B,

$$\tau_B := \inf\{n \geq 1 : X_n \in B\}.$$

If B is a singleton then we can repeat the classification of B as in the previous section. However, this does not work well for Markov chains that are truly real-valued, as it could happen that then $\mathbb{P}_B\{\tau_B < \infty\} = 0$. For that reason we introduce a classification of a general Borel set B with respect to X_n that reduces to the classification presented in the last section if B is a singleton.

Definition 1.15 A set B is called *positive recurrent* if $\mathbb{E}_x\tau_B < \infty$ for all $x \in B$.

Definition 1.16 A set B is called *non-positive* if it is not positive recurrent; more precisely, if either $\mathbb{P}_x\{\tau_B = \infty\} > 0$, or $\mathbb{P}_x\{\tau_B < \infty\} = 1$ and $\mathbb{E}_x\tau_B = \infty$ for some $x \in B$.

Definition 1.17 A set B is called *recurrent* if τ_B is finite a.s. for all initial states $x \in B$.

Definition 1.18 A set B is called *null recurrent* if τ_B is finite a.s. and $\mathbb{E}_x\tau_B = \infty$ for all initial states $x \in B$.

Definition 1.19 A set B is called *transient* if $\mathbb{P}_x\{\tau_B < \infty\} < 1$ for all initial states $x \in B$.

Definition 1.20 A measure π is called *invariant* for $\{X_n\}$ if

$$\pi(B) = \int_S P(x,B)\pi(dx) \quad \text{for all } B \in \mathcal{B}(S).$$

In [111] Lamperti showed that if $S = \mathbb{R}^+$, $\limsup X_n = \infty$, and $\mathbb{E}|\xi(x)|^{2+\delta}$ is bounded for some $\delta > 0$ then

- $2xm_1(x) \le m_2(x) + O(x^{-\delta})$ implies that some neighborhood of zero is recurrent,
- $2xm_1(x) \ge (1+\varepsilon)m_2(x)$, for some $\varepsilon > 0$ and all sufficiently large x, implies that any compact set is transient.

In [113] Lamperti proved that $2xm_1(x) + m_2(x) \le -\varepsilon$ is sufficient for the positive recurrence of any compact set and that $2xm_1(x) + m_2(x) \ge \varepsilon$ implies the non-positivity of any compact set (for the case of either null recurrence or transience). These criteria were improved later by Menshikov et al. [124]. Instead of the existence of moments of order $2+\delta$ they assumed that $\mathbb{E}\xi^2(x)\log^{2+\delta}(1+|\xi(x)|)$ is bounded. Moreover, they obtained a more precise classification of positive recurrence, null recurrence, and transience which involves iterated logarithms.

In the next section we discuss classical random walks to show the difference between them and Lamperti processes. That is followed by a couple of sections devoted to two types of specific processes – nearest neighbour Markov chains and diffusion processes – where many characteristics of interest may be computed in closed form by following quite elementary calculations; this provides the basic intuition needed to approach general Markov chains with asymptotically zero drift.

In Section 1.6 we describe our approach to general Markov chains with asymptotically zero drift.

1.3 Random Walks

Let us consider a fundamental example of Markov chains, random walks. We start by recalling some important asymptotic results, which will be extended to Lamperti's Markov chains later.

Definition 1.21 *A random walk* with initial state x is a sequence of partial sums, $S_0 = x$,

$$S_n := S_{n-1} + \xi_n = x + \xi_1 + \cdots + \xi_n, \quad n \ge 1,$$

where the ξ_n are independent and identically distributed random variables.

Any random walk is a Markov chain with transition kernel

$$P(x,B) = \mathbb{P}\{\xi_1 \in B - x\}, \quad x \in \mathbb{R}, \quad B \in \mathcal{B}(\mathbb{R}).$$

It is a *space-homogeneous* Markov chain because all its jumps $\xi(x)$, $x \in \mathbb{R}$, are distributed as ξ_1. Roughly speaking, it is a process with continuous statistics in the sense that there are no boundary effects in this model.

If $\mathbb{E}|\xi_1| < \infty$ then the strong law of large numbers holds, that is,

$$S_n/n \to \mathbb{E}\xi_1 \quad \text{a.s. as } n \to \infty.$$

This implies, in particular, that if $\mathbb{E}\xi_1 > 0$ then the set $(-\infty, \widehat{x}\,]$ is transient for all $\widehat{x} \in \mathbb{R}$. If $\mathbb{E}\xi_1 < 0$ then the set $(-\infty, \widehat{x}\,]$ is positive recurrent. It is also well known that in the case $\mathbb{E}\xi_1 = 0$ the random walk S_n is null recurrent, that is, any bounded set is null recurrent.

In addition, if $\mathbb{E}\xi_1^2 < \infty$ then the central limit theorem holds, that is,

$$\frac{S_n - n\mathbb{E}\xi_1}{\sqrt{n\mathbb{V}\mathrm{ar}\,\xi_1}} \Rightarrow N_{0,1} \quad \text{as } n \to \infty.$$

The simplest process with discontinuous statistics – that is, with boundary effects – is a random walk delayed at zero, which is defined next.

Definition 1.22 *A random walk delayed at zero (a Lindley recursion)* is a stochastic process $W = \{W_n, n \geq 0\}$ such that, for all $n \geq 1$,

$$W_n = (W_{n-1} + \xi_n)^+ := \max(0, W_{n-1} + \xi_n),$$

where the ξ_n are independent and identically distributed random variables that are independent of $W_0 \geq 0$.

The process W is a Markov chain with transition kernel

$$P(x,B) = \mathbb{P}\{(x+\xi_1)^+ \in B\}, \quad x \in \mathbb{R}^+, \quad B \in \mathcal{B}(\mathbb{R});$$

this is a particular example of a Markov chain that is asymptotically homogeneous in space, defined below, because its jumps satisfy the following weak (and in total-variation distance) convergence:

$$\xi(x) =_{\mathrm{st}} (x+\xi_1)^+ - x \Rightarrow \xi_1 \quad \text{as } x \to \infty.$$

Definition 1.23 We say that a Markov chain $\{X_n\}$ is *asymptotically homogeneous in space* if

$$\xi(x) \Rightarrow \xi \quad \text{as } x \to \infty, \tag{1.3}$$

for some random variable ξ. Equivalently, $P(x, x+\cdot) \Rightarrow \mathbb{P}\{\xi \in \cdot\}$.

Let $W_0 = 0$. Then

$$W_n \;=\; \max(0, \xi_n, \xi_n + \xi_{n-1}, \xi_n + \xi_{n-1} + \xi_{n-2}, \ldots, \xi_n + \xi_{n-1} + \cdots + \xi_1);$$

hence, for all n, W_n is equal in distribution to the maximum

$$\begin{aligned} M_n &:= \max(0, \xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3, \ldots, \xi_1 + \xi_2 + \cdots + \xi_n) \\ &= \max_{0 \le k \le n} S_k, \quad \text{where } S_0 = 0. \end{aligned}$$

An application of the Lindley recursion $\{W_n\}$ is the waiting time process in the single-server queue system with $\xi = \sigma - \tau$, where σ represents the typical service time and τ the typical inter-arrival time. Among applications of the process of maxima M_n is the collective risk process, with $\xi = X - c\tau$, where X represents the typical claim size, τ the typical inter-arrival time, and c the premium rate; here $\mathbb{P}\{M_\infty > x\}$ represents the ruin probability given the initial reserve $x > 0$.

If $\mathbb{E}\xi_1 > 0$ then $\{W_n\}$ is a transient Markov chain (any bounded set is transient), which satisfies the central limit theorem provided that $\mathbb{E}\xi_1^2 < \infty$:

$$\frac{W_n - n\mathbb{E}\xi_1}{\sqrt{n \mathbb{V}\mathrm{ar}\, \xi_1}} \Rightarrow N_{0,1} \quad \text{as } n \to \infty.$$

If $\mathbb{E}\xi_1 = 0$ then $\{W_n\}$ is null recurrent (any bounded set is null recurrent), and, by the functional central limit theorem (Donsker's theorem),

$$\frac{W_n}{\sqrt{n \mathbb{V}\mathrm{ar}\, \xi_1}} \Rightarrow \sup_{t \le 1} B(t) \quad \text{as } n \to \infty,$$

where $B(t)$ is a Brownian motion; see, e.g. Billingsley [16, Section 10].

If $\mathbb{E}\xi_1 < 0$ then $\{W_n\}$ is positive recurrent (any bounded set is positive recurrent) and possesses a unique invariant probability measure, say π_W. This measure is the distribution of $M_\infty := \max_{n \ge 0} S_n$, and the distribution of W_n converges to π_W in the total-variation metric, that is,

$$\sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}\{W_n \in B\} - \pi_W(B)| \;\to\; 0 \quad \text{as } n \to \infty.$$

The distribution π_W is explicitly known in few cases only. The tail behaviour of π_W is understood very well; it depends heavily on the existence of positive exponential moments of ξ_1. For that reason the following classes of distributions are introduced:

Definition 1.24 We say that a distribution F is *light-tailed* if

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) < \infty \quad \text{for some } \lambda > 0.$$

A random variable ξ is called *light-tailed* if its distribution is light-tailed.

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Definition 1.25 We say that a distribution F is *heavy-tailed* if

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) = \infty \quad \text{for all } \lambda > 0.$$

A random variable ξ is called *heavy-tailed* if its distribution is heavy-tailed.

Definition 1.26 We say that a function $g(x)$ is *long-tailed* if, for any fixed y, $g(x+y) \sim g(x)$ as $x \to \infty$. A distribution F with right unbounded support is called *long-tailed* if $F(x,\infty)$ is a long-tailed function.

Any long-tailed distribution is necessarily heavy-tailed.

Definition 1.27 A distribution F on $\mathbb{R}^+$ is called *subexponential* if

$$(F * F)(x,\infty) \sim 2F(x,\infty) \quad \text{as } x \to \infty.$$

A distribution F of a random variable ξ is called *subexponential* if the distribution of ξ^+ is subexponential.

Any subexponential distribution is necessarily long-tailed and hence heavy-tailed; see e.g. [67, Lemma 3.2].

In order to describe the tail behaviour of π_W, let us introduce $\varphi(\lambda) = \mathbb{E}e^{\lambda\xi_1}$ and $\beta = \sup\{\lambda \ge 0 : \varphi(\lambda) \le 1\}$. Given $\mathbb{P}\{\xi_1 > 0\} > 0$, it follows that $\beta < \infty$. It turns out that the asymptotic behavior of $\mathbb{P}\{M_\infty > x\}$ depends heavily on the values of β and $\varphi(\beta)$. The following three different cases are considered:

(i) $\beta > 0$ and $\varphi(\beta) = 1$, the Cramér case;
(ii) $\beta = 0$, the heavy-tailed case where all positive exponential moments of ξ_1 are infinite;
(iii) $\beta > 0$ and $\varphi(\beta) < 1$, the intermediate case.

In the Cramér case, under the additional assumption $\varphi'(\beta - 0) < \infty$, for some $c \in (0,1)$ we have

$$\mathbb{P}\{M_\infty > x\} \sim ce^{-\beta x} \quad \text{as } x \to \infty;$$

this result goes back to H. Cramér, see e.g. [38] or [63, Chapter XII]. In Chapter 10, a similar exponential asymptotics of invariant probabilities of this type is proven for a broad class of Markov chains on $\mathbb{R}$ that are asymptotically homogeneous in space and have asymptotically negative drift.

In the heavy-tailed case, the tail asymptotics for M_∞ is only available under subexponential-type conditions, namely,

$$\mathbb{P}\{M_\infty > x\} \sim \frac{1}{|\mathbb{E}\xi_1|}\int_x^\infty \mathbb{P}\{\xi_1 > y\}dy \quad \text{as } x \to \infty$$

if and only if the integrated tail distribution F_I on $\mathbb{R}^+$ defined by its tail

$$\overline{F}_I(x) := \min\Big(1, \int_x^\infty \mathbb{P}\{\xi_1 > y\}dy\Big)$$

is subexponential; see e.g. [67, Theorem 5.12].

In the intermediate case, we have $\mathbb{E}e^{\beta M_\infty} < \infty$. In addition, if the function $e^{\beta x}\mathbb{P}\{\xi_1 > x\}$ is long-tailed then

$$\mathbb{P}\{M_\infty > x\} \sim c\mathbb{P}\{\xi_1 > x\} \quad \text{as } x \to \infty,$$

for some $c \in (0,\infty)$ (in the lattice case x must be taken as a multiple of the lattice step), if and only if the distribution of the random variable ξ_1^+ belongs to the so-called class $\mathcal{S}(\beta)$; see [14, Theorem 1] and [102, Theorem 2]. In that case $c = \mathbb{E}e^{\beta M_\infty}/(1-\varphi(\beta))$.

So, the invariant measure of $\{W_n\}$ is light-tailed if and only if the distribution of ξ_1 is light-tailed. As we will see in what follows, for Markov chains with asymptotically zero drift the situation is very different – the invariant measure is always heavy-tailed apart from degenerate cases.

1.4 Nearest Neighbour Markov Chains

In this section we discuss nearest neighbour Markov chains, which represent one of the two classes of Markov chains for which either the invariant measure, in the case of positive recurrence, or the Green function, in the case of transience, is available in closed form. A closed form makes possible the direct analysis of such Markov chains: their classification and the tail asymptotics of their invariant probabilities or of Green function. This discussion should shed some light on what we may expect for general Markov chains. Another class is provided by diffusion processes, which are discussed in the next section.

Definition 1.28 A Markov chain $\{X_n\}$ on $\mathbb{Z}^+$ is called a *nearest neighbour* (*skip-free* or *continuous*) *Markov chain*, if $\xi(x)$ takes only the values -1, 1, or 0, with probabilities $p_-(x)$, $p_+(x)$, and $p_0(x) = 1 - p_-(x) - p_+(x)$ respectively, with $p_-(0) = 0$.

Let

$$p_+(x) = p + \varepsilon_+(x) \quad \text{and} \quad p_-(x) = p - \varepsilon_-(x), \quad p \le 1/2,$$

where the probabilities are assumed to be neither 0 nor 1, so that we have an irreducible Markov chain.

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Assume that $\varepsilon_\pm(x) \to 0$ as $x \to \infty$, which corresponds to the case of asymptotically zero drift, $m_1(x) = \varepsilon_+(x) + \varepsilon_-(x) \to 0$ as $x \to \infty$. Then the second moment of the jumps is convergent: $m_2(x) \to 2p$ as $x \to \infty$.

1.4.1 Positive Recurrence

To find a sufficient condition for the positive recurrence of $\{X_n\}$, let us consider a test function $L(y) = y^2$. Its drift at all states $x \geq 1$ equals

$$\begin{aligned}\mathbb{E}L(x+\xi(x)) - L(x) &= 2x\mathbb{E}\xi(x) + \mathbb{E}\xi^2(x)\\ &= 2(\varepsilon_+(x)+\varepsilon_-(x))x + 2p + \varepsilon_+(x) - \varepsilon_-(x),\end{aligned}$$

so the chain is positive recurrent if

$$\limsup_{x\to\infty}(\varepsilon_+(x)+\varepsilon_-(x))x < -p, \tag{1.4}$$

see e.g. Lamperti [111] or Section 2.2. Let us denote the stationary probabilities of $\{X_n\}$ by $\pi(x)$, $x \in \mathbb{Z}^+$.

Proposition 1.29 *Under the condition* (1.4)*, for some* $c_1 \in \mathbb{R}$*,*

$$\pi(x) \sim \exp\left(\frac{1}{p}\sum_{k=1}^{x}(\varepsilon_+(k)+\varepsilon_-(k)) + c_1\right) \quad \textit{as } x \to \infty, \tag{1.5}$$

provided that

$$\sum_{k=0}^{\infty}\varepsilon^2(k) < \infty, \tag{1.6}$$

where $\varepsilon(k) := \max(|\varepsilon_-(k)|, |\varepsilon_+(k)|)$.

Proof If the chain $\{X_n\}$ is positive recurrent then its stationary probabilities $\pi(x)$, $x \in \mathbb{Z}^+$, satisfy the equations

$$\begin{aligned}\pi(0) &= \pi(0)p_0(0) + \pi(1)p_-(1),\\ \pi(x) &= \pi(x-1)p_+(x-1) + \pi(x)p_0(x) + \pi(x+1)p_-(x+1), \quad x \geq 1,\end{aligned}$$

which are equivalent to

$$\begin{aligned}\pi(0)p_+(0) &= \pi(1)p_-(1),\\ \pi(x+1)p_-(x+1) - \pi(x)p_+(x) &= \pi(x)p_-(x) - \pi(x-1)p_+(x-1)\\ &\ \ \vdots\\ &= \pi(1)p_-(1) - \pi(0)p_+(0) \;=\; 0,\end{aligned}$$

which yields $\pi(x)p_-(x) = \pi(x-1)p_+(x-1)$ for all $x \geq 1$. Hence we obtain the following solution:

$$\pi(x) = \pi(0)\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)}, \quad x \geq 1, \tag{1.7}$$

where

$$\pi(0) = \left(1 + \sum_{x=1}^{\infty}\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)}\right)^{-1}.$$

So $\{X_n\}$ is positive recurrent if and only if

$$\sum_{x=1}^{\infty}\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)} < \infty;$$

see Harris [77] or Karlin and Taylor [87, pp. 86–87], in which these calculations are carried out for the case where $p_0(k) = 0$ for all $k \geq 1$.

Since $\varepsilon_\pm(k) \to 0$,

$$\begin{aligned}\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)} &= \frac{p_+(0)}{p_+(x)}\prod_{k=1}^{x}\frac{1+\varepsilon_+(k)/p}{1-\varepsilon_-(k)/p}\\ &\sim \frac{p_+(0)}{p}\prod_{k=1}^{x}\frac{1+\varepsilon_+(k)/p}{1-\varepsilon_-(k)/p} \quad \text{as } x \to \infty.\end{aligned}$$

The logarithm of the product on the right-hand side equals

$$\begin{aligned}&\sum_{k=1}^{x}\big(\log(1+\varepsilon_+(k)/p) - \log(1-\varepsilon_-(k)/p)\big)\\ &\qquad = \frac{1}{p}\sum_{k=1}^{x}\big(\varepsilon_+(k)+\varepsilon_-(k)\big) + \sum_{k=1}^{x}\delta(k),\end{aligned} \tag{1.8}$$

where $\delta(k) = O(\varepsilon^2(k))$ as $k \to \infty$, for $\varepsilon(k) := \max(|\varepsilon_-(k)|, |\varepsilon_+(k)|)$. Hence, for some $c_1 \in \mathbb{R}$,

$$\pi(x) \;=\; \pi(0)\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)} \sim e^{\frac{1}{p}\sum_{k=1}^{x}(\varepsilon_+(k)+\varepsilon_-(k))+c_1} \quad \text{as } x \to \infty,$$

provided (1.6) holds. □

Let us consider a couple of examples with expressions for specific ε. In what follows we need the following results on the harmonic series and generalised harmonic series.

Proposition 1.30 *For the truncated harmonic series,*

$$\sum_{x=1}^{n} \frac{1}{x} = \log n + \gamma + O(1/n) \quad \textit{as } n \to \infty, \tag{1.9}$$

where γ is the Euler constant.

For the truncated generalised harmonic series, for any $\alpha \in (0,1)$,

$$\sum_{x=1}^{n} \frac{1}{x^\alpha} = \frac{n^{1-\alpha}}{1-\alpha} + \gamma_\alpha + O(1/n^\alpha) \quad \textit{as } n \to \infty. \tag{1.10}$$

The first example of expressions for ε concerns a drift of order $-\mu/x$.

Example 1.31 If $\varepsilon_+(x) \sim -\mu_+/x$ and $\varepsilon_-(x) \sim -\mu_-/x$ as $x \to \infty$ in such a way that

$$\sum_{x=0}^{\infty} \left| \varepsilon_+(x) + \varepsilon_-(x) + \frac{\mu_+ + \mu_-}{x} \right| < \infty$$

then (1.4) yields a positive recurrence of the chain provided that $\mu := \mu_+ + \mu_- > p$ and (1.5) implies an asymptotic equivalence, for some $c_2 \in \mathbb{R}$:

$$\pi(x) \sim e^{-(\mu/p)\log x + c_2} = \frac{e^{c_2}}{x^{\mu/p}} \quad \text{as } x \to \infty. \tag{1.11}$$

In Chapter 8 the power asymptotics of invariant probabilities of this type are extended to a broad class of Markov chains on $\mathbb{R}$ with asymptotically zero drift of order $-\mu/x$.

The second example of expressions for ε concerns a drift of order $-\mu/x^\alpha$, $\alpha \in (0,1)$.

Example 1.32 If $\varepsilon_+(x) \sim -\mu_+/x^\alpha$ and $\varepsilon_-(x) \sim -\mu_-/x^\alpha$ as $x \to \infty$, for some $\mu_+, \mu_- > 0$ and $\alpha \in (1/2, 1)$, in such a way that

$$\sum_{x=0}^{\infty} \left| \varepsilon_+(x) + \varepsilon_-(x) + \frac{\mu_+ + \mu_-}{x^\alpha} \right| < \infty$$

then the series $\sum \varepsilon^2(x)$ is again convergent and we observe a Weibullian asymptotic behaviour of the invariant probabilities,

$$\pi(x) \sim c_3 \exp\Big(-(\mu_+ + \mu_-)x^{1-\alpha}/p(1-\alpha)\Big) \quad \text{as } x \to \infty. \tag{1.12}$$

If now $\alpha \in (1/3, 1/2]$ then the series (1.6) diverges and the quadratic terms in (1.8) make a significant contribution to the asymptotic behaviour of the invariant probabilities,

$$\pi(x) \sim c_4 \exp\Bigl(-\frac{\mu_+ + \mu_-}{p(1-\alpha)} x^{1-\alpha} + \frac{\mu_-^2 - \mu_+^2}{(2\alpha-1)2p^2} x^{1-2\alpha}\Bigr) \quad \text{as } x \to \infty.$$

If $\alpha \in (1/4, 1/3]$ then we need to keep the cubic terms in the Taylor expansion of the logarithm, which adds a further correction term of order $x^{1-3\alpha}$ to the exponential function, and so on.

General Markov chains on $\mathbb{R}$ with asymptotically zero drift of order $-\mu/x^\alpha$, $\alpha \in (0,1)$, are considered in Chapter 9, where Weibullian-type asymptotics of invariant probabilities are proven.

1.4.2 Transience

Let a nearest neighbour Markov chain $\{X_n\}$ be irreducible and transient. Then $\mathbb{P}_x\{\tau_x < \infty\} < 1$ for all x and hence the renewal measure (Green function)

$$\begin{aligned} h_{x_0}(x) &:= \sum_{n=0}^{\infty} \mathbb{P}_{x_0}\{X_n = x\} \\ &= \mathbb{E}_{x_0} \sum_{n=0}^{\infty} \mathbb{I}\{X_n = x\} \end{aligned}$$

is finite for all x_0, $x \in \mathbb{Z}^+$, because

$$\begin{aligned} h_{x_0}(x) &= \mathbb{P}_{x_0}\{X_k = x \text{ for some } k\} \sum_{n=0}^{\infty} \mathbb{P}_x\{X_n = x\} \\ &= \mathbb{P}_{x_0}\{X_k = x \text{ for some } k\} \frac{1}{1 - \mathbb{P}_x\{\tau_x < \infty\}} \; < \; \infty. \end{aligned}$$

Since we are considering a Markov chain that jumps by 1 only, $h_{x_0}(x) = h_x(x)$ for all $x_0 \le x$. In the next result we find $h_{x_0}(x)$ in closed form.

Proposition 1.33 *Under the condition*

$$\sum_{u=1}^{\infty} \prod_{z=1}^{u} \frac{p_-(z)}{p_+(z)} < \infty, \tag{1.13}$$

the following representations hold true:

$$\begin{aligned} h_{x_0}(x) &= \frac{1}{p_+(x)} \sum_{u=x \vee x_0}^{\infty} \prod_{z=x+1}^{u} \frac{p_-(z)}{p_+(z)} \\ &= \frac{1}{p_-(x)} \sum_{u=x \vee x_0}^{\infty} \prod_{z=x}^{u} \frac{p_-(z)}{p_+(z)}. \end{aligned}$$

Proof We first look for a function $g(x,z) \geq 0$ such that, for all x, the process

$$Z_n = g(x, X_n) - \sum_{k=0}^{n-1} \mathbb{I}\{X_k = x\}, \quad n \geq 0, \tag{1.14}$$

is a martingale, which is the case if g satisfies the following system of equations:

$$g(x,0) = p_0(0)g(x,0) + p_+(0)g(x,1) - \mathbb{I}\{x = 0\},$$
$$g(x,y) = p_-(y)g(x,y-1) + p_0(y)g(x,y) + p_+(y)g(x,y+1) - \mathbb{I}\{y = x\},$$

for $y \geq 1$. Take $g(x,0) = g(x,1) = \cdots = g(x,x) = 0$. Then for $y = x$ we get

$$g(x,x+1) = g(x,x+1) - g(x,x) \;=\; \frac{1}{p_+(x)},$$

and, for $y \geq x+1$,

$$\begin{aligned} g(x,y+1) - g(x,y) &= \frac{p_-(y)}{p_+(y)}(g(x,y) - g(x,y-1)) \\ &= \prod_{z=x+1}^{y} \frac{p_-(z)}{p_+(z)}(g(x,x+1) - g(x,x)) \\ &= \frac{1}{p_+(x)} \prod_{z=x+1}^{y} \frac{p_-(z)}{p_+(z)}. \end{aligned}$$

Therefore, for $y \geq x+1$,

$$\begin{aligned} g(x,y) \;=\; \sum_{u=x}^{y-1}(g(x,u+1) - g(x,u)) &= \frac{1}{p_+(x)} \sum_{u=x}^{y-1} \prod_{z=x+1}^{u} \frac{p_-(z)}{p_+(z)} \\ &= \frac{1}{p_-(x)} \sum_{u=x}^{y-1} \prod_{z=x}^{u} \frac{p_-(z)}{p_+(z)}, \end{aligned}$$

which is increasing in y. This sequence is bounded under the condition (1.13). Then

$$g(x,\infty) \;:=\; \lim_{y\to\infty} g(x,y) = \frac{1}{p_+(x)} \sum_{u=x}^{\infty} \prod_{z=x+1}^{u} \frac{p_-(z)}{p_+(z)} \;<\; \infty.$$

The sequence (1.14) is a martingale, so for all n, x, and x_0,

$$g(x,x_0) \;=\; \mathbb{E}_{x_0} Z_0 \;=\; \mathbb{E}_{x_0} Z_n = \mathbb{E}_{x_0} g(x, X_n) - \mathbb{E}_{x_0} \sum_{k=0}^{n-1} \mathbb{I}\{X_k = x\}$$

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and hence

$$\sum_{k=0}^{n-1} \mathbb{P}_{x_0}\{X_k = x\} = \mathbb{E}_{x_0} g(x, X_n) - g(x, x_0) \ < \ g(x, \infty) \ < \ \infty.$$

The finiteness of the Green function implies the transience of $\{X_n\}$; hence $X_n \to \infty$ a.s. as $n \to \infty$. Thus, we get the following explicit representation for the renewal measure:

$$\begin{aligned} h_{x_0}(x) \ &= \ g(x, \infty) - g(x, x_0) = \frac{1}{p_+(x)} \sum_{u=x\vee x_0}^{\infty} \prod_{z=x+1}^{u} \frac{p_-(z)}{p_+(z)} \\ &= \frac{1}{p_-(x)} \sum_{u=x\vee x_0}^{\infty} \prod_{z=x}^{u} \frac{p_-(z)}{p_+(z)}. \end{aligned}$$

□

Now let us derive some asymptotics for $h_{x_0}(x)$ as $x \to \infty$.

Proposition 1.34 *Assume that*

$$\frac{2m_1(x)}{m_2(x)} \ = \ \frac{2(\varepsilon_+(x) + \varepsilon_-(x))}{2p + \varepsilon_+(x) - \varepsilon_-(x)} \sim r(x) \quad \text{as } x \to \infty, \tag{1.15}$$

where $r(x)$ is a differentiable decreasing function such that $r'(x)/r^2(x)$ has a limit at infinity. Then

$$h_{x_0}(x) \sim \frac{1}{pr(x)} \frac{1}{1 + \lim_{y\to\infty} r'(y)/r^2(y)} \quad \text{as } x \to \infty.$$

Proof We have

$$\prod_{z=x}^{u} \frac{p_-(z)}{p_+(z)} = \exp\left\{\sum_{z=x}^{u} \log \frac{1 - \varepsilon_-(z)/p}{1 + \varepsilon_+(z)/p}\right\}.$$

The asymptotic equivalence (1.15) is equivalent to

$$\log \frac{1 - \varepsilon_-(x)/p}{1 + \varepsilon_+(x)/p} \sim -r(x) \quad \text{as } x \to \infty.$$

Fix an $\varepsilon > 0$. Then, for all sufficiently large x, we can write

$$-(1+\varepsilon)r(x) \ \leq \ \log \frac{1 - \varepsilon_-(x)/p}{1 + \varepsilon_+(x)/p} \ \leq \ -(1-\varepsilon)r(x).$$

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Therefore, for such x, we have the following upper bound:

$$\begin{aligned} h_{x_0}(x) &\le \frac{1}{p_-(x)} \sum_{u=x}^{\infty} \exp\Big\{-(1-\varepsilon)\sum_{z=x}^{u} r(z)\Big\} \\ &\le \frac{1}{p_-(x)} \sum_{u=x}^{\infty} \exp\Big\{-(1-\varepsilon)\int_x^{u+1} r(z)dz\Big\} \\ &\le \frac{1}{p_-(x)} \int_x^{\infty} \exp\Big\{-(1-\varepsilon)\int_x^{u} r(z)dz\Big\}du, \end{aligned}$$

as $r(z)$ is a decreasing function. Setting

$$U_\varepsilon(x) = \int_x^{\infty} \exp\Big\{-(1-\varepsilon)\int_0^{u} r(z)dz\Big\}du$$

we observe that

$$\int_x^{\infty} \exp\Big\{-(1-\varepsilon)\int_x^{u} r(z)dz\Big\}du = \frac{U_\varepsilon(x)}{-U_\varepsilon'(x)}.$$

By L'Hôpital's rule and the equality $U_\varepsilon''(x) = -(1-\varepsilon)r(x)U_\varepsilon'(x)$ we have

$$\begin{aligned} \lim_{x\to\infty} \frac{U_\varepsilon(x)}{-U_\varepsilon'(x)/r(x)} &= \lim_{x\to\infty} \frac{U_\varepsilon'(x)}{-U_\varepsilon''(x)/r(x) + U_\varepsilon'(x)r'(x)/r^2(x)} \\ &= \frac{1}{1-\varepsilon+\lim_{x\to\infty} r'(x)/r^2(x)}. \end{aligned}$$

Therefore,

$$\limsup_{x\to\infty} h_{x_0}(x)r(x) \le \frac{1}{p}\frac{1}{1-\varepsilon+\lim_{x\to\infty} r'(x)/r^2(x)}.$$

Similarly, starting from the inequalities

$$\begin{aligned} h_{x_0}(x) &\ge \frac{1}{p_+(x)} \sum_{u=x}^{\infty} \exp\Big\{-(1+\varepsilon)\sum_{z=x+1}^{u} r(z)\Big\} \\ &\ge \frac{1}{p_+(x)} \sum_{u=x}^{\infty} \exp\Big\{-(1+\varepsilon)\int_x^{u} r(z)dz\Big\} \\ &\ge \frac{1}{p_+(x)} \int_x^{\infty} \exp\Big\{-(1+\varepsilon)\int_x^{u} r(z)dz\Big\}du, \end{aligned}$$

we get the lower bound

$$\liminf_{x\to\infty} h_{x_0}(x)r(x) \ge \frac{1}{p}\frac{1}{1+\varepsilon+\lim_{x\to\infty} r'(x)/r^2(x)}.$$

Since $\varepsilon > 0$ is arbitrary we arrive at the claim of the theorem. □

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Example 1.35 Assume that $\varepsilon_+(x) \sim \mu_+/x$ and $\varepsilon_-(x) \sim \mu_-/x$ as $x \to \infty$. If $\mu := \mu_+ + \mu_- > p$ then (1.15) is valid, with $r(x) = \mu/px$, $r'(x)/r^2(x) \to -p/\mu$, and we deduce that

$$h_{x_0}(x) \sim \frac{x}{\mu - p} \quad \text{as } x \to \infty.$$

Example 1.36 Assume that $\varepsilon_+(x) \sim \mu_+/x^\alpha$ and $\varepsilon_-(x) \sim \mu_-/x^\alpha$ as $x \to \infty$. If $\mu := \mu_+ + \mu_- > 0$ and $\alpha \in (0,1)$ then (1.15) is valid with $r(x) = \mu/px^\alpha$, $r'(x)/r^2(x) \to 0$, and we deduce a Weibullian asymptotics for the renewal measure at infinity,

$$h_{x_0}(x) \sim \frac{x^\alpha}{\mu} \sim \frac{1}{m_1(x)} \quad \text{as } x \to \infty.$$

These two examples demonstrate the kind of asymptotic behaviour of the renewal measure that we could expect for general Markov chains; see Chapters 4 and 6.

We conclude this section by showing that the condition (1.13) is also necessary for the transience of nearest neigbour Markov chains. The transience of $\{X_n\}$ implies that, for all x, the sequence $\sum_{k=0}^{n-1} \mathbb{I}\{X_k = x\}$ monotonically converges a.s. and in L_1 as $n \to \infty$. Therefore, the sequence (1.14) satisfies $\mathbb{E} \min_n Z_n > -\infty$. This allows us to apply the martingale convergence theorem: Z_n converges almost surely to an integrable random variable Z_∞. Combining this with the convergence of $\sum_{k=0}^{n-1} \mathbb{I}\{X_k = x\}$, we infer that $g(x, X_n)$ converges almost surely too. If we assume now that (1.13) is not valid then

$$g(x,y) \uparrow g(x,\infty) = \infty \quad \text{as } y \to \infty,$$

and the irreducibility of $\{X_n\}$ would imply that

$$\limsup_{n\to\infty} g(x, X_n) = \infty \quad \text{a.s.}$$

This would contradict the convergence of $g(x, X_n)$, hence (1.13) is necessary for the transience of $\{X_n\}$.

An alternative approach to the classification of nearest neighbour Markov chains may be found in Karlin and Taylor [87, Section 3.7].

1.4.3 Harmonic Functions and *h*-Transforms

Consider $\{X_n\}$ killed at hitting zero, by setting $p_-(1) = 0$. The corresponding transition kernel is s*ubstochastic*, which means that each row sums to a value not greater than 1. Let us construct a harmonic function for this kernel, that is, a non-negative solution V to the system of linear equations

$$V(x) = p_+(x)V(x+1) + p_0(x)V(x) + p_-(x)V(x-1), \quad x \geq 1, \quad (1.16)$$

with initial condition $V(0) = 0$.

Lemma 1.37 *For all $x \geq 1$,*

$$V(x) = V(1) \sum_{y=0}^{x-1} \prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)}. \quad (1.17)$$

Proof Let τ_y be the first hitting time of y, that is,

$$\tau_y := \inf\{n \geq 1 : X_n = y\}.$$

Then the equations (1.16) with initial condition $V(0) = 0$ are equivalent to

$$V(x) = \mathbb{E}_x\{V(X_1); \ \tau_0 > 1\}, \quad x \geq 1, \quad (1.18)$$

which defines a harmonic function for the chain $\{X_n\}$ killed at hitting zero.

It is clear that (1.16) can be rewritten in the form

$$p_+(x)[V(x+1) - V(x)] = p_-(x)[V(x) - V(x-1)].$$

Consequently,

$$V(x+1) - V(x) = [V(1) - V(0)] \prod_{k=1}^{x} \frac{p_-(k)}{p_+(k)}, \quad x \geq 1. \quad (1.19)$$

Recalling that $V(0) = 0$, we then obtain the harmonic function V for the chain $\{X_n\}$ killed at hitting zero in closed form:

$$V(x) = \sum_{y=0}^{x-1} [V(y+1) - V(y)] = V(1) \sum_{y=0}^{x-1} \prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)}.$$

□

The existence of a positive harmonic function allows us to transform a strictly substochastic transition kernel for the chain $\{X_n\}$ killed at hitting zero into a stochastic transition kernel. For every $x \geq 1$, define

$$\widehat{p}_+(x) := \frac{V(x+1)}{V(x)} p_+(x), \quad \widehat{p}_0(x) = p_0(x), \ \text{ and } \widehat{p}_-(x) := \frac{V(x-1)}{V(x)} p_-(x).$$

The new transition kernel $\widehat{P}$ is stochastic because, as follows from (1.16),

$$\widehat{p}_-(x) + \widehat{p}_0(x) + \widehat{p}_+(x) = 1 \quad \text{for all } x \geq 1.$$

This transformation is called Doob's h-transform, for a Markov chain killed at hitting zero. Let $\{\widehat{X}_n\}$ be a Markov chain on $\{1, 2, \ldots\}$ with transition kernel $\widehat{P}$.

Lemma 1.38 *The chain $\{\widehat{X}_n\}$ is always transient.*

Proof As shown in the previous subsection, it suffices to show that (1.13) holds for the transition probabilities $\widehat{P}$. We first apply the definition of $\widehat{P}$:

$$\sum_{u=1}^{\infty}\prod_{z=2}^{u}\frac{\widehat{p}_-(z)}{\widehat{p}_+(z)} = \sum_{u=1}^{\infty}\prod_{z=2}^{u}\frac{V(z-1)}{V(z+1)}\frac{p_-(z)}{p_+(z)}$$
$$= \sum_{u=1}^{\infty}\frac{V(1)V(2)}{V(u)V(u+1)}\prod_{z=2}^{u}\frac{p_-(z)}{p_+(z)}.$$

It follows from (1.19) that

$$\frac{1}{V(u)} - \frac{1}{V(u+1)} = \frac{V(u+1)-V(u)}{V(u)V(u+1)} = \frac{V(1)}{V(u)V(u+1)}\prod_{z=1}^{u}\frac{p_-(z)}{p_+(z)}.$$

Therefore,

$$\sum_{u=1}^{\infty}\prod_{z=2}^{u}\frac{\widehat{p}_-(z)}{\widehat{p}_+(z)} = \frac{p_+(1)}{p_-(1)}V(2)\sum_{u=1}^{\infty}\left(\frac{1}{V(u)} - \frac{1}{V(u+1)}\right)$$
$$\le \frac{p_+(1)}{p_-(1)}\frac{V(2)}{V(1)} < \infty,$$

which is equivalent to the transience of the transformed chain $\{\widehat{X}_n\}$. □

A standard application of Doob's h-transform is a random walk conditioned to stay positive. Let $\{X_n\}$ be a simple symmetric random walk on $\mathbb{Z}$, that is, $p_-(x) = p_+(x) = 1/2$ for all $x \in \mathbb{Z}$. Then it follows from (1.17) that $V(x) = xV(1)$. As a result the transformed chain $\{\widehat{X}_n\}$ has transition probabilities

$$\widehat{p}_-(x) = \frac{x-1}{2x} = \frac{1}{2} - \frac{1}{2x}, \quad \widehat{p}_+(x) = \frac{x+1}{2x} = \frac{1}{2} + \frac{1}{2x}, \quad x \ge 1.$$

It is immediate from these formulae that the transformed chain has an asymptotically zero drift and unit second moment of jumps.

If the original Markov chain $\{X_n\}$ is recurrent then one can use the h-transform to connect the stationary measure π of $\{X_n\}$ with the Green function of $\{\widehat{X}_n\}$. The following representation for the invariant measure π via a cycle structure (generated by the returning time to the state 0) of the Markov chain $\{X_n\}$ is well known – see e.g. [126, Theorem 10.4.9] – for $x \ge 1$:

$$\pi(x) = \pi(0)\sum_{n=1}^{\infty}\mathbb{P}_0\{X_n = x,\ \tau_0 > n\}$$
$$= \pi(0)p_+(0)\sum_{n=0}^{\infty}\mathbb{P}_1\{X_n = x,\ \tau_0 > n\}.$$

Noting that $\mathbb{P}_1\{X_n = x,\ \tau_0 > n\} = \frac{V(1)}{V(x)}\mathbb{P}_1\{\widehat{X}_n = x\}$ for all $x, n \geq 1$, we obtain

$$\pi(x) = \frac{\pi(0)p_+(0)V(1)}{V(x)}\widehat{h}_1(x), \tag{1.20}$$

where

$$\widehat{h}_1(x) := \sum_{n=0}^{\infty}\mathbb{P}_1\{\widehat{X}_n = x\}, \quad x \geq 1.$$

Let us consider a couple of examples; we first discuss a drift of order $-\mu/x$.

Example 1.39 Let $\varepsilon_+(x) \sim -\mu_+/x$ and $\varepsilon_-(x) \sim -\mu_-/x$ as $x \to \infty$ in such a way that

$$\sum_{x=0}^{\infty}\left|\varepsilon_+(x) + \varepsilon_-(x) + \frac{\mu_+ + \mu_-}{x}\right| < \infty.$$

Let $\mu := \mu_+ + \mu_- > p$, so the chain is positive recurrent. As follows from (1.19), for all $x \geq 1$,

$$\begin{aligned}
V(x+1) - V(x) &= [V(1) - V(0)]\prod_{k=1}^{x}\frac{p_-(k)}{p_+(k)} \\
&= [V(1) - V(0)]\ \exp\left(\sum_{k=1}^{x}(\log p_-(k) - \log p_+(k))\right) \\
&= [V(1) - V(0)]\ \exp\left(\sum_{k=1}^{x}(\log(1 - \varepsilon_-(k)/p) - \log(1 + \varepsilon_+(k)/p))\right).
\end{aligned}$$

As in (1.5), we conclude that there is an asymptotic relation, for some c_1,

$$\begin{aligned}
V(x+1) - V(x) &\sim [V(1) - V(0)]\exp\left(-\frac{1}{p}\sum_{k=1}^{x}(\varepsilon_-(k) + \varepsilon_+(k)) + c_1\right) \\
&\sim [V(1) - V(0)]\exp\left(\frac{\mu_- + \mu_+}{p}\log x + c_2\right) \\
&\sim c_3 x^{\mu/p} \quad \text{as } x \to \infty.
\end{aligned}$$

Therefore, as $x \to \infty$,

$$\frac{V(x+1)}{V(x)} = 1 + \frac{V(x+1) - V(x)}{V(x)} = 1 + \frac{\mu/p + 1}{x} + o(1/x),$$

and

$$\frac{V(x-1)}{V(x)} = 1 - \frac{V(x) - V(x-1)}{V(x)} = 1 - \frac{\mu/p + 1}{x} + o(1/x).$$

Hence, the transition probabilities of the transformed Markov chain satisfy the relations

$$\widehat{p}_+(x) := \frac{V(x+1)}{V(x)} p_+(x) \;=\; p + \frac{\mu_- + p}{x} + o(1/x),$$
$$\widehat{p}_-(x) := \frac{V(x-1)}{V(x)} p_-(x) \;=\; p - \frac{\mu_+ + p}{x} + o(1/x).$$

It follows from Example 1.35 with $\widehat{\mu}_+ = \mu_- + p$ and $\widehat{\mu}_- = \mu_+ + p$ that

$$\widehat{h}_1(x) \sim \frac{x}{\widehat{\mu}_+ + \widehat{\mu}_- - p} \;=\; \frac{x}{\mu + p},$$

which, on being substituted into (1.20), implies, as $x \to \infty$,

$$\pi(x) = c_3 \frac{\widehat{h}_1(x)}{V(x)} \;\sim\; \frac{c_4}{x^{\mu/p}},$$

which coincides with the expression (1.11).

This relation between the stationary measure of a nearest neighbour Markov chain and the Green function of the transformed chain may be extended to the general case. We follow this approach in Chapter 8 to derive the power asymptotics of invariant probabilities of this type for a broad class of Markov chains on $\mathbb{R}$, those with asymptotically zero drift of order $-\mu/x$.

The second example concerns a drift of order $-\mu/x^\alpha$, $\alpha \in (0,1)$.

Example 1.40 Let $\varepsilon_+(x) \sim -\mu_+/x^\alpha$ and $\varepsilon_-(x) \sim -\mu_-/x^\alpha$ as $x \to \infty$ for some $\mu_+, \mu_- > 0$ and $\alpha \in (1/2, 1)$, in such a way that

$$\sum_{x=0}^{\infty} \left| \varepsilon_+(x) + \varepsilon_-(x) + \frac{\mu_+ + \mu_-}{x^\alpha} \right| < \infty.$$

Similarly to the last example, for some c_5,

$$V(x+1) - V(x) \sim [V(1) - V(0)] \exp\left(-\frac{1}{p} \sum_{k=1}^{x} (\varepsilon_-(k) + \varepsilon_+(k)) + c_5 \right)$$
$$\sim c_6 \exp\left(\frac{\mu_- + \mu_+}{p(1-\alpha)} x^{1-\alpha} \right) \quad \text{as } x \to \infty.$$

Therefore, as $x \to \infty$,

$$\frac{V(x+1)}{V(x)} = 1 + \frac{\mu_+ + \mu_-}{p x^\alpha} + o(1/x)$$

and

$$\frac{V(x-1)}{V(x)} = 1 - \frac{\mu_+ + \mu_-}{p x^\alpha} + o(1/x).$$

Hence, the transition probabilities of the transformed Markov chain satisfy the relations

$$\widehat{p}_+(x) := \frac{V(x+1)}{V(x)} p_+(x) \;=\; p + \frac{\mu_-}{x^\alpha} + O(1/x^{2\alpha}),$$
$$\widehat{p}_-(x) := \frac{V(x-1)}{V(x)} p_-(x) \;=\; p - \frac{\mu_+}{x^\alpha} + O(1/x^{2\alpha}).$$

It follows from Example 1.35 with $\widehat{\mu}_+ = \mu_-$ and $\widehat{\mu}_- = \mu_+$ that

$$\widehat{h}_1(x) \sim \frac{x^\alpha}{\widehat{\mu}_+ + \widehat{\mu}_-},$$

which, on being substituted into (1.20), implies a Weibullian asymptotic behaviour of the invariant probabilities, as $x \to \infty$:

$$\pi(x) = c_7 \frac{\widehat{h}_1(x)}{V(x)} \;\sim\; c_8 \exp\left(-\frac{\mu_- + \mu_+}{p(1-\alpha)} x^{1-\alpha}\right),$$

which coincides with the expression (1.12).

General Markov chains on $\mathbb{R}$ with asymptotically zero drift of order $-\mu/x^\alpha$, $\alpha \in (0,1)$, are considered in Chapter 9 where we again follow the above approach to derive the Weibullian-type asymptotics of invariant probabilities.

1.4.4 Down-Crossing Probabilities for Transient Chain

Let $\{X_n\}$ be transient, that is, its probability of hitting the origin, $\mathbb{P}_x\{\tau_0 < \infty\}$, is less than 1 for all $x \geq 1$. The goal of the following calculations is to find this probability.

The function $V(x)$ computed in (1.17) is increasing and bounded provided that the condition (1.13) holds. As it has already been noticed in (1.18), the sequence $V(X_{n\wedge\tau_0})$ is a bounded non-negative martingale, so, by the optional stopping theorem (see e.g. [56, Section 4.7]),

$$\begin{aligned} V(x) \;&=\; \mathbb{E}_x V(X_0) = \mathbb{E}_x V(X_{\tau_0}) \\ &\qquad\qquad = V(0)\mathbb{P}_x\{\tau_0 < \infty\} + V(\infty)\mathbb{P}_x\{\tau_0 = \infty\} \end{aligned}$$

and hence

$$\mathbb{P}_x\{\tau_0 < \infty\} = \frac{V(\infty) - V(x)}{V(\infty) - V(0)} \;=\; \frac{\sum_{y=x}^\infty \prod_{k=1}^y \frac{p_-(k)}{p_+(k)}}{\sum_{y=0}^\infty \prod_{k=1}^y \frac{p_-(k)}{p_+(k)}}.$$

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Owing to the left-continuity of the Markov chain, similarly we get, for all $0 \le \widehat{x} < x$,

$$\mathbb{P}_x\{\tau_{\widehat{x}} < \infty\} = \frac{V(\infty) - V(x)}{V(\infty) - V(\widehat{x})} = \frac{\sum_{y=x}^{\infty} \prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)}}{\sum_{y=\widehat{x}}^{\infty} \prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)}}. \tag{1.21}$$

Example 1.41 In the case where $\varepsilon_+(x) \sim \mu_+/x$ and $\varepsilon_-(x) \sim \mu_-/x$ as $x \to \infty$, $\mu := \mu_+ + \mu_- > p$, and

$$\sum_{x=0}^{\infty} \left| \varepsilon_+(x) + \varepsilon_-(x) - \frac{\mu}{x} \right| < \infty,$$

then similarly to (1.11) we derive that

$$\prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)} \sim c_5 y^{-\mu/p} \quad \text{as } y \to \infty,$$

where $c_5 > 0$. Therefore, (1.21) implies that there exists a function $c(\widehat{x}) \to 1$ as $\widehat{x} \to \infty$ such that

$$\mathbb{P}_x\{\tau_{\widehat{x}} < \infty\} \sim c(\widehat{x})(\widehat{x}/x)^{\mu/p-1} \quad \text{as } x \to \infty, \text{ uniformly for all } \widehat{x} < x.$$

In particular,

$$\mathbb{P}_x\{\tau_{\widehat{x}} < \infty\} \sim (\widehat{x}/x)^{\mu/p-1} \quad \text{as } \widehat{x},\ x \to \infty,\ x > \widehat{x}.$$

Compare this result with Theorem 3.2 and Corollary 3.3, where a general transient Markov chain with a drift of order μ/x is studied.

Example 1.42 Assume that $\varepsilon_+(x) \sim \mu_+/x^\alpha$ and $\varepsilon_-(x) \sim \mu_-/x^\alpha$ as $x \to \infty$. If $\mu := \mu_+ + \mu_- > 0$, $\alpha \in (1/2, 1)$, and

$$\sum_{x=0}^{\infty} \left| \varepsilon_+(x) + \varepsilon_-(x) - \frac{\mu}{x^\alpha} \right| < \infty$$

then the series $\sum \varepsilon^2(x)$ is convergent and we obtain

$$\prod_{k=1}^{y} \frac{p_-(k)}{p_+(k)} \sim c_6 e^{-\mu y^{1-\alpha}/p(1-\alpha)} \quad \text{as } y \to \infty,$$

where $c_6 > 0$. Therefore, (1.21) implies a Weibullian asymptotic behaviour of the down-crossing probability, that is, there exists a function $c(\widehat{x}) \to 1$ as $\widehat{x} \to \infty$ such that

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$$\mathbb{P}_x\{\tau_{\widehat{x}} < \infty\} \sim c(\widehat{x}) \frac{\sum_{u=x}^{\infty} \exp\left(-\mu u^{1-\alpha}/p(1-\alpha)\right)}{\sum_{u=\hat{x}}^{\infty} \exp\left(-\mu u^{1-\alpha}/p(1-\alpha)\right)}$$
$$\sim c(\widehat{x}) \left(\frac{x}{\hat{x}}\right)^{\alpha} \exp\left(\mu(\widehat{x}^{1-\alpha} - x^{1-\alpha})/p(1-\alpha)\right)$$

as $x \to \infty$ uniformly for all $\widehat{x} < x$. In particular,

$$\mathbb{P}_x\{\tau_{\widehat{x}} < \infty\} \sim \left(\frac{x}{\hat{x}}\right)^{\alpha} \exp\Big(\mu(\widehat{x}^{1-\alpha} - x^{1-\alpha})/p(1-\alpha)\Big) \text{ as } \widehat{x},\ x \to \infty,\ x > \widehat{x}.$$

Compare this result with Theorem 3.7, where a general transient Markov chain with a drift of order μ/x^{α}, $\alpha \in (1/2, 1)$, is studied.

1.5 Heuristics Coming from Diffusion Processes

1.5.1 Diffusion with Bounded Smooth Infinitesimal Parameters

Another example where various characteristics are available in closed form is provided by diffusion processes on $\mathbb{R}$ that are continuous-time Markov processes with continuous paths. If they are sampled at non-random equally spaced time epochs they give us examples of Markov chains for which some characteristics are explicitly calculable.

Let us start with a result that demonstrates that the existence of an invariant probability measure for a diffusion process is equivalent to its positive recurrence.

Lemma 1.43 *For a diffusion process $\{X(t)\}$ with diffusion coefficient everywhere positive the following are equivalent:*

(i) *there is a stationary version of the process $\{X(t)\}$;*
(ii) *the process $\{X(t)\}$ is positive recurrent, that is, $\mathbb{E}_x\tau_y < \infty$ for all states x and y, where $\tau_y := \inf\{t : X(t) = y\}$.*

Proof Let $\{X(t)\}$ possess an invariant probability measure π. Then the same is true for the slotted Markov chain $X_n = X(n)$, $n \in \mathbb{Z}^+$. Since the diffusion coefficient is everywhere positive, the jumps of $\{X_n\}$ are absolutely continuous with positive density function, so the chain $\{X_n\}$ is ψ-irreducible; see [126, Proposition 4.2.2]. Therefore, the existence of an invariant probability measure for $\{X_n\}$ implies the positive recurrence of any compact set B of positive Lebesgue measure, in the sense that $\mathbb{E}_x\tau_B < \infty$ for all x. Hence, B is positive recurrent for $\{X(t)\}$ too, which implies the positive recurrence of the diffusion process due to the continuity of its paths.

Vice versa, let $\{X(t)\}$ be positive recurrent. Then, for any two fixed distinct states x and y, the stopping time

$$\tau := \min\{t : X(t) = x \text{ and } X(s) = y \text{ for some } s < t\}$$

is finite on average given $X(0) = x$, $\mathbb{E}_x\tau < \infty$. In addition, $\tau > 0$. For that reason a measure

$$\begin{aligned}\mu(B) &:= \mathbb{E}_x \int_0^\tau \mathbb{I}\{X(t) \in B\}\,dt \\ &= \int_0^\infty \mathbb{P}_x\{X(t) \in B,\ \tau > t\}\,dt\end{aligned}$$

is non-zero and finite, $\mu(\mathbb{R}) = \mathbb{E}_x\tau \in (0, \infty)$. Let us show that it is invariant for $\{X(t)\}$, that is, for any $s > 0$ and any bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have

$$\int_{\mathbb{R}} \varphi(z)\mu(dz) = \int_{\mathbb{R}} \mathbb{E}\{\varphi(X(s)) \mid X(0) = z\}\mu(dz).$$

Indeed, the difference between the right- and left-hand side integrals equals

$$\begin{aligned}&\int_{\mathbb{R}} \mathbb{E}\{\varphi(X(s)) - \varphi(z) \mid X(0) = z\}\mu(dz) \\ &\quad= \int_{\mathbb{R}} \mathbb{E}\{\varphi(X(t+s)) - \varphi(X(t)) \mid X(t) = z\} \int_0^\infty \mathbb{P}_x\{X(t) \in dz,\ \tau > t\}\,dt \\ &\quad= \int_0^\infty \mathbb{E}_x\{\varphi(X(t+s)) - \varphi(X(t)),\ \tau > t\}\,dt,\end{aligned}$$

because $\{\tau > t\} = \overline{\{\tau \le t\}} \in \sigma(X_u,\ u \le t)$. Since

$$\begin{aligned}\int_0^\infty \mathbb{E}_x\{\varphi(X(t+s)),\ \tau > t\}\,dt &= \mathbb{E}_x \int_0^\tau \varphi(X(t+s))dt \\ &= \mathbb{E}_x \int_s^{\tau+s} \varphi(X(t))dt,\end{aligned}$$

we get

$$\begin{aligned}&\int_0^\infty \mathbb{E}_x\{\varphi(X(t+s)) - \varphi(X(t)),\ \tau > t\}\,dt \\ &\qquad= \mathbb{E}_x \int_s^{\tau+s} \varphi(X(t))dt - \mathbb{E}_x \int_0^\tau \varphi(X(t))dt \\ &\qquad= \mathbb{E}_x \int_\tau^{\tau+s} \varphi(X(t))dt - \mathbb{E}_x \int_0^s \varphi(X(t))dt \\ &\qquad= 0,\end{aligned}$$

by the Markov property, since $X(\tau) = x$. □

Consider a diffusion process $X = \{X(t)\}$ on $\mathbb{R}$ with smooth drift $\mu(x)$ and diffusion coefficient $\sigma^2(x) > 0$. In the case of a stationary diffusion process, the invariant density function $p(x)$ solves the stationary Kolmogorov forward equation

$$0 = -\frac{d}{dx}(\mu(x)p(x)) + \frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)p(x)),$$

which has the following solution:

$$p(x) = \frac{c}{\sigma^2(x)} \exp\left(\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy\right), \quad c > 0. \tag{1.22}$$

It follows that a diffusion process possesses a probabilistic invariant distribution – is positive recurrent – if and only if

$$\text{the function } \frac{1}{\sigma^2(x)} \exp\left(\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy\right) \quad \text{is integrable at } \pm\infty. \tag{1.23}$$

It is also known that the half-line $(-\infty, 0]$ is recurrent for a diffusion process, in the sense that $\mathbb{P}_x\{X(t) \le 0 \text{ for some } t\} = 1$ for all $x > 0$ if

$$\text{the function } \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy\right) \quad \text{is not integrable at } \infty; \tag{1.24}$$

see, e.g. [88, Chapter 15, Theorem 7.3] or [34, Section 4.1]; and, vice versa, it is transient in the sense that $\mathbb{P}_x\{X(t) > 0 \text{ for all } t > 0\} > 0$ for all $x > 0$ if

$$\text{the function } \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy\right) \quad \text{is integrable at } \infty, \tag{1.25}$$

see e.g. [88, Chapter 15, Lemma 6.1].

As one can see, the classification of diffusion processes relies heavily on the asymptotic behaviour of the ratio $2\mu(x)/\sigma^2(x)$ at infinity. In particular, if

$$\mu(x) \sim -\mu/x \text{ and } \sigma^2(x) \to \sigma^2 > 0 \qquad \text{as } x \to \infty \tag{1.26}$$

for some $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ then:

- integrability at infinity in (1.23) holds for $2\mu > \sigma^2$;
- non-integrability at infinity in (1.24) holds for $2\mu > -\sigma^2$;
- integrability at infinity in (1.25) holds for $2\mu < -\sigma^2$.

Knowledge of the invariant probability density function in closed form, (1.22), allows us to analyse its asymptotic behaviour under various regularity conditions of the drift and diffusion coefficients at infinity.

Example 1.44 Let $\{X(t)\}$ possess a probabilistic invariant measure and let (1.26) hold with $2\mu > \sigma^2$. If

$$\int_1^\infty \left|\frac{\mu(x)}{\sigma^2(x)} + \frac{\mu}{\sigma^2 x}\right| dx < \infty$$

then (1.22) yields the following asymptotic equivalence, for some $c_1 > 0$,

$$p(x) \sim \frac{c_1}{x^{2\mu/\sigma^2}} \quad \text{as } x \to \infty.$$

Example 1.45 Let $\{X(t)\}$ possess a probabilistic invariant measure. If $\mu(x) \sim -\mu/x^\alpha$ and $\sigma^2(x) \to \sigma^2 > 0$ as $x \to \infty$ for some $\mu > 0$ and $\alpha \in (0,1)$, in such a way that

$$\int_1^\infty \left|\frac{\mu(x)}{\sigma^2(x)} + \frac{\mu}{\sigma^2 x^\alpha}\right| dx < \infty,$$

then

$$p(x) \sim c_2 \exp\left(-2\mu x^{1-\alpha}/\sigma^2(1-\alpha)\right) \quad \text{as } x \to \infty.$$

Let $\{X(t)\}$ be a diffusion process satisfying the condition (1.25), so that the negative half-line $(-\infty, 0]$ is transient. A harmonic function $h(x)$ for such a diffusion process with transition kernel $P(t,x,dy)$, that is, a solution to the equation

$$\Big(\frac{\sigma^2(x)}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}\Big)h(x) = 0, \tag{1.27}$$

is computable in closed form as follows:

$$h(x) = \int_x^\infty \exp\left(-\int_0^z \frac{2\mu(y)}{\sigma^2(y)}dy\right) dz, \quad x \in \mathbb{R}. \tag{1.28}$$

This is a positive decreasing function. By Itô's formula, the process $\{h(X(t))\}$ is a martingale; hence we can apply Doob's h-transform, which returns a new stochastic transition kernel

$$\widehat{P}(t,x,dy) := \frac{h(y)}{h(x)} P(t,x,dy).$$

Let us consider a diffusion process $\widehat{X} = \{\widehat{X}(t)\}$ with this transition kernel. The drift coefficient of $\widehat{X}$ is

$$\begin{aligned}
\widehat{\mu}(x) &= \lim_{t\to 0} \frac{1}{t} \int (y-x)\frac{h(y)}{h(x)} P(t,x,dy) \\
&= \lim_{t\to 0} \frac{1}{t} \int (y-x)\Big(1 + \frac{h'(x)}{h(x)}(y-x) + O((y-x)^2)\Big) P(t,x,dy) \\
&= \mu(x) + \frac{h'(x)}{h(x)}\sigma^2(x),
\end{aligned} \tag{1.29}$$

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and, since $h'(x) < 0$, $\widehat{\mu}(x) < \mu(x)$. The diffusion coefficient does not change: $\widehat{\sigma}^2(x) = \sigma^2(x)$.

If, for some $\widetilde{c} > 3$,

$$\frac{2\mu(x)}{\sigma^2(x)} \geq \frac{\widetilde{c}}{x} \quad \text{for all sufficiently large } x,$$

then, under some mild additional condition,

$$-h'(x) \geq c_2 h(x)/x \qquad \text{for some } c_2 > 0,$$

and the set $(-\infty, 0]$ is positive recurrent for the transformed chain $\{\widehat{X}(t)\}$. Indeed, in this case

$$h(x) \leq \int_x^\infty \exp\left(c_3 - \int_1^z \frac{\widetilde{c}}{y} dy\right) dz \;=\; c_4 x^{1-\widetilde{c}},$$

hence the function

$$\begin{aligned}
\exp\left(\int_0^x \frac{2\widehat{\mu}(y)}{\widehat{\sigma}^2(y)} dy\right) &= \exp\left(\int_0^x \frac{2\mu(y)}{\sigma^2(y)} dy + \int_0^x 2\frac{h'(y)}{h(y)} dy\right) \\
&= \frac{h^2(x)}{h^2(0)} \exp\left(\int_0^x \frac{2\mu(y)}{\sigma^2(y)} dy\right) \;=\; -\frac{h^2(x)}{h'(x)} \frac{1}{h^2(0)} \\
&\leq \frac{x h(x)}{c_2} \;\leq\; c_4 x x^{1-\widetilde{c}}/c_2
\end{aligned}$$

is integrable at infinity because $\widetilde{c} > 3$ and the condition (1.23) for positive recurrence is met.

If, for some $\widetilde{c} \in (1, 3]$ and a function $p(x)$ that is absolutely integrable at infinity, we have

$$\frac{2\mu(x)}{\sigma^2(x)} = \frac{\widetilde{c}}{x} + p(x),$$

then the diffusion process $\{X(t)\}$ is transient by the criterion (1.25) and the transformed process $\{\widehat{X}(t)\}$ is null recurrent, because in this case

$$\begin{aligned}
h'(x) \;\sim\; -\exp\left(c_5 - \int_1^x \frac{\widetilde{c}}{y} dy\right) &= -e^{c_5} x^{-\widetilde{c}} \text{and} \\
h(x) &= -\int_x^\infty h'(z) dz \;\sim\; c_6 x^{1-\widetilde{c}}.
\end{aligned}$$

Thus the function

$$\exp\left(\int_0^x \frac{2\widehat{\mu}(y)}{\widehat{\sigma}^2(y)} dy\right) = -\frac{h^2(x)}{h'(x)} \;\sim\; c_7 x^{2-\widetilde{c}}$$

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is not integrable at infinity because $\widetilde{c} \in (1,3]$ and hence $\{\widehat{X}(t)\}$ is not positive recurrent, by (1.23), but it is still recurrent, by (1.24), because the function

$$\exp\left(-\int_0^x \frac{2\widehat{\mu}(y)}{\widehat{\sigma}^2(y)}dy\right) = -\frac{h'(x)}{h^2(x)} \sim \frac{x^{\widetilde{c}-2}}{c_7}$$

is not integrable at infinity either.

Conversely, let us consider a recurrent diffusion process $\{X(t)\}$ when $\tau = \tau_{(-\infty,0]} = \min\{t \geq 0 : X(t) \leq 0\}$ is finite with probability 1. Consider the process $Y(t) := X(t \wedge \tau)$, which is the original process stopped at the time of leaving the positive half-line. Its harmonic function solves (1.27) with $h(0) = 1$,

$$h(x) = 1 + \int_0^x \exp\left(-\int_0^z \frac{2\mu(y)}{\sigma^2(y)}dy\right) dz, \quad x \geq 0. \tag{1.30}$$

It is an increasing function tending to infinity as $x \to \infty$, due to the recurrence condition (1.24). By Itô's formula, the process $\{h(Y(t))\}$ is a martingale; hence we can apply Doob's h-transform, which returns a new stochastic transition kernel

$$\widehat{P}_Y(t,x,dy) := \frac{h(y)}{h(x)} P_Y(t,x,dy).$$

Let us consider a diffusion process $\{\widehat{Y}(t)\}$ with this transition kernel. The drift coefficient of $\{\widehat{Y}(t)\}$ is given in (1.29). Since the function $h(x)$ is increasing, $\widehat{\mu}(x) > \mu(x)$. The increase in the drift is so strong that the process $\{\widehat{Y}(t)\}$ is transient. Indeed, the function

$$\begin{aligned}\exp\left(-\int_0^x \frac{2\widehat{\mu}(y)}{\widehat{\sigma}^2(y)}dy\right) &= \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy - \int_0^x 2\frac{h'(y)}{h(y)}dy\right) \\ &= \frac{1}{h^2(x)} \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma^2(y)}dy\right) \\ &= \frac{h'(x)}{h^2(x)} = \left(\frac{-1}{h(x)}\right)'\end{aligned}$$

is integrable at infinity because $h(x) \to \infty$ and, therefore, the condition (1.25) for transience is met:

$$\int_z^\infty \exp\left(-\int_0^x \frac{2\widehat{\mu}(y)}{\widehat{\sigma}^2(y)}dy\right) dx = \frac{1}{h(z)} < \infty.$$

We follow up the idea of these calculations related to harmonic functions and changes of measure for diffusion processes in our tail analysis of invariant measures of Markov chains in Chapters 8 and 9.

1.5.2 Green Function for Transient Diffusion

Let $\{X(t)\}$ be a transient diffusion on $\mathbb{R}$ (or $\mathbb{R}^+$) with the following generator:

$$A = \mu(x)\frac{d}{dx} + \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2}.$$

We consider a regular diffusion, in the sense of properties (i)–(iii) of [135, Chapter VII.3]. For its transience it is sufficient to assume that the following function,

$$U(x) := \int_x^\infty \exp\left\{-\int_0^v \frac{2\mu(y)}{\sigma^2(y)}dy\right\} dv, \tag{1.31}$$

is finite for all x, see (1.25); this function solves the homogeneous equation

$$AU = 0. \tag{1.32}$$

In this case $X(t) \to \infty$ a.s. and we are interested in the continuous-time analogue of the renewal (Green) function,

$$H_y(x, x+h] := \int_0^\infty \mathbb{P}_y\{X(t) \in (x, x+h]\}\, dt, \quad h > 0.$$

By Proposition 1.6 in Revuz and Yor [135, Chapter VII.1], the process

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a local martingale for a wide class of functions f. This suggests the following method for computing the renewal measure of $X(t)$. Fix x and h. Suppose we can find a bounded function $f(z) = f_{h,x}(z)$ such that $f(z) \to 0$ as $z \to \infty$ and

$$Af(z) = -\mathbb{I}\{z \in (x, x+h]\}. \tag{1.33}$$

Then the optional stopping theorem, see e.g. [56, Section 4.7], and a.s. convergence of $X(t) \to \infty$ as $t \to \infty$ give us the equality

$$f(y) = \mathbb{E}_y f(X(0)) = \mathbb{E}_y\left[\int_0^\infty \mathbb{I}\{X(t) \in (x, x+h]\}\, dt\right] = H_y(x, x+h],$$

which allows us to analyse H_y.

So, we need to solve the ordinary differential equation (1.33). To this end, consider

$$m(x) := \int_0^x \frac{2dv}{-U'(v)\sigma^2(v)} = \int_0^x \frac{2}{\sigma^2(v)} \exp\left\{\int_0^v \frac{2\mu(y)}{\sigma^2(y)}dy\right\} dv$$

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and also

$$G_x(z) \;:=\; \begin{cases} U(z)m(z) + \int_z^x U(v)m(dv), & z \le x, \\ U(z)m(x), & z > x. \end{cases}$$

We have

$$\frac{d}{dz} G_x(z) \;=\; \begin{cases} U'(z)m(z), & z \le x, \\ U'(z)m(x), & z > x \end{cases}$$

and

$$\frac{d^2}{dz^2} G_x(z) \;=\; \begin{cases} U''(z)m(z) - 2/\sigma^2(z), & z \le x, \\ U''(z)m(x), & z > x, \end{cases}$$

where we consider the left second derivative at $z = x$, which, together with (1.32), implies that

$$AG_x(z) = \begin{cases} -1, & z \le x, \\ 0, & z > x, \end{cases}$$

and hence the function

$$f(z) \;=\; G_{h,x}(z) := G_{x+h}(z) - G_x(z) \tag{1.34}$$

solves (1.33).

Alternatively, one can notice that $U(x)$ is the scale function, that $m(x)$ corresponds to the speed measure, and that (see [135, Chapter VII, Theorem 3.12])

$$AG_x(z) = \frac{d}{dm(z)} \left(\frac{dG_x(z)}{-dU(z)} \right).$$

Thus, it follows from (1.34) that, for $y < x$,

$$H_y(x, x+h] = f(y) \;=\; \int_x^{x+h} U(v)m(dv) \;=\; \int_x^{x+h} \frac{2U(v)dv}{-U'(v)\sigma^2(v)}.$$

More formally one can obtain the last equality from Corollary 3.8 and Exercise 3.20 in [135, Chapter VII.3].

If the function $W(v) := U(v)/U'(v)\sigma^2(v)$ is long-tailed at infinity, see Definition 1.26, then we get the following local renewal theorem for a process $X(t)$ starting at y:

$$H_y(x, x+h] \;\sim\; \frac{2U(x)}{-U'(x)\sigma^2(x)} h \quad \text{as } x \to \infty.$$

Assume that

$$\frac{2\mu(x)}{\sigma^2(x)} \sim r(x) \quad \text{as } x \to \infty, \tag{1.35}$$

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for some differentiable function $r(x)$ such that the quotient $r'(x)/r^2(x)$ has a limit at infinity. Hence, we can apply L'Hôpital's rule and the equality $U'' = -rU'$ to obtain

$$\lim_{x\to\infty}\frac{U(x)}{-U'(x)/r(x)} = \lim_{x\to\infty}\frac{U'(x)}{-U''(x)/r(x)+U'(x)r'(x)/r^2(x)}$$
$$= \frac{1}{1+\lim_{x\to\infty} r'(x)/r^2(x)}.$$

Therefore, for any fixed $h > 0$,

$$H_y(x,x+h] \sim \frac{2}{\sigma^2(x)r(x)}\frac{1}{1+\lim_{y\to\infty} r'(y)/r^2(y)}h \quad \text{as } x\to\infty.$$

Example 1.46 If $\mu(x) \sim \mu/x$ and $\sigma^2(x) \to \sigma^2 > 0$ as $x\to\infty$ with $2\mu > \sigma^2$ then (1.35) is satisfied by $r(x) = 2\mu/\sigma^2 x$, $r'(x)/r^2(x) \to -\sigma^2/2\mu$, and we obtain

$$H_y(x,x+h] \sim \frac{2h}{2\mu-\sigma^2}x \quad \text{as } x\to\infty.$$

Example 1.47 If $\mu(x) \sim \mu/x^\alpha$, $\mu > 0$, $\alpha \in (0,1)$, and $\sigma^2(x) \to \sigma^2 > 0$ as $x \to \infty$ then (1.35) is satisfied by $r(x) = 2\mu/\sigma^2 x^\alpha$, $r'(x)/r^2(x) \to 0$, and we find that

$$H_y(x,x+h] \sim \frac{h}{\mu}x^\alpha \sim \frac{h}{\mu(x)} \quad \text{as } x\to\infty.$$

Note that this asymptotic behaviour of the renewal function does not depend on the diffusion coefficient, as for a process with constant positive drift.

1.5.3 Bessel Processes

A Bessel process is an important example of a diffusion process with asymptotically zero drift whose various probabilistic characteristics can be calculated in closed form; this provides some intuition for what can be expected for Markov chains. The simplest version of a Bessel process is defined as the Euclidean norm $\|B^{(d)}(t)\|$ of a d-dimensional Brownian motion $B^{(d)}(t)$, and it solves the stochastic differential equation

$$dX(t) = dY(t) + \frac{d-1}{2}\frac{dt}{X(t)} = dY(t) + \frac{2\nu+1}{2}\frac{dt}{X(t)}, \tag{1.36}$$

where the process $Y(t)$ is a one-dimensional Brownian motion. The parameter $\nu = (d-2)/2$ is called the *index* of X. By the same stochastic differential equation we define a Bessel process with an arbitrary index $\nu \in \mathbb{R}$. A Bessel

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process with a non-integer dimension naturally appears as the norm of a multi-dimensional Brownian motion in a cone, and the dimension is determined by the cone's geometry; see Corollary 3 in [54] and its proof.

In other words, X is a diffusion process with drift $(2\nu+1)/2x$ and diffusion coefficient 1. The intrinsic property of a Bessel process is that its drift is singular at the origin, which makes it impossible to apply the results of the last subsection.

The drift of the squared Bessel process $X^2(t)$ at any state equals $2\nu+2$, and this gives rise to the following classification; see e.g. [21, Appendix 1.21].

- If $\nu > 0$ then the process $\{X(t)\}$ is transient and there is a unique strong solution to equation (1.36). The case where the index $\nu = 0$ corresponds to the process $\sqrt{B_1^2 + B_2^2}$, which is null recurrent but the origin is never visited; hence there is again a unique strong solution to the equation (1.36).
- If $-1 \leq \nu < 0$ then the hitting time of the origin from any state $x > 0$ is finite with probability 1 and has infinite mean. In the case $-1 < \nu < 0$, the origin is a repelling (instantaneously reflecting) state for X, so there is a weak solution to (1.36) which is not unique. In the case where the index is -1, the origin is an absorbing state.
- If $\nu < -1$ then the hitting time of the origin from any state $x > 0$ has finite mean $x^2/|2\nu+2|$ and the origin is an absorbing state for $\{X(t)\}$, so there is no weak solution to (1.36).

In the first case, where $\nu \geq 0$, the transition density of $\{X(t)\}$ is well known, see e.g. [21, Appendix 1.21], and is given by the equalities

$$\begin{aligned} p_t(x,y) &= \frac{1}{t}\frac{y^{\nu+1}}{x^\nu} e^{-(x^2+y^2)/2t} I_\nu(xy/t), \\ p_t(0,y) &= \frac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}, \end{aligned} \tag{1.37}$$

where $I_\nu(z)$ is a modified Bessel function. The same formula is still valid for $\nu \in (-1,0)$ if we reflect the process $\{X(t)\}$ each time it reaches the origin.

In the positive recurrent case $\nu < -1$ or in the null recurrent case $\nu \in (-1,0)$, if we kill the process at 0 then the transition probability density function of $\{X(t)\}$ is

$$p_t(x,y) = \frac{1}{t}\frac{y^{\nu+1}}{x^\nu} e^{-(x^2+y^2)/2t} I_{|\nu|}(xy/t).$$

If $\nu \geq 0$ or $\nu \in (-1,0)$ and the process $\{X(t)\}$ is reflected each time it reaches the origin, the probability density function of $X(t)$ given $X(0) = 0$ is given by

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$$p_t(x) \;=\; p_t(0,x) = \frac{1}{2^{\nu}\Gamma(\nu+1)}\frac{x^{2\nu+1}}{t^{\nu+1}}e^{-x^2/2t}. \tag{1.38}$$

In both cases the probability density function of $X^2(t)/t$ is

$$\frac{1}{2^{\nu+1}\Gamma(\nu+1)}x^{\nu}e^{-x/2},$$

which is a Gamma density function with mean $2(\nu+1)$ and variance $4(\nu+1)$.

In the transient case $\nu > 0$ we can write down the Green function h_0 of $\{X(t)\}$ in closed form by integrating (1.38):

$$h_0(y) = \int_0^{\infty} p_t(0,y)dt = \frac{y^{2\nu+1}}{2^{\nu}\Gamma(\nu+1)}\int_0^{\infty}\frac{1}{t^{\nu+1}}e^{-y^2/2t}dt \;=\; \frac{y}{\nu},$$

which indicates the asymptotic behaviour of the renewal measure that we can expect for transient Markov chains with drift of order c/x at infinity; see Section 4.8 for results in this area.

It follows from the representation of the α-potential density G_α of X in [21, Appendix 1.21] that, for all $x \ge 0$,

$$h_x(y) = \int_0^{\infty} p_t(x,y)dt = \frac{1}{\nu}\frac{y^{2\nu+1}}{\max(x,y)^{2\nu}},$$

which implies that the first hitting time $\tau_{[0,\,y]}$ of the compact set $[0,y]$ is finite with probability

$$\begin{aligned}\mathbb{P}_x\{\tau_{[0,\,y]} < \infty\} &= \mathbb{P}_x\{X(t) = y \text{ for some } t\}\\ &= \frac{h_x(y)}{h_y(y)} \;=\; \left(\frac{y}{x}\right)^{2\nu} \quad \text{for } x > y;\end{aligned} \tag{1.39}$$

this kind of result for transient Markov chains is discussed in Chapter 3.

For any ν, the function $h(x) = x^{-2\nu}$ is harmonic for $\{X(t)\}$ as it solves the equation

$$\Big(\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}\Big)h(x) = 0.$$

By Itô's formula, the process $\{h(X(t))\}$ is a local martingale. Let $y > 0$. If $\nu > 0$ then $h(x)$ is bounded on $[y,\infty)$ and if $\nu < 0$ then it is bounded on $[0,y]$. So, in either case we can apply the optional stopping time theorem (see e.g. [56, Section 4.7]) for martingales and conclude that, for $\nu > 0$ and $x > y$,

$$\begin{aligned}h(x) &= h(y)\mathbb{P}_x\{X(t) = y \text{ for some } t\} + h(\infty)\mathbb{P}_x\{X(t) \neq y \text{ for all } t\}\\ &= h(y)\mathbb{P}_x\{\tau_{[0,\,y]} < \infty\},\end{aligned}$$

which agrees with (1.39).

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If $\nu \le -1$, which corresponds to the origin being an absorbing state, then, for $x < y$,

$$\begin{aligned} h(x) &= h(y)\mathbb{P}_x\{X(t) = y \text{ for some } t\} + h(0)\mathbb{P}_x\{X(t) \neq y \text{ for all } t\} \\ &= h(y)\mathbb{P}_x\Big\{\sup_{t\ge 0} X(t) \ge y\Big\}, \end{aligned}$$

which implies that

$$\mathbb{P}_x\Big\{\sup_{t\ge 0} X(t) \ge y\Big\} = \frac{h(x)}{h(y)} = \Big(\frac{x}{y}\Big)^{2|\nu|}.$$

For recurrent Markov chains, the tail distribution of the trajectory supremum until the time of the first entry to a neighborhood of the origin is described in Theorem 8.26.

In conclusion, let us establish a link to Markov chains by sampling the process $\{X(t)\}$ at integer time epochs and getting a Markov chain $X_n := X(n)$ in this way; in the null recurrent case we assume a reflecting boundary condition. This Markov chain is of Lamperti type with mean drift $m_1(x)$ and second moment of jumps $m_2(x)$ satisfying the relations

$$m_1(x) \sim \frac{\nu + 1/2}{x} =: \frac{c}{x} \quad \text{and} \quad m_2(x) \to 1 \quad \text{as } x \to \infty. \tag{1.40}$$

Indeed, it follows from (1.37) that

$$\begin{aligned} \mathbb{E}_x X(1) &= \int_0^\infty \frac{y^{\nu+2}}{x^\nu} e^{-(x^2+y^2)/2} I_\nu(xy)dy \\ &= \frac{e^{-x^2/2}}{x^\nu} \int_0^\infty y^{\nu+2} e^{-y^2/2} I_\nu(xy)dy \\ &= \frac{e^{-x^2/2}}{x^\nu} \frac{\Gamma(\nu+3/2)}{\frac{x}{2}\Gamma(\nu+1)} e^{x^2/4} 2^{\nu/2} M_{-\nu/2-1,\,\nu/2}(x^2/2), \end{aligned}$$

where $M_\cdot(\cdot)$ is the Whittaker function; see [74, Formula 6.643(2)]. As $x \to \infty$,

$$M_{-\nu/2-1,\,\nu/2}(x^2/2) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)} e^{x^2/4} (x^2/2)^{\nu/2+1} \Big(1 + \frac{2\nu+1}{2x^2} + O(1/x^4)\Big),$$

which gives

$$\mathbb{E}_x X(1) = x\Big(1 + \frac{2\nu+1}{2x^2} + O(1/x^4)\Big) \quad \text{as } x \to \infty;$$

this in turn yields the first relation in (1.40). In a similar way we conclude that the asymptotic behaviour of the higher moments of jumps, for any fixed $j \ge 1$, is given by

$$\mathbb{E}_x X^{2j}(1) = x^{2j} + 2j(\nu + j)x^{2j-2} + O(x^{2j-4}) \quad \text{as } x \to \infty. \tag{1.41}$$

Choosing $j = 1$ and using the formula for the first moment of $X(1)$, one obtains the second convergence in (1.40).

If the Bessel process $\{X(t)\}$ is transient or null recurrent, that is, if $\nu > -1$, then it follows from the distribution property of the Bessel process $\{X(t)\}$ discussed above that, for all n, X_n^2/n has a Γ-distribution with mean $2(\nu+1)$ and variance $4(\nu+1)$. In Sections 4.5 and 4.6 we discuss the convergence of X_n^2/n to a Γ-distribution for a general transient or null recurrent Markov chain with asymptotic drift of order c/x.

1.6 General Approach and Plan of the Book

One of the most popular examples of Markov chains with asymptotically zero drift is a driftless random walk conditioned to stay positive. This process is an h-transform of a random walk killed at leaving $\mathbb{R}^+$. If the second moment of the original random walk is finite then the transformed process has a drift of order $1/x$, that is, $xm_1(x) \to c_1 > 0$. But the second moment of the transformed process is finite if and only if the third moment of the original walk is finite; see the calculations in Section 11.1. Therefore, Lamperti's criterion for transience is not always applicable to this type of chain.

This observation motivates us to look for appropriate conditions for transience, null recurrence, and positive recurrence in terms of the truncated moments and tail probabilities of the jumps $\xi(x)$. For any $s > 0$ we denote the s-truncation of the kth moment of the jump at state x by

$$m_k^{[s]}(x) := \mathbb{E}\{\xi^k(x);\ |\xi(x)| \le s\}.$$

Another reason for considering truncated moments comes from the case where the drift function decays more slowly than $1/x$, say as $1/x^\beta$ with β between 0 and 1. In that case it is not practical to assume the boundedness or even the existence of a full second moment of jumps, whereas an appropriate restriction on the growth of a truncated second moment is a rational approach; see e.g. Section 5.1.

In Chapter 2 we introduce a classification of Markov chains with asymptotically zero drift, which relies on the relation between $m_1^{[s(x)]}$ and $m_2^{[s(x)]}$. Additional assumptions are expressed in terms of truncated moments of higher order and the tail probabilities of jumps. Another, more important, contrast with previous results on recurrence and transience is the fact that we do not use concrete Lyapunov test functions (such as x^2, $\log^a x$ or $x^2 \log x\ \log\log x$). Instead, we construct an abstract Lyapunov function that is motivated by

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the harmonic function of a diffusion process with drift $m_1(x)$ and diffusion coefficient $m_2(x)$; see Section 1.5 above.

The asymptotic behaviour of transient Markov chains and the tail analysis of recurrent Markov chains is discussed in Chapters 3–6 and 8–9 respectively. In Chapter 7, motivated by the exponential change-of-measure approach suggested by Cramér in the 1920s for the study of large deviations of sums of independent random variables in the context of risk processes, we suggest the following general strategy for the study of positive recurrent Markov chains with asymptotically zero drift.

- First, apply an appropriate Doob's h-transform to a process $\{X_n\}$ killed at its time of entry to the half-line $(-\infty, \widehat{x}]$ for some $\widehat{x} \in \mathbb{R}$, in order to change the sign of the drift from negative to positive so that we get a transition kernel that generates a transient embedded Markov chain. With necessity an appropriate change of measure is generated by a subexponential function, either regularly varying or Weibullian-type at infinity.
- Second, apply the limit results to the transient Markov chain that is obtained.
- Third, apply an inverse change of measure that makes it possible to identify the tail and local asymptotics of both the stationary and pre-stationary distributions of the original positive recurrent Markov chain.

In Chapter 10 we show that our approach also works for Markov chains with asymptotically negative drift bounded away from zero. We consider Markov chains that are asymptotically homogeneous in space, that is, Markov chains with jumps satisfying $\xi(x) \Rightarrow \xi$ as $x \to \infty$. This means that far from the origin one can approximate $\{X_n\}$ by a random walk, which makes it natural to apply an exponential change of measure in a similar way to how this is done for sums of independent random variables. We study the tail asymptotic behaviour of the stationary and pre-stationary distributions of $\{X_n\}$ in the case where the limiting random variable ξ has negative mean and satisfies the Cramér condition. It turns out that the tail behaviour of these distributions depends on the rate of convergence of $\xi(x)$ to ξ.

In Chapter 11 we consider some important applications of our results. Processes with asymptotically zero drift appear naturally in various stochastic models such as random billiards, see Menshikov et al. [125], and random polymers, see Alexander [5], Alexander and Zygouras [6], and De Coninck et al. [41]).

Such chains appear when we study critical and near-critical branching processes. In critical branching processes one typically observes a linearly growing second moment of jumps, but, on considering the square root of the

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process, one obtains bounded second moments and a drift that decreases to zero. Thus we can apply our theorems to this transformation. As a result we get limit theorems for population-size-dependent processes with migration of particles. To the best of our knowledge, there is no paper in the literature in which a combination of size dependence and migration has been considered.

We have also found that processes with asymptotically zero drift can be used in the study of risk processes with a reserve-dependent premium rate. More precisely, we have derived upper and lower bounds for ruin probabilities in the case when the premium rate approaches from above – as the risk reserve grows – the critical value for a model with constant rate.

Besides these two main examples we consider also random walks conditioned to stay positive and reflected random walks.