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RESEARCH ARTICLE

Motivic versions of mass formulas by Krasner, Serre and Bhargava

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Abstract
We prove motivic versions of mass formulas by Krasner, Serre, and Bhargava concerning (weighted) counts of extensions of local fields.

Contents

1. Introduction

The aim of this article is to prove motivic versions of mass formulas by Krasner [Kra62, Kra66], Serre [Ser78], and Bhargava [Bha07]. For a nonarchimedean local field K with residue field $\mathbb{F}_q$ and for a positive integer n, Serre proved the formula

$$\sum_L \frac{q^{-\mathbf{d}_{L/K}}}{|\operatorname{Aut}(L)|} = q^{1-n}, \tag{1.1}$$

where L runs over isomorphism classes of totally ramified extensions of K with $[L : K] = n$, $\operatorname{Aut}(L)$ is the group of K-automorphisms and $\mathbf{d}_{L/K}$ is the discriminant exponent of L/K. Using Serre's formula, Bhargava proved a similar formula

$$\sum_L \frac{q^{-\mathbf{d}_{L/K}}}{|\operatorname{Aut}(L/K)|} = \sum_{j=0}^{n-1} P(n, n-j)q^{-j}, \tag{1.2}$$

this time L running over isomorphism classes of étale K-algebras of degree n. Here $P(n,i)$ denotes the number of partitions of n into exactly i positive integers.

In [Yas17], the author proved these formulas by a different approach which was based upon observation from [WY15] which relates Bhargava's formula with the Hilbert scheme of points. In this approach, we can prove Bhargava's formula first and deduce Serre's formula from it, using the relationship between the two formulas, obtained by Kedlaya [Ked07] using the exponential formula.

Before Serre obtained his formula, Krasner had obtained a formula for the number of degree-n extensions of K with a prescribed discriminant exponent as well as one restricted to totally ramified extensions. Thus, Krasner's result is a refinement of that of Serre and we can also obtain Serre's result as a consequence of that of Krasner. Under the condition that K has characteristic $p > 0$, the situation that we focus on in the present paper, the most interesting case is when $p \mid n, m-n+1 > 0$ and $p \nmid (m-n+1)$ with m the prescribed discriminant exponent. Under these conditions, the number of degree-n totally ramified extensions of K in a fixed algebraic closure of K with discriminant exponent m is

$$n(q-1)q^{\lfloor (m-n+1)/p \rfloor}.$$

We may rewrite this formula as

$$\sum_L \frac{1}{|\operatorname{Aut}(L)|} = (q-1)q^{\lfloor (m-n+1)/p \rfloor}, \tag{1.3}$$

where L runs over isomorphism classes of such extensions of K(instead of counting subfields of $\overline{K}$).

To formulate motivic versions of these formulas, we consider the P-moduli space Δ_n (resp. Δ_n°) of degree-n étale covers (resp. connected covers) of the punctured formal disk $\operatorname{Spec} k(\!(t)\!)$ with k denoting a field, constructed in [TY23]. The notion of P-moduli space is even coarser than the one of coarse moduli space, but enough to define motivic integrals that we consider below. For details, see Section (3). The discriminant exponent defines a constructible function $\mathbf{d}\colon \Delta_n \to \mathbb{Z}$ as well as its restriction to Δ_n°. We can define the integral

$$\int_{\Delta_n} \mathbb{L}^{-\mathbf{d}} := \sum_{m=0}^{\infty} [\mathbf{d}^{-1}(m)]\mathbb{L}^{-m}$$

and similarly the integral $\int_{\Delta_n^\circ} \mathbb{L}^{-\mathbf{d}}$ in a version of the complete Grothendieck ring of varieties, denoted by $\widehat{\mathcal{M}}_k^\heartsuit$ (for the definition of this ring, see Definition 7.1). Motivic versions of formulas (1.1) and (1.2) by Serre and Bhargava are formulated as follows:

Theorem 1.1 (Theorem 7.4 and Corollary 7.5). *We have the following equalities in* $\widehat{\mathcal{M}}_k^\heartsuit$*:*

$$\int_{\Delta_n^\circ} \mathbb{L}^{-\mathbf{d}} = \mathbb{L}^{1-n},$$
$$\int_{\Delta_n} \mathbb{L}^{-\mathbf{d}} = \sum_{j=0}^{n-1} P(n, n-j)\mathbb{L}^{-j}.$$

The second equality of the theorem is equivalent to [Yas24a, Corollary 1.5] via the correspondence between the discriminant exponent and the Artin conductor [WY15], except for a slight difference in the Grothendieck rings being used. The proof in [Yas24a] is obtained by translating the proof of Bhargava's formula in [Yas17] into the motivic setting.

We also prove a motivic version of Krasner's formula (1.3). Let $\Delta_n^{(m)} := \mathbf{d}^{-1}(m) \subset \Delta_n^\circ$, the locus of étale covers with discriminant exponent m. This is a constructible subset.

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Theorem 1.2 (see Theorem 7.6 for the full statement of the result). *If k has characteristic $p > 0$ and if $p \mid n$, $m - n + 1 > 0$ and $p \nmid (m - n + 1)$, then we have*

$$[\Delta_n^{(m)}] = (\mathbb{L} - 1)\mathbb{L}^{\lfloor (m-n+1)/p \rfloor}$$

in $\widehat{\mathcal{M}}_k^{\heartsuit}$.

The author used this result as a working hypothesis in a previous paper [Yas16, Section 12] to verify a certain duality in mass formulas. He also applies this theorem to a study of quotient singularities in another paper [Yas24b].

We now explain the outline of the proof of Theorem 1.1. Our strategy is to translate Serre's arguments, which use p-adic measures, to the motivic setting by using the theory of motivic integration, a theory pioneered by Kontsevich [Kon95] and Denef–Loeser [DL99]. We use a version of the theory over a complete discrete valuation ring established by Sebag [Seb04]. We consider the space of Eisenstein polynomials of degree n with coefficients in $k[\![t]\!]$, denoted by $\mathfrak{Eis}$. We regard this as a subspace of the arc space $\mathrm{J}_\infty(\mathbb{A}^n_{k[\![t]\!]})$ of the affine space $\mathbb{A}^n_{k[\![t]\!]}$. Through the natural map $\mathfrak{Eis} \to \Delta_n^\circ$, we relate the integral $\int_{\Delta_n^\circ} \mathbb{L}^{-\mathbf{d}}$ with the motivic volume of $\mathfrak{Eis}/\mathbb{G}_m$, the quotient of $\mathfrak{Eis}$ by a natural action of $\mathbb{G}_m$. This motivic volume is easy to compute, leading to the motivic version of Serre's formula. The motivic version of Bhargava's formula easily follows from that of Serre. The motivic version of Krasner's formula is obtained by an explicit description of the locus of those Eisenstein polynomials which give extensions of a prescribed discriminant exponent.

Throughout the paper, we work over a field k of characteristic $p \geq 0$. A *k-variety* means a separated scheme of finite type over k. We follow the convention that when $p = 0$, then p is coprime to any positive integer n and we write $p \nmid n$. The symbol K means an extension of k, unless otherwise noted. In this paper, every ring is assumed to be commutative. For a ring R, we denote by $R[\![t]\!]$ the ring of power series with coefficients in R and by $R(\!(t)\!)$ its localization by t, which is nothing but the ring of Laurent power series with coefficients in R. When K is a field and A is an étale $K(\!(t)\!)$-algebra, we denote by $\mathcal{O}_A$ the integral closure of $K[\![t]\!]$ in A. For a finite group G, a *G-torsor* means an étale G-torsor.

2. Discriminants

Let R be a ring and let S be an R-algebra which is free of rank n as an R-module. For each element $s \in S$, the map $S \longrightarrow S$, $x \mapsto sx$ is an R-linear map and its trace $\mathrm{Tr}(s)$ is defined as an element of R. The *discriminant* $D(s_1, \ldots, s_n)$ of an R-module basis $s_1, \ldots, s_n \in S$ is defined to be the determinant of the $n \times n$ matrix $(\mathrm{Tr}(s_i s_j))_{i,j}$ with entries in R. It is known that the ideal generated by $D(s_1, \ldots, s_n)$ is independent of the choice of basis and that S is étale over R if and only if $(D(s_1, \ldots, s_n)) = R$ or equivalently $D(s_1, \ldots, s_n)$ is an invertible element of R.

Consider the case $R = \mathbb{Z}[Y_1, \ldots, Y_n]$, the n-variate polynomial ring with integer coefficients, and

$$S = R[x]/(x^n + Y_1 x^{n-1} + \cdots + Y_{n-1}x + Y_n).$$

We define the *discriminant polynomial* $F(Y_1, \ldots, Y_n) \in R$ to be the discriminant $D(1, x, \ldots, x^{n-1})$ of the R-basis $1, x, \ldots, x^{n-1}$ of S. For any ring R' and an R'-algebra

$$S' = R'[x]/(x^n + y_1 x^{n-1} + \cdots + y_{n-1}x + y_n) \quad (y_1, \ldots, y_n \in R'),$$

we have $D(1, x_{S'}, \ldots, x_{S'}^{n-1}) = F(y_1, \ldots, y_n)$, where $x_{S'}$ is the image of x in S'. The R'-algebra S' is étale if and only if $F(y_1, \ldots, y_n)$ is an invertible element of R'.

Next consider the case where $R = K(\!(t)\!)$ with K a field and

$$S = R[x]/(x^n + y_1 x^{n-1} + \cdots + y_{n-1}x + y_n) \quad (y_1, \ldots, y_n \in R).$$

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The polynomial in the last equality is called an *Eisenstein polynomial*if $\operatorname{ord} y_i > 0$ for every i and $\operatorname{ord} y_n = 1$. If this is the case, from [Ser79, p. 19], S is a discrete valuation field with uniformizer x and has residue field isomorphic to K. The field extension S/R is separable if and only if $F(y_1, \dots, y_n) \neq 0$. When S/R is separable, its *discriminant exponent* is defined to be

$$\mathbf{d}_{S/R} := \operatorname{ord} F(y_1, \dots, y_n) \in \mathbb{Z}_{\geq 0}.$$

We often write $\mathbf{d}_{S/R}$ simply as $\mathbf{d}_S$, omitting R.

3. Strong P-moduli spaces

In this section, we recall results from [TY23]. Let $\mathbf{Aff}/k$ be the category of affine schemes over k and let $\mathbf{ACF}/k$ be its full subcategory consisting of spectra $\operatorname{Spec} K$ with K an algebraically closed field. We identify a k-scheme X with the associated functor

$$(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set},\ T \mapsto \operatorname{Hom}_k(T, X).$$

For a functor $Z\colon (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$, we let $Z_F\colon (\mathbf{ACF}/k)^{\mathrm{op}} \to \mathbf{Set}$ be its restriction to $(\mathbf{ACF}/k)^{\mathrm{op}}$.

Definition 3.1. Let Y be a k-scheme and let X be a functor $(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$ (e.g., a k-scheme). A *P-morphism* $f\colon Y \to X$ is a natural transformation $Y_F \to X_F$ such that there exist morphisms of $h\colon Z \to Y$ and $g\colon Z \to X$ of k-schemes such that h is surjective and locally of finite type and the following diagram is commutative:

$$\begin{array}{ccc} Z_F & & \\ {\scriptstyle h_F}\downarrow & \searrow^{g_F} & \\ Y_F & \xrightarrow[f]{} & X_F \end{array} \tag{3.1}$$

We denote by $\operatorname{Hom}_k^P(Y, X)$ the set of P-morphisms over k from Y to X. We denote by X^P the functor

$$(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set},\ T \mapsto \operatorname{Hom}_k^P(T, X).$$

A P-morphism $f\colon Y \to X$ is said to be a *P-isomorphism*if there exists a P-morphism $g\colon X \to Y$ such that both $f \circ g$ and $g \circ f$ are the identities.

For a scheme X, we denote its underlying point set by $|X|$. This set is identified with the set of equivalence classes of geometric points $\operatorname{Spec} K \to X$; two geometric points $\operatorname{Spec} K \to X$ and $\operatorname{Spec} K' \to X$ are equivalent if they fit into the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K'' & \longrightarrow & \operatorname{Spec} K' \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & X \end{array}$$

with K'' also an algebraically closed field. The set $|X|$ is equipped with the Zariski topology. A P-morphism $f\colon Y \to X$ induces a map $|f|\colon |Y| \to |X|$ in the obvious way.

Lemma 3.2 [TY23, Lemmas 4.7 and 4.32]. *Let Y and X be separated schemes locally of finite type over k. Let $f\colon Y \to X$ be a P-morphism.*

1. *There exist finite-type morphisms $h\colon Z \to Y$ and $g\colon Z \to X$ such that diagram (3.1) is commutative and h is geometrically bijective (that is, for every algebraically closed field K, $h(K)\colon Z(K) \to Y(K)$ is bijective.)*

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2. *If* $\Gamma_f : Y \to Y \times_k X$ *is the graph of f, then* $\mathrm{Im}(|\Gamma_f|)$ *is a locally constructible subset of* $|Y \times_k X|$.
3. *f is a P-isomorphism if and only if it is geometrically bijective.*

Definition 3.3. Keeping the assumption of Lemma 3.2, we denote the locally constructible subset $\mathrm{Im}(|\Gamma_f|)$ again by Γ_f. For a point $x\colon \operatorname{Spec} K \to X$ with K any field, the *fiber* $f^{-1}(x)$ is defined to be the constructible subset

$$(\Gamma_f \times_{X,x} \operatorname{Spec} K) \cap \mathrm{pr}_X^{-1}(x) \subset Y \otimes_k K.$$

Here $\Gamma_f \times_{X,x} \operatorname{Spec} K$ means the preimage of Γ_f by the morphism

$$\mathrm{id} \times x\colon Y \otimes_k K \to Y \times_k X.$$

Definition 3.4. We define the *category of P-schemes over* k, denoted by P-$\mathbf{Sch}/k$, to be the category having k-schemes as objects and P-morphisms over k as morphisms.

Definition 3.5. Let $\mathcal{F}\colon (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$ be a functor. A *strong P-moduli space* of $\mathcal{F}$ is a k-scheme X given with a morphism $\pi\colon \mathcal{F} \to X^P$ such that the induced morphism $\pi\colon \mathcal{F}^P \to X^P$ is an isomorphism.

If it exists, a strong P-moduli space is unique up to a unique P-isomorphism. By definition, if X is a strong P-moduli space of $\mathcal{F}$, then for every algebraically closed field K, the map $\mathcal{F}(K) \to X(K)$ is bijective.

Theorem 3.6 [TY23, Theorem 8.9]. *For a finite group G, the functor*

$$\mathcal{F}_G\colon (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set},\ \operatorname{Spec} R \mapsto \{G\text{-torsors over } \operatorname{Spec} R(\!(t)\!)\}/\cong$$

has a strong P-moduli space which is a countable coproduct of affine k-varieties. Here a G-torsor means an étale G-torsor.

Definition 3.7. For a ring R, we say that a finite étale $R(\!(t)\!)$-algebra A is of *degree n* (resp. *of discriminant exponent m*, *connected*) if for every point $\operatorname{Spec} K \to \operatorname{Spec} R$, the induced $K(\!(t)\!)$-algebra $A \otimes_{R(\!(t)\!)} K(\!(t)\!)$ is of degree n (resp. of discriminant exponent m, a field). For $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$, we define the following functors:

$$\begin{aligned}\mathcal{F}_n\colon (\mathbf{Aff}/k)^{\mathrm{op}} &\to \mathbf{Set}\\ \operatorname{Spec} R &\mapsto \{\text{finite étale } R(\!(t)\!)\text{-algebras of degree } n\}/\cong,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_n^\circ\colon (\mathbf{Aff}/k)^{\mathrm{op}} &\to \mathbf{Set}\\ \operatorname{Spec} R &\mapsto \{\text{connected finite étale } R(\!(t)\!)\text{-algebras of degree } n\}/\cong\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_n^{(m)}\colon (\mathbf{Aff}/k)^{\mathrm{op}} &\to \mathbf{Set}\\ \operatorname{Spec} R &\mapsto \left\{\begin{matrix}\text{connected finite étale } R(\!(t)\!)\text{-algebras of}\\ \text{degree } n \text{ and discriminant exponent} m\end{matrix}\right\}/\cong.\end{aligned}$$

Corollary 3.8. *The functor* $\mathcal{F}_n$, $\mathcal{F}_n^\circ$ *and* $\mathcal{F}_n^{(m)}$ *have strong P-moduli spaces which are coproducts of countably many affine k-varieties.*

Proof. Since $\mathcal{F}_n$ is isomorphic to $\mathcal{F}_{S_n}$, Theorem 3.6 shows that $\mathcal{F}_n$ has a strong P-moduli space which is a coproduct of countably many affine k-varieties. From [TY23, Lemma 8.7 and Theorem 8.9], $\mathcal{F}_n^\circ$ also has a strong P-moduli space which is a coproduct of countably many affine k-varieties. From [WY15], the discriminant exponent function $\mathbf{d}\colon \mathcal{F}_n \to \mathbb{Z}_{\geq 0}$, which is identified with the Artin conductor

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$\mathbf{a}\colon \mathcal{F}_{S_n} \to \mathbb{Z}_{\geq 0}$, is a special case of v-function $\mathbf{v}\colon \mathcal{F}_{S_n} \to \mathbb{Z}_{\geq 0}$. From [TY23, Theorem 9.8], $\mathbf{d}$ is a locally constructible function. Hence, the property "$\mathbf{d} = m$" is locally constructible. From [TY23, Theorem 8.9], $\mathcal{F}_n^{(m)}$ also has a P-moduli spaces which is a coproduct of countably many affine k-varieties. □

Definition 3.9. We denote by Δ_n° and $\Delta_n^{(m)}$ P-moduli spaces of $\mathcal{F}_n^\circ$ and $\mathcal{F}_n^{(m)}$, which are coproducts of countably many affine k-varieties.

By the definition of strong P-moduli space, for each connected finite étale $R(\!(t)\!)$-algebra of degree n and discriminant exponent m, we have the induced P-morphism $\operatorname{Spec} R \to \Delta_n^{(m)}$.

Remark 3.10. Let us write $\Delta_n^\circ = \coprod_{i\in I} W_i$, where I is a countable set and W_i are k-varieties. Then, locally constructible subsets and constructible subsets of Δ_n° are characterized as follows. A subset $C \subset \Delta_n^\circ$ is locally constructible if and only if for every i, $C \cap W_i$ is a constructible subset of W_i. A locally constructible subset $C \subset \Delta_n^\circ$ is constructible if it is quasicompact or equivalently if it is contained in $\bigcup_{i\in I_0} W_i$ for a finite subset $I_0 \subset I$. Note that whether a subset $C \subset \Delta_n^\circ$ is locally constructible (resp. constructible) or not is independent of the choice of P-moduli space: if $(\Delta_n^\circ)'$ is another P-moduli space of $\mathcal{F}_n^\circ$ and $C' \subset (\Delta_n^\circ)'$ is the subset corresponding to C, then C is locally constructible (resp. constructible) if and only if so is C'.

4. The space of Eisenstein polynomials

In this section, we construct the space of Eisenstein polynomials as a subspace of an arc space and study its properties. We refer the reader to [CLNS18] for details on arc spaces, in particular, from the viewpoint of motivic integration.

Let $V := \mathbb{A}^n_{k[\![t]\!]} = \operatorname{Spec} k[\![t]\!][x_1, \dots, x_n]$ and let $\mathrm{J}_\infty V$ and $\mathrm{J}_l V$, $l \in \mathbb{Z}_{\geq 0}$, be its arc scheme and jet schemes. Namely, for a k-algebra R, we have

$$(\mathrm{J}_\infty V)(R) = \operatorname{Hom}_{k[\![t]\!]}(\operatorname{Spec} R[\![t]\!], V),$$
$$(\mathrm{J}_l V)(R) = \operatorname{Hom}_{k[\![t]\!]}(\operatorname{Spec} R[\![t]\!]/(t^{l+1}), V).$$

For $l', l \in \mathbb{Z}_{\geq 0}$ with $l' \geq l$, we have truncation morphisms

$$\pi_l\colon \mathrm{J}_\infty V \to \mathrm{J}_l V \text{ and } \pi_l^{l'}\colon \mathrm{J}_{l'} V \to \mathrm{J}_l V.$$

Definition 4.1. For a k-algebra R, R-points of $\mathrm{J}_\infty V$ correspond to n-tuples of power series $y = (y_1, \dots, y_n) \in R[\![t]\!]^n$. We let them correspond also to polynomials

$$f_y(x) := x^n + y_1 x^{n-1} + \cdots + y_{n-1} x + y_n \in R[\![t]\!][x].$$

We let A_y be the $R(\!(t)\!)$-algebra $R(\!(t)\!)[x]/(f_y(x))$.

As we saw in Section 2, when R is a field K, the extension $A_y/K(\!(t)\!)$ is separable if and only if $F(y) \neq 0$.

Definition 4.2. For indeterminates $Y_{i,j}$ $(1 \leq i \leq n, j \geq 0)$ and for integers $m \geq 0$, we define polynomials $F_m(Y_{i,j}) \in \mathbb{Z}[Y_{i,j}; j \leq m]$ by

$$F\left(\sum_{j\geq 0} Y_{1,j} t^j, \dots, \sum_{j\geq 0} Y_{n,j} t^j\right) = \sum_{m\geq 0} F_m(Y_{i,j}) t^m.$$

Definition 4.3. Let $\mathfrak{Eis} \subset \mathrm{J}_\infty V$ (resp. $\mathfrak{Eis}^{\mathrm{sep}}, \mathfrak{Eis}^{(m)}$) to be the locus of Eisenstein polynomials (resp. separable Eisenstein polynomials, Eisenstein polynomials whose discriminants have order m).

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If we write $y_i = \sum_{j\in\mathbb{Z}_{\geq 0}} y_{i,j}t^j$, then the above loci are described as

$$\begin{aligned}\mathfrak{Eis} &= \{(y_{i,j}) \mid \text{for every } i, y_{i,0} = 0 \text{ and } y_{n,1} \neq 0\},\\ \mathfrak{Eis}^{\mathrm{sep}} &= \{(y_{i,j}) \in \mathfrak{Eis} \mid \text{for some } m, F_m(y_{i,j}) \neq 0\},\\ \mathfrak{Eis}^{(m)} &= \{(y_{i,j}) \in \mathfrak{Eis} \mid \text{for } m' < m, F_{m'}(y_{i,j}) = 0 \text{ and } F_m(y_{i,j}) \neq 0\}.\end{aligned}$$

These are locally closed subsets of $\mathrm{J}_\infty V$. We have $\mathfrak{Eis}^{\mathrm{sep}} = \bigsqcup_{m\geq 0} \mathfrak{Eis}^{(m)}$. For $y \in \mathfrak{Eis}^{\mathrm{sep}}(K)$, the associated extension $A_y/K(\!(t)\!)$ is separable and totally ramified. Let $\varpi \in A_y = K(\!(t)\!)[x]/(f_y(x))$ denote the image of x, which is a uniformizer of A_y. As is well-known, the discriminant exponent $\mathbf{d}_{A_y/K(\!(t)\!)}$ of $A_y/K(\!(t)\!)$ is also equal to

$$n \operatorname{ord} f_y'(\varpi) = n \operatorname{ord}\left(\sum_{i=0}^{n}(n-i)y_i\varpi^{n-i-1}\right) \tag{4.1}$$

with $y_0 = 1$ (for example, see [Ser79, Proposition 6 on p. 50 and Corollary 2 on p. 56]). Here we denote the unique extension of the valuation $\operatorname{ord}\colon K(\!(t)\!) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to A_y again by ord. This equality shows that if $p \nmid n$, then $\mathbf{d}_{A_y/K(\!(t)\!)} = n-1$ and hence $\mathfrak{Eis}^{(m)} = \emptyset$ for $m \neq n-1$. For n with $p \mid n$, we have the following explicit description of $\mathfrak{Eis}^{(m)}$.

Proposition 4.4. *Suppose that $p > 0$ and $p \mid n$. Let*

$$y = (y_1, \dots, y_n) \in K[\![t]\!]^n = \mathfrak{Eis}(K).$$

We have $y \in \mathfrak{Eis}^{(m)}(K)$ if and only if for every $i \in \{1, \dots, n-1\}$ with $p \nmid (n-i)$, the inequality

$$\operatorname{ord} y_i \geq \left\lceil \frac{m-n+1+i}{n} \right\rceil$$

holds and the equality in this inequality holds if $m + i + 1 \in n\mathbb{Z}$.

Proof. In the situation of the proposition, we have

$$\mathbf{d}_{A_y/K(\!(t)\!)} = n \operatorname{ord}\left(\sum_{\substack{0\leq i<n\\ p\nmid(n-i)}} (n-i)y_i\varpi^{n-i-1}\right).$$

For i with $p \nmid (n-i)$, we have

$$n \operatorname{ord}\left((n-i)y_i\varpi^{n-i-1}\right) \equiv n-i-1 \mod n\mathbb{Z}.$$

In particular, for distinct i's, these values are different to one another. Therefore,

$$n \operatorname{ord} f_y'(\varpi) = \min\{n-i-1+n \operatorname{ord} y_i \mid 0 \leq i < n,\ p \nmid (n-i)\}.$$

Moreover, if $\mathbf{d}_{A_y/K(\!(t)\!)} = m$, the minimum is attained at i with $m+i+1 \in n\mathbb{Z}$. This proves the proposition. ◻

Definition 4.5. For $l \in \mathbb{Z}_{\geq 0}$, we let $\mathfrak{Eis}_l^{(m)} \subset \mathrm{J}_l V$ be the image of $\mathfrak{Eis}^{(m)}$ by π_l.

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Corollary 4.6. *Let $c := m-n+1$. Suppose that $p \mid n$, $p \nmid c$ and $c \geq 0$. For $l \geq \lfloor (c+n-1)/n \rfloor = \lfloor m/n \rfloor$, $\mathfrak{Eis}_l^{(m)}$ is a locally closed subset of $\mathrm{J}_l V$. Moreover, if we give it the reduced scheme structure, then*

$$\mathfrak{Eis}_l^{(m)} \cong \mathbb{G}_m^2 \times \mathbb{A}_k^{nl-c+\lfloor c/p \rfloor -1}.$$

In particular, $\mathfrak{Eis}_l^{(m)}$ is an affine variety.

Proof. The jet scheme $\mathrm{J}_l V$ is the affine space $\mathbb{A}_k^{n(l+1)}$ with the coordinates $y_{i,j}$, $1 \leq i \leq n$, $0 \leq j \leq l$. The subset $\mathfrak{Eis}_l^{(m)}$ of it is defined by

$$\begin{cases} y_{i,0} = 0 & (1 \leq i \leq n), \\ y_{n,1} \neq 0, & \\ y_{i,j} = 0 & \left(i < n,\ p \nmid (n-i),\ j < \left\lceil \frac{c+i}{n} \right\rceil\right), \\ y_{i,\frac{c+i}{n}} \neq 0 & (i < n,\ p \nmid (n-i),\ c+i \in n\mathbb{Z}), \end{cases}$$

which shows that $\mathfrak{Eis}_l^{(m)}$ is a locally closed subset. The last two conditions can be rephrased as:

$$\begin{cases} y_{i,j} = 0 & \left(i < n,\ p \nmid (n-i),\ c+i \notin n\mathbb{Z},\ j \leq \left\lfloor \frac{c+i}{n} \right\rfloor\right) \\ y_{i,j} = 0 & \left(i < n,\ p \nmid (n-i),\ c+i \in n\mathbb{Z},\ j < \left\lfloor \frac{c+i}{n} \right\rfloor\right) \\ y_{i,\frac{c+i}{n}} \neq 0 & (i < n,\ p \nmid (n-i),\ c+i \in n\mathbb{Z}). \end{cases}$$

Thus, we have

$$\mathfrak{Eis}_l^{(m)} \cong \mathbb{G}_m^2 \times \mathbb{A}_k^{nl-s-1},$$

where

$$s = \sum_{\substack{1 \leq i < n \\ p \nmid (n-i)}} \left\lfloor \frac{c+i}{n} \right\rfloor.$$

Let us write $n = pn'$. From Hermite's identity (for example, see [ST03, Chapter 12]), we have

$$\begin{aligned} s &= \sum_{i=1}^{n-1} \left\lfloor \frac{c}{n} + \frac{i}{n} \right\rfloor - \sum_{i=1}^{n'-1} \left\lfloor \frac{c}{n} + \frac{i}{n'} \right\rfloor \\ &= \left\lfloor n \frac{c}{n} \right\rfloor - \left\lfloor n' \frac{c}{n} \right\rfloor \\ &= c - \left\lfloor \frac{c}{p} \right\rfloor. \end{aligned}$$

□

Lemma 4.7. *For a k-algebra R and for a point $y \in \mathfrak{Eis}^{(m)}(R)$, A_y is étale over $R(\!(t)\!)$.*

Proof. Since the leading coefficient $F_m(y_{i,j})$ of $F(y_1, \ldots, y_n)$ is an invertible element of R, $F(y) \in R(\!(t)\!)$ is invertible and $A_y/R(\!(t)\!)$ is étale. □

Since $\mathfrak{Eis}^{(m)}$ itself is an affine scheme having the coordinate ring

$$S = k[y_{i,j}, y_{n,1}^{-1}, F_m^{-1} \mid i \in \{1, \ldots, n\}, j \in \mathbb{Z}_{\geq 0}]/(F_{m'} \mid m' < m),$$

we have the corresponding étale algebra over $S(\!(t)\!)$ and the induced P-morphism

$$\psi = \psi^{(m)} : \mathfrak{Eis}^{(m)} \to \Delta_n^{(m)}.$$

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For each point $y \in \mathfrak{Eis}^{(m)}(R)$, the composition P-morphism

$$\operatorname{Spec} R \xrightarrow{y} \mathfrak{Eis}^{(m)} \to \Delta_n^{(m)}$$

is the P-morphism associated to the $R(\!(t)\!)$-algebra A_y.

Remark 4.8. For $y \in \mathfrak{Eis}(R)$, even if $K(\!(t)\!) \to K(\!(t)\!)[x]/(f_y)$ is étale for every point $\operatorname{Spec} K \to \operatorname{Spec} R$ with K a field, the map $R(\!(t)\!) \to R(\!(t)\!)[x]/(f_y)$ is not generally étale as the following example exists. Suppose that k has characteristic two. Let $R = k[s]$ and $f = x^2 + (st + (s+1)t^2)x + t \in R(\!(t)\!)[x]$. Then $R(\!(t)\!)[x]/(f)$ is not étale over $R(\!(t)\!)$. Indeed, $df/dx = st + (s+1)t^2$ is not a unit in $R(\!(t)\!)$. Hence it is not a unit in $R(\!(t)\!)[x]/(f)$ either. On the other hand, for any point $\operatorname{Spec} K \to \operatorname{Spec} R$, since df/dx is a unit in $K(\!(t)\!)$, the induced map $K(\!(t)\!) \to K(\!(t)\!)[x]/(f)$ is étale. This explains why we need to decompose $\mathfrak{Eis}^{\mathrm{sep}}$ into subsets $\mathfrak{Eis}^{(m)}$ to have a map to Δ_n°.

For $y \in \mathfrak{Eis}_l^{(m)}(R) \subset (R[\![t]\!]/(t^{l+1}))^n$, let $\tilde{y} \in \mathfrak{Eis}^{(m)}(R) \subset R[\![t]\!]^n$ be its canonical lift given by

$$\tilde{y}_{i,j} = \begin{cases} y_{i,j} & (j \le m) \\ 0 & (j > m). \end{cases}$$

For $l \ge \lfloor m/n \rfloor$, the assignment $y \mapsto \widetilde{y}$ defines a morphism $\mathfrak{Eis}_l^{(m)} \to \mathfrak{Eis}^{(m)}$, which is a section of $\pi_l|_{\mathfrak{Eis}^{(m)}} : \mathfrak{Eis}^{(m)} \to \mathfrak{Eis}_l^{(m)}$. We define the P-morphism

$$\psi_l = \psi_l^{(m)} : \mathfrak{Eis}_l^{(m)} \to \Delta_n^{\circ},\ y \mapsto A_{\tilde{y}},$$

which is the composition of the section $\mathfrak{Eis}_l^{(m)} \to \mathfrak{Eis}^{(m)}$ and $\psi : \mathfrak{Eis}^{(m)} \to \Delta_n^{(m)}$.

We need the following lemma, which is a variant of Fontaine's [Fon85, Prop. 1.5].

Lemma 4.9. *Let $L/K(\!(t)\!)$ be a finite separable extension and $E/K(\!(t)\!)$ any algebraic extension. Let $\mathcal{O}_L$ and $\mathcal{O}_E$ be the integral closures of $K[\![t]\!]$ in L and E, respectively. Let l be an integer with $l > \mathbf{d}_{L/K(\!(t)\!)}$. Suppose that there exists a $K[\![t]\!]$-algebra homomorphism $\eta : \mathcal{O}_L \to \mathcal{O}_E/t^l\mathcal{O}_E$. Then there exists a $K(\!(t)\!)$-embedding $L \to E$.*

Proof. The proof is also similar to the one of [Fon85, Proposition 1.5]. Let us write $\mathcal{O}_L = K[\![t]\!][\alpha]$ and let $f(X)$ be the minimal polynomial of α over $K(\!(t)\!)$, which is of degree $n = [L : K(\!(t)\!)]$. We embed L and E into an algebraic closure Ω of $K(\!(t)\!)$ and denote the extension of the valuation ord to Ω again by ord. Let $\beta \in \mathcal{O}_E$ be a lift of $\eta(\alpha)$. Then $\operatorname{ord} f(\beta) \ge l$. Let $\alpha = \alpha_1, , \dots, \alpha_n \in \Omega$ be the conjugates of α. Since $f(\beta) = \prod_{i=1}^n (\beta - \alpha_i)$, we have

$$\sup_i \operatorname{ord}(\beta - \alpha_i) \ge \frac{\operatorname{ord}(f(\beta))}{n} \ge \frac{l}{n} > \frac{\mathbf{d}_L}{n}. \tag{4.2}$$

Recall that the discriminant of $L/K(\!(t)\!)$ is the ideal generated by $\prod_{i \ne j}(\alpha_j - \alpha_i)$. Suppose that $\operatorname{ord}(\alpha_1 - \alpha_2) = \sup_{i \ne j} \operatorname{ord}(\alpha_i - \alpha_j)$. If σ_i, $1 \le i \le n$, are $K(\!(t)\!)$-automorphisms of Ω with $\sigma_i(\alpha) = \alpha_i$. Then,

$$\mathbf{d}_L \ge \sum_{i \ne j} \operatorname{ord}(\alpha_i - \alpha_j) \ge \sum_{i=1}^n \operatorname{ord}(\sigma_i(\alpha_1) - \sigma_i(\alpha_2)) \ge n \operatorname{ord}(\alpha_1 - \alpha_2). \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$\sup_i \operatorname{ord}(\beta - \alpha_i) > \sup_{i \ne j} \operatorname{ord}(\alpha_i - \alpha_j).$$

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From Krasner's lemma [Lan94, II, S2, Proposition3], for i with $\mathrm{ord}(\beta-\alpha_i)=\sup_i \mathrm{ord}(\beta-\alpha_i)$, we have $K(\!(t)\!)(\alpha_i)\subset K(\!(t)\!)(\beta)\subset E$. The composite map

$$L \xrightarrow{\sim} K(\!(t)\!)(\alpha_i) \hookrightarrow K(\!(t)\!)(\beta) \hookrightarrow E$$

is a $K(\!(t)\!)$-embedding. □

Corollary 4.10. *Let L and L' be finite separable field extensions of $K(\!(t)\!)$ of the same degree. Suppose that for some $l > \mathbf{d}_L$, there is a $K[\![t]\!]$-isomorphism $\mathcal{O}_L/t^l\mathcal{O}_L \xrightarrow{\sim} \mathcal{O}_{L'}/t^l\mathcal{O}_{L'}$. Then there exists a $K(\!(t)\!)$-isomorphism $L \xrightarrow{\sim} L'$.*

Proof. From the last proposition, there exists a $K(\!(t)\!)$-embedding $L \to L'$. Because of their degrees, it is an isomorphism. □

Corollary 4.11. *For $l \geq m$, the two morphisms $\psi^{(m)}, \psi_l^{(m)} \circ \pi_l \colon \mathfrak{Eis}^{(m)} \to \Delta_n^\circ$ induce the same P-morphism.*

Proof. For a geometric point $y \in \mathfrak{Eis}^{(m)}(K)$, the field extension $A_y/K(\!(t)\!)$ has discriminant exponent m. Since

$$\mathcal{O}_{A_y}/t^{l+1}\mathcal{O}_{A_y} = (K[\![t]\!]/(t^{l+1}))[x]/(f_y(x)),$$

if $y, y' \in \mathfrak{Eis}^{(m)}(K)$ map to the same point of $\mathfrak{Eis}_l^{(m)}(K)$ for $l \geq m$, then A_y and $A_{y'}$ are isomorphic over $K(\!(t)\!)$. This proves the corollary. □

Corollary 4.12. *The locally closed subset $\Delta_n^{(m)} \subset \Delta_n^\circ$ is quasicompact and constructible.*

Proof. We first note that from Remark 3.10, the assertion is independent of the choice of the P-moduli space Δ_n° of the functor $\mathcal{F}_n^\circ$. For an algebraically closed field K, every totally ramified extension $A/K(\!(t)\!)$ is associated to some Eisenstein polynomial. This shows that

$$\mathrm{Im}(\psi^{(m)}) = \mathrm{Im}(\psi_l^{(m)}) = \Delta_n^{(m)}.$$

Since the P-morphism $\psi_l^{(m)}$ is represented by a morphism $Y \to \Delta_n^\circ$ of k-schemes with Y a k-variety, its image is quasicompact and constructible. □

Definition 4.13. Let $\mathbb{G}_m$ denote the multiplicative group $\mathrm{Spec}\, k[s, s^{-1}]$ over the base field k. We define a grading on $k[x_1, \ldots, x_n]$ by $\deg(x_i) = i$ which induces a $\mathbb{G}_m$-action on V and one on $\mathrm{J}_\infty V$.

Lemma 4.14. *The maps $\psi^{(m)}$ and $\psi_l^{(m)}$ are $\mathbb{G}_m$-invariant.*

Proof. Let $y = (y_1, \ldots, y_n) \in \mathfrak{Eis}^{(m)}(K) \subset K[\![t]\!]^n$ and $b \in K^* = \mathbb{G}_m(K)$. Let

$$y' := by = (by_1, b^2y_2, \ldots, b^n y_n).$$

We have an isomorphism $A_y \to A_{y'}$, $x \mapsto b^{-1}x$. This shows that $\psi^{(m)}$ is $\mathbb{G}_m$-invariant. Similarly for $\psi_l^{(m)}$. □

Proposition 4.15. *Let $A\colon \mathrm{Spec}\, K \to \Delta_n^{(m)}$ be a point with K any field and let $l \geq m$. The fiber $\psi_l^{-1}(A)$ is a closed subset of $\mathfrak{Eis}_l^{(m)} \otimes_k K$.*

Proof. Since the proof is a little long and technical, we divide it into several steps.

A setup. By base change, we may assume that $K = k$ and that they are algebraically closed. Then, the point $A \in \Delta_n^{(m)}(k)$ is identified with an étale $k(\!(t)\!)$-algebra of degree n. Let $Z = \mathrm{Spec}\, R$ be an affine

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smooth irreducible curve over k with a distinguished closed point $0 \in \operatorname{Spec} R$ and let $y\colon Z \to \mathfrak{Eis}_l^{(m)}$ be a k-morphism such that an open dense subset $Z^\circ \subset Z$ maps into $\psi_l^{-1}(A)$. We will show that the distinguished point 0 also maps into $\psi_l^{-1}(A)$, which implies the proposition. Let $A_y = R(\!(t)\!)[x]/(f_y(x))$ be the algebra corresponding to y. Each point $z \in Z(k)$ induces a homomorphism $R(\!(t)\!) \to k(\!(t)\!)$, which in turn induces an algebra $A_{y,z}/k(\!(t)\!)$. For $z \in Z^\circ(k)$, we have $A_{y,z} \cong A$. Our goal is to show that $A_{y,0} \cong A$.

Strategy. Our strategy to achieve this goal is as follows. For a group G of a special form, we construct some G-torsor $\operatorname{Spec} B_y \to \operatorname{Spec} R(\!(t)\!)$ which factors through $\operatorname{Spec} A_y$. We show that for each point $z \in Z(k)$, the "fiber" $B_{y,z}$ is isomorphic to $(\widetilde{A_{y,z}})^\nu$ for some $\nu \in \mathbb{Z}_{>0}$, where $\widetilde{A_{y,z}}$ denotes a Galois closure of $A_{y,z}/k(\!(t)\!)$. For such a group G, we have a fine moduli stack $\widetilde{\Delta}_G$ of G-torsors over $\operatorname{Spec} k(\!(t)\!)$ and have the morphism $Z \to \widetilde{\Delta}_G$ corresponding to the above G-torsor. We will see that the image of the map $Z(k) \to \widetilde{\Delta}_G(k)/\cong$ is finite and hence the map is constant. This implies the desired conclusion.

Construction of B_y. We begin to construct a G-torsor as above. Let $C_y/R(\!(t)\!)$ and $C_{y,0}/k(\!(t)\!)$ be the S_n-torsors corresponding to the degree-n étale algebras $A_y/R(\!(t)\!)$ and $A_{y,0}/k(\!(t)\!)$, respectively (for example, see [Ser58, Section 1.5]). If we identify S_{n-1} with the stabilizer of 1 for the action $S_n \curvearrowright \{1, \dots, n\}$, then $(C_y)^{S_{n-1}} = A_y$ and $(C_{y,0})^{S_{n-1}} = A_{y,0}$. Let $\operatorname{Spec} B'_{y,0} \subset \operatorname{Spec} C_{y,0}$ be a connected component and let $G \subset S_n$ be its stabilizer so that $\operatorname{Spec} B'_{y,0} \to \operatorname{Spec} k(\!(t)\!)$ is a G-torsor and $B'_{y,0}/k(\!(t)\!)$ is a Galois closure of $A_{y,0}/k(\!(t)\!)$. Since the residue field k of $k(\!(t)\!)$ is algebraically closed, G coincides with its inertia group (often denoted by G_0). From [Ser79, p. 68, Corollary 4], $G = G_0$ is the semidirect product $H \rtimes C$ of a p-group H and a cyclic group C of order coprime to the characteristic of k. If G' denotes the stabilizer of 1 for the action $G \curvearrowright \{1, \dots, n\}$, then $A_{y,0} = (B'_{y,0})^{G'}$.

Let $u \in A_y$ be the image of x by the map

$$R(\!(t)\!)[x] \to A_y = R(\!(t)\!)[x]/(f_y(x))$$

and let $\overline{u}$ be the image of u in $A_{y,0}$. Then, $A_{y,0} = k(\!(t)\!)[\overline{u}]$ and $B'_{y,0} = k(\!(t)\!)[g\overline{u} \mid g \in G]$. Let $B'_y := R(\!(t)\!)[gu \mid g \in G] \subset C_y$. Then B'_y is a finitely generated torsion-free $R(\!(t)\!)$-module. Note that $R[\![t]\!]$ is a Noetherian domain of dimension 2. Moreover, $R[\![t]\!]$ is excellent [Val75] and [Mat87, Theorem 19.5]. It follows that the localization $R(\!(t)\!) = R[\![t]\!]_t$ is an excellent Dedekind domain. Since B'_y is a torsion-free $R(\!(t)\!)$-module and $B'_{y,0}$ is a free $k(\!(t)\!)$-module of rank $|G|$, B'_y is a locally free $R(\!(t)\!)$-module of rank $|G|$. Moreover, the subring $B'_y \subset C_y$ is stable under the G-action.

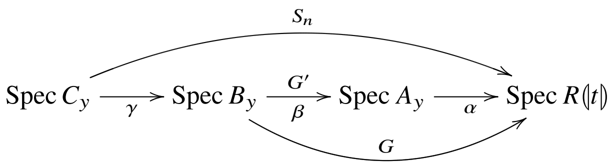
From [Gro65, Scholie 7.8.3], the ring B'_y is excellent. Therefore, the normalization B_y of B'_y is finitely generated as a B'_y-module as well as an $R(\!(t)\!)$-module. By a similar reasoning as above, B_y is a locally free $R(\!(t)\!)$-module of rank $|G|$. We also see that B_y is a locally free A_y-module of rank $|G'| = |G|/n$. We have natural morphisms of Dedekind schemes

$$\operatorname{Spec} C_y \xrightarrow{\gamma} \operatorname{Spec} B_y \xrightarrow{\beta} \operatorname{Spec} A_y \xrightarrow{\alpha} \operatorname{Spec} R(\!(t)\!). \tag{4.4}$$

Morphisms α, β, γ as well as their compositions are flat. The morphism $\alpha \circ \beta$ is G-invariant and the morphism β is G'-invariant. We know that α and $\alpha \circ \beta \circ \gamma$ are étale. From [Gro67, Proposition 17.7.7], $\alpha \circ \beta$ is étale. From [Gro67, Proposition 17.3.4], β is also étale.

Proving that $\operatorname{Spec} B_y \to \operatorname{Spec} R(\!(t)\!)$ is a G-torsor. We now claim that the G-action on $\operatorname{Spec} B_y$ is free. Each element $g \in G \setminus \{1\}$ acts nontrivially on $B'_{y,0}$ and hence also on B'_y and B_y. If the G-action on $\operatorname{Spec} B_y$ is not free, then for some $g \in G \setminus \{1\}$, the quotient morphism $\operatorname{Spec} B_y \to (\operatorname{Spec} B_y)/\langle g \rangle$ is ramified. But, this is impossible due to [Gro67, Proposition 17.3.3(v)] and the fact that the morphism $\alpha \circ \beta\colon \operatorname{Spec} B_y \to \operatorname{Spec} R(\!(t)\!)$ is étale and factors through $(\operatorname{Spec} B_y)/\langle g \rangle$. This shows the claim. We have shown that $\operatorname{Spec} B_y \to \operatorname{Spec} R(\!(t)\!)$ is an étale finite morphism of degree $|G|$ and G-invariant for a free G-action on $\operatorname{Spec} B_y$. We conclude that this is a G-torsor. By the same argument, we also see that

$\operatorname{Spec} B_y \to \operatorname{Spec} A_y$ is a G'-torsor. In summary, we have shown that some compositions of morphisms in (4.4) are torsors for groups displayed in the following diagram:



“Fibers” of B_y are powers of Galois closures. For a point $z \in Z(k)$, base changing (4.4) with $\operatorname{Spec} k(\!(t)\!) \to \operatorname{Spec} R(\!(t)\!)$, we get a sequence of finite étale morphisms

$$\operatorname{Spec} C_{y,z} \xrightarrow{\gamma_z} \operatorname{Spec} B_{y,z} \xrightarrow{\beta_z} \operatorname{Spec} A_{y,z} \xrightarrow{\alpha_z} \operatorname{Spec} k(\!(t)\!).$$

If $\widetilde{A_{y,z}}/k(\!(t)\!)$ denotes a Galois closure of $A_{y,z}/k(\!(t)\!)$, then $C_{y,z} \cong (\widetilde{A_{y,z}})^u$ for some $u \in \mathbb{Z}_{>0}$. Since $\operatorname{Spec} B_{y,z} \to \operatorname{Spec} k(\!(t)\!)$ is a G-torsor, we can write $B_{y,z} \cong D^v$ for some Galois extension $D/k(\!(t)\!)$. Then, D is an intermediate field of $\widetilde{A_{y,z}}/A_{y,z}$ such that $D/k(\!(t)\!)$ is Galois, which shows that $D = \widetilde{A_{y,z}}$ and $B_{y,z} \cong (\widetilde{A_{y,z}})^v$.

Completing the proof by using a moduli stack. Since G is the semidirect product $H \rtimes C$ of a p-group H and a tame cyclic group C, we can use the moduli stack $\widetilde{\Delta}_G$ of G-torsors over $\operatorname{Spec} k(\!(t)\!)$ constructed in [TY20] (denoted by Δ_G in the cited paper). The stack $\widetilde{\Delta}_G$ is written as the inductive limit of Deligne–Mumford stacks $\mathcal{X}_n$, $n \in \mathbb{Z}_{\geq 0}$ of finite type, where transition morphisms $\mathcal{X}_n \to \mathcal{X}_{n+1}$ are representable and universally injective. In particular, we have

$$\begin{aligned}\{G\text{-torsors over } \operatorname{Spec} k(\!(t)\!)\}/\cong &= \Delta_G(k)/\cong \\ &= \bigcup_n \mathcal{X}_n(k)/\cong.\end{aligned}$$

Since Z is quasicompact, the morphism $Z \to \widetilde{\Delta}_G$ corresponding to the G-torsor $\operatorname{Spec} B_y \to \operatorname{Spec} R(\!(t)\!)$ factors through a morphism $Z \to \mathcal{X}_n$ for some n. Recall that for $z \in Z^\circ(k)$, we have an $k(\!(t)\!)$-isomorphism $A_{y,z} \cong A$. Therefore, for $z \in Z^\circ(k)$, $\widetilde{A_{y,z}} \cong \widetilde{A}$, where $\widetilde{A}$ is a Galois closure of $A/k(\!(t)\!)$. It follows that for $z \in Z^\circ(k)$, $B_{y,z} \cong (\widetilde{A})^v$ as a $k(\!(t)\!)$-algebra. There are at most finitely many G-torsor structures which can be given to the morphism $\operatorname{Spec}(\widetilde{A})^v \to \operatorname{Spec} k(\!(t)\!)$. It follows that the image of the map $Z^\circ(k) \to \mathcal{X}_n(k)/\cong$ is a finite set. Since $(Z \setminus Z^\circ)(k)$ is a finite set, the image of the map $Z(k) \to \mathcal{X}_n(k)/\cong$ is also finite. Since Z is irreducible, we conclude that the map $Z(k) \to \mathcal{X}_n(k)/\cong$ is constant. Hence, the G-torsors $B_{y,z}/k(\!(t)\!)$, $z \in Z(k)$ are isomorphic to one another. It follows that étale $k(\!(t)\!)$-algebras $A_{y,z} = (B_{y,z})^{G'}$, $z \in Z(k)$ are isomorphic to one another, and hence all of them are isomorphic to A. In particular, $A_{y,0} \cong A$ as desired. □

5. The space of uniformizers

Let $A/K(\!(t)\!)$ be a totally ramified extension of degree n and discriminant exponent m with a chosen uniformizer ϖ. We construct a map associating an Eisenstein polynomial to an arbitrary uniformizer of A. To do so, we introduce some more notation. Let $\widetilde{A}$ be a Galois closure of $A/K(\!(t)\!)$ with Galois group G. We define groups

$$\begin{aligned} E &:= \{g \in G | \forall a \in A,\ g(a) = a\}, \\ \widetilde{H} &:= \{g \in G \mid g(A) = A\}, \\ H &:= \widetilde{H}/E = \operatorname{Aut}(A/K(\!(t)\!)). \end{aligned}$$

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Let $\sigma_1, \ldots, \sigma_n \colon A \hookrightarrow \tilde{A}$ be the $K(\!(t)\!)$-embeddings. Let $s_i(u_1, \ldots, u_n)$, $i = 1, \ldots, n$, be elementary symmetric polynomials of n variables with degree i and put $s_0 = 1$ by convention.

With the above notation, to a uniformizer $\varpi' \in A$, we associate the polynomial

$$f_{\varpi'}(x) := \sum_{i=0}^{n} (-1)^i s_i(\sigma_1(\varpi'), \ldots, \sigma_n(\varpi')) x^{n-i} \in K[\![t]\!][x].$$

Note that the coefficients $(-1)^i s_i(\sigma_1(\varpi'), \ldots, \sigma_n(\varpi'))$ are elements of $K[\![t]\!]$, since they are invariant under the G-action. If we denote the unique extension of the valuation ord to A again by ord, then

$$\operatorname{ord} s_n(\sigma_1(\varpi'), \ldots, \sigma_n(\varpi')) = \operatorname{ord} \prod_{i=1}^{n} \sigma_i(\varpi') = \sum_{i=1}^{n} \operatorname{ord} \sigma_i(\varpi') = n \cdot \frac{1}{n} = 1.$$

We also see that $s_i(\sigma_1(\varpi'), \ldots, \sigma_n(\varpi'))$ has positive order for every $i > 0$. There exists a $K(\!(t)\!)$-isomorphism

$$K(\!(t)\!)[x]/(f_{\varpi'}) \to A,\ x \mapsto \varpi'.$$

Thus, we have obtained the map

$$\varpi' \mapsto f_{\varpi'}(x)$$

sending a uniformizer to an Eisenstein polynomial. Conversely, if an Eisenstein polynomial $f(x) \in K[\![t]\!][x]$ defines an extension $K(\!(t)\!)[x]/(f(x))$ admitting a $K(\!(t)\!)$-isomorphism $\rho \colon K(\!(t)\!)[x]/(f(x)) \xrightarrow{\sim} A$, then $f(x)$ is recovered as $f_{\rho(x)}$. Namely, the map

$$\{\text{uniformizers of } A\} \to \{y \in \mathfrak{Eis}^{(m)}(K) \mid A_y \cong A\},\ \varpi' \mapsto f_{\varpi'} \tag{5.1}$$

is surjective.

Next we realize this map as a map of arc spaces as follows. Let

$$W := \operatorname{Spec} K[\![t]\!][w_0, \ldots, w_{n-1}] = \mathbb{A}^n_{K[\![t]\!]}$$

and let $\mathfrak{O}_A := \mathrm{J}_\infty W$. For an extension L/K, we can identify $\mathfrak{O}_A(L)$ with the integer ring $\mathcal{O}_{A_L}$ of $A_L := A \otimes_{K(\!(t)\!)} L(\!(t)\!)$ by the bijection

$$\begin{aligned} (\mathrm{J}_\infty W)(L) &\to \mathcal{O}_{A_L}, \\ \gamma = (\gamma_0, \ldots, \gamma_{n-1}) &\mapsto \varpi_\gamma := \sum_{a=0}^{n-1} \gamma_a \varpi^a. \end{aligned}$$

Let $\mathfrak{Ut}_A, \mathfrak{Uf}_A, \mathfrak{M}_A \subset \mathfrak{O}_A$ be the locally closed subschemes corresponding to the groups of units, the sets of uniformizers and the maximal ideals of $\mathcal{O}_{A_L}$ by the above bijection. For each $l \in \mathbb{Z}_{\geq 0}$, let $\mathfrak{O}_{A,l} := \mathrm{J}_l W$, the l-jet scheme, which corresponds to $\mathcal{O}_A/t^{l+1}\mathcal{O}_A$, and let $\pi_l \colon \mathfrak{O}_A \to \mathfrak{O}_{A,l}$ be the truncation map. We put

$$\mathfrak{Ut}_{A,l} := \pi_l(\mathfrak{Ut}_A) \text{ and } \mathfrak{Uf}_{A,l} := \pi_l(\mathfrak{Uf}_A).$$

Using the fixed uniformizer $\varpi \in A$, for each $1 \leq i \leq n$, we define the polynomial

$$S_i(w_0, \ldots, w_{n-1}) := s_i\left(\sum_{a=0}^{n-1} w_a \sigma_1(\varpi^a), \ldots, \sum_{a=0}^{n-1} w_a \sigma_n(\varpi^a)\right) \in \mathcal{O}_{\tilde{A}}[w_0, \ldots, w_{n-1}].$$

This is invariant by the G-action, hence in fact belongs to $K[\![t]\!][w_0, \ldots, w_{n-1}]$.

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Definition 5.1. We define a $K[\![t]\!]$-morphism

$$\phi\colon W = \operatorname{Spec} K[\![t]\!][w_0, \dots, w_{n-1}] \to V_K = \operatorname{Spec} K[\![t]\!][x_1, \dots, x_n]$$

by $\phi^*(x_i) = (-1)^i S_i$. We write the associated maps of arc spaces and jet schemes as

$$\phi_l\colon \mathrm{J}_l W = \mathfrak{O}_{A,l} \to \mathrm{J}_l V_K \quad (0 \le l \le \infty).$$

We define gradings on $K[\![t]\!][w_0, \dots, w_{n-1}]$ and $K[\![t]\!][x_1, \dots, x_n]$ by

$$\deg w_i = 1 \text{ and } \deg x_i = i,$$

respectively. With these gradings, the map ϕ^* of coordinate rings is degree-preserving and ϕ is equivariant for the corresponding actions of $\mathbb{G}_{m,K} = \mathbb{G}_m \otimes_k K$. It follows that ϕ_l, $0 \le l \le \infty$ are also equivariant for the induced actions of $\mathbb{G}_{m,K}$ on jet/arc spaces.

Lemma 5.2. *Let L/K be a field extension. The map ϕ_∞ sends $\gamma \in \mathfrak{Uf}_A(L)$ to the point of $\mathfrak{Eis}^{(m)}(L)$ corresponding to f_{ϖ_γ}. In particular, we have $\phi_\infty(\mathfrak{Uf}_A) = \psi^{-1}(A)$ and $\phi_l(\mathfrak{Uf}_{A,l}) = \psi_l^{-1}(A)$ for $l \ge m$.*

Proof. The first assertion follows from construction. For a field extension L/K, let $A_L := A \otimes_{K(\!(t)\!)} L(\!(t)\!)$. As a base change of map (5.1), we get a surjection

$$\{\text{uniformizers of } A_L\} \to \{y \in \mathfrak{Eis}^{(m)}(L) \mid A_y \cong A_L\},\ \varpi' \mapsto f_{\varpi'}.$$

This shows $\phi_\infty(\mathfrak{Uf}_A) = \psi^{-1}(A)$. We get

$$\phi_l(\mathfrak{Uf}_{A,l}) = \phi_l(\pi_l(\mathfrak{Uf}_A)) = \pi_l(\phi_\infty(\mathfrak{Uf}_A)) = \pi_l(\psi^{-1}(A)) = \psi_l^{-1}(A),$$

where the last equality follows from Corollary 4.11. □

Let $\sigma \in H = \operatorname{Aut}(A/K(\!(t)\!))$ and for each $0 \le a < n$, let us write

$$\sigma(\varpi^a) = \sum_{b=0}^{n-1} c_{ab}^\sigma \varpi^b \quad (c_{ab}^\sigma \in K[\![t]\!]).$$

Let σ act on $K[\![t]\!][w_0, \dots, w_{n-1}]$ by $w_b \mapsto \sum_{a=0}^{n-1} c_{ab}^\sigma w_a$. This gives an H-action on W.

Lemma 5.3. *This H-action on W commutes with the $\mathbb{G}_{m,K}$-action and the morphism ϕ is H-invariant.*

Proof. The first assertion follows from the fact that the automorphism of $K[\![t]\!][w_0, \dots, w_{n-1}]$ induced by $\sigma \in H$ is linear and degree-preserving. To show the second assertion, it is enough to show that S_i are H-invariant. For $\sigma, \sigma' \in H$, we have

$$\begin{aligned}\sum_{b=0}^{n-1} \sigma(w_b)\sigma'(\varpi^b) &= \sum_{b=0}^{n-1}\sum_{a=0}^{n-1} c_{ab}^\sigma w_a \sigma'(\varpi^b) \\ &= \sum_{a=0}^{n-1} w_a \sum_{b=0}^{n-1} c_{ab}^\sigma \sigma'(\varpi^b) \\ &= \sum_{a=0}^{n-1} w_a (\sigma'\sigma)(\varpi^a).\end{aligned}$$

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Since the s_i are symmetric polynomials and the sequences $(\sigma\sigma_1)(\varpi^a), \dots, (\sigma\sigma_m)(\varpi^a)$ and $\sigma_1(\varpi^a), \dots, \sigma_m(\varpi^a)$ are permutations of each other, we have

$$\begin{aligned} S_i(\sigma(w_0), \dots, \sigma(w_{n-1})) &= s_i\left(\sum_{a=0}^{n-1} \sigma(w_a)\sigma_1(\varpi^a), \dots, \sum_{a=0}^{n-1} \sigma(w_a)\sigma_n(\varpi^a)\right) \\ &= s_i\left(\sum_{a=0}^{n-1} w_a(\sigma\sigma_1)(\varpi^a), \dots, \sum_{a=0}^{n-1} w_a(\sigma\sigma_n)(\varpi^a)\right) \\ &= S_i(w_0, \dots, w_{n-1}). \end{aligned}$$

Thus ϕ is H-invariant. □

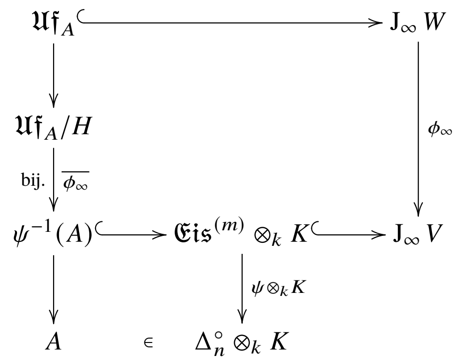
The last lemma implies that the morphism $\phi_\infty \colon \mathrm{J}_\infty W \to \mathrm{J}_\infty V_K$ is also H-invariant and so is its restriction $\mathfrak{Uf}_A \to \psi^{-1}(A)$. Note that from Proposition 4.15, $\psi_l^{-1}(A) \subset \mathrm{J}_l V_K$, $l \geq m$ and $\psi^{-1}(A) = \pi_l^{-1}(\psi_l^{-1}(A)) \subset \mathrm{J}_\infty V_K$ are closed subsets. We obtain morphisms

$$\overline{\phi_\infty} \colon \mathfrak{Uf}_A/H \to \psi^{-1}(A) \text{ and } \overline{\phi_l} \colon \mathfrak{Uf}_{A,l}/H \to \psi_l^{-1}(A) \quad (l \in \mathbb{Z}_{\geq 0}).$$

Lemma 5.4. *For each extension L/K, the map $\mathfrak{Uf}_A(L)/H \to \psi^{-1}(A)(L)$ is bijective.*

Proof. For a uniformizer ϖ', the associated Eisenstein polynomial is also written as $f_{\varpi'} = \prod_{i=1}^n (x - \sigma_i(\varpi'))$. Therefore, if $f_{\varpi'} = f_{\varpi''}$, then ϖ' and ϖ'' are conjugate and $\varpi'' = \sigma(\varpi')$ for some $\sigma \in H$. Thus $\mathfrak{Uf}_A(L)/H \to \psi^{-1}(A)(L)$ is injective. The surjectivity follows from Lemma 5.2. □

We can summarize relations among various spaces that we obtained so far in the following diagram.



Let e be the ramification index of $\widetilde{A}/K(\!(t)\!)$ and let $\mathfrak{D}_{\widetilde{A}/K(\!(t)\!)}$ and $\mathfrak{D}_{A/K(\!(t)\!)}$ be the differents of $A/K(\!(t)\!)$ and $\widetilde{A}/K(\!(t)\!)$, which are principal ideals of $\mathcal{O}_A$ and $\mathcal{O}_{\widetilde{A}}$, respectively. We denote the normalized valuation on $\widetilde{A}$ by $v_{\widetilde{A}}$ and the unique extension of the valuation ord to $\widetilde{A}$ again by ord. The two valuations are related by $e \cdot v_{\widetilde{A}} = \mathrm{ord}$.

Lemma 5.5. *We have*

$$\frac{v_{\tilde{A}}(\mathfrak{D}_{\tilde{A}/K(\!(t)\!)})}{e} = \mathrm{ord}\,\mathfrak{D}_{\tilde{A}/K(\!(t)\!)} \leq n \cdot \mathrm{ord}\,\mathfrak{D}_{A/K(\!(t)\!)} = m.$$

Proof. The left equality is obvious. The right equality follows from the fact that the discriminant is the norm of the different and a formula for valuations of norms (for example, [Neu99, Chapter II, (4.8) and Chapter III, (2.9)]). The Galois closure $\tilde{A}$ is obtained as the composite of all conjugates of A in an algebraic closure of $K(\!(t)\!)$. This shows $\mathfrak{D}_{\tilde{A}/K(\!(t)\!)} \mid (\mathfrak{D}_{A/K(\!(t)\!)})^n$. Indeed Toyama [Tôy55] proved the corresponding result for the composite of two number fields. The same argument applies to the

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case of local fields and the generalization to the composite of an arbitrary number of local fields is straightforward. It follows that $\operatorname{ord} \mathfrak{D}_{\tilde{A}/K(\!(t)\!)} \le n \cdot \operatorname{ord} \mathfrak{D}_{A/K(\!(t)\!)}$. □

Lemma 5.6. *For $l \ge m$, the H-action on $\mathfrak{Uf}_{A,l}$ is free.*

Proof. By taking a base change, we may assume that K is algebraically closed, in particular, perfect. Then, we can apply the theory of ramification groups explained in [Ser79, Chapter IV]; we follow the notation from this reference. From [Ser79, p. 64, Proposition 4], for $1 \ne g \in G$, $i_G(g) \le v_{\tilde{A}}(\mathfrak{D}_{\tilde{A}/K(\!(t)\!)})$. Let $a = em$ and let G_a be the a-th ramification group (lower numbering) consisting of g with $i_G(g) \ge a+1$. Using Lemma 5.5, we see that if $g \ne 1$, then $i_G(g) \le a$ and hence $g \notin G_a$. Thus, we conclude that $G_a = 1$. Let $\varphi_{\tilde{A}/K(\!(t)\!)}$ be the Herbrand function so that $G_a = G^{\varphi_{\tilde{A}/K(\!(t)\!)}(a)}$, where the right-hand side is a higher ramification group with the upper numbering. From [Ser79, p.64, Proposition 4 and p. 74, Lemma 3], we have

$$\begin{aligned}\varphi_{\tilde{A}/K(\!(t)\!)}(a) &= \frac{1}{e}\sum_{g\in G}\min\{i_G(g), a+1\} - 1\\ &= \frac{1}{e}\sum_{g\ne 1} i_G(g) + \frac{a+1}{e} - 1\\ &= \frac{v_{\tilde{A}}(\mathfrak{D}_{\tilde{A}/K(\!(t)\!)})}{e} + \frac{a+1}{e} - 1\\ &\le m + \frac{em+1}{e} - 1\\ &\le 2m.\end{aligned}$$

Thus $G^{2m} = 1$. From [Del84, Proposition A.6.1] and [Ser79, p.61, Lemma 1], if $\sigma\colon A \hookrightarrow \tilde{A}$ is a $K(\!(t)\!)$-embedding different from the chosen embedding $A \hookrightarrow \widetilde{A}$, which we denote by σ_0, and if $\varpi' \in \mathcal{O}_A$ is a uniformizer, then

$$\begin{aligned}2\cdot \operatorname{ord}(\sigma(\varpi') - \varpi') &\le \sum_{\tau\ne\sigma_0}\operatorname{ord}(\tau(\varpi') - \varpi') + \sup_{\tau\ne\sigma_0}\operatorname{ord}(\tau(\varpi') - \varpi')\\ &< 2m+1.\end{aligned}$$

and hence $\operatorname{ord}(\sigma(\varpi') - \varpi') < m+1$. In particular, for $1 \ne h \in H$, if $l \ge m$, then

$$h(\varpi') \ne \varpi' \mod t^{l+1}.$$

This shows the lemma. □

6. Bundles having "almost affine spaces" as fibers

We keep the notation from the last section. In particular, A denotes a totally ramified extension of $K(\!(t)\!)$ of degree n and discriminant exponent m. Let $\mathcal{J} \subset K[\![t]\!][w_0, \dots, w_{n-1}]$ be the Jacobian ideal of the morphism $\phi\colon W \to V$. Namely, $\mathcal{J}$ is the principal ideal generated by the determinant of the Jacobian matrix

$$\frac{\partial(S_1, \dots, S_n)}{\partial(w_0, \dots, w_{n-1})}.$$

Its order function

$$\operatorname{ord}_{\mathcal{J}}\colon \mathrm{J}_\infty W \to \mathbb{Z}_{\ge 0} \cup \{\infty\}$$

is defined as follows: for a point $\gamma \in \mathrm{J}_\infty W$ represented by a morphism $\operatorname{Spec} L \to \mathrm{J}_\infty W$ with L a field, if $\gamma' \colon \operatorname{Spec} L[[t]] \to W$ is the corresponding arc and if we can write $(\gamma')^{-1}\mathcal{J} = (t^n) \subset L[[t]]$, then $\operatorname{ord}_{\mathcal{J}}(\gamma) = n$. Here, we followed the convention that $(0) = (t^\infty)$.

Lemma 6.1. *The restriction* $\operatorname{ord}_{\mathcal{J}}|_{\mathfrak{Uf}_A}$ *of* $\operatorname{ord}_{\mathcal{J}}$ *to* $\mathfrak{Uf}_A$ *takes the constant value m.*

Proof. Since S_i is the composite of $s_i(x_1, \dots, x_n)$ and $x_j = \sum_a w_a \sigma_j(\varpi^a)$, the above Jacobian matrix can be written as

$$\left.\frac{\partial(s_1, \dots, s_n)}{\partial(x_1, \dots, x_n)}\right|_{x_i = \sum_{a=0}^{n-1} w_a \sigma_i(\varpi^a)} \times \frac{\partial\left(\sum_{a=0}^{n-1} w_a \sigma_1(\varpi^a), \dots, \sum_{a=0}^{n-1} w_a \sigma_n(\varpi^a)\right)}{\partial(w_0, \dots, w_{n-1})}. \tag{6.1}$$

Concerning the first factor, as was proved in [Ser78, p. 1036], we have

$$\det \frac{\partial(s_1, \dots, s_n)}{\partial(x_1, \dots, x_n)} = \prod_{i<j} (x_i - x_j).$$

For $\gamma = (\gamma_0, \dots, \gamma_{n-1}) \in \mathfrak{Uf}_A$, substituting γ_a for w_a in

$$\left.\left(\prod_{i<j}(x_i - x_j)\right)\right|_{x_i = \sum_{a=0}^{n-1} w_a \sigma_i(\varpi^a)},$$

we get

$$\prod_{i<j} (\sigma_i(\varpi_\gamma) - \sigma_j(\varpi_\gamma)).$$

From [Keu23, the proof of Proposition 1.29], its square is the discriminant of $1, \varpi_\gamma, \dots, \varpi_\gamma^{n-1}$.

On the other hand, concerning the second factor of (6.1), we have

$$\frac{\partial\left(\sum_{a=0}^{n-1} w_a \sigma_1(\varpi^a), \dots, \sum_{a=0}^{n-1} w_a \sigma_n(\varpi^a)\right)}{\partial(w_0, \dots, w_{n-1})} = (\sigma_i(\varpi^a))_{i,a}.$$

The discriminant of $A/k((t))$ with respect to the basis $1, \varpi, \dots, \varpi^{n-1}$ is the square of $\det(\sigma_i(\varpi^a))_{i,a}$ [Keu23, Proposition 1.28]. We have showed that the determinants of the two factors in (6.1) both have order $m/2$, which shows $\operatorname{ord}_{\mathcal{J}}(\gamma) = m$. □

Lemma 6.2. *For an extension L/K and for $l \geq 2m$, every fiber of $\mathfrak{Uf}_{A,l}(L)/H \to \mathfrak{Eis}_l^{(m)}(L)$ is contained in a fiber of $\mathfrak{Uf}_{A,l}(L)/H \to \mathfrak{Uf}_{l-m}(L)/H$.*

Proof. Suppose that two points $b_l, b_l' \in \mathfrak{Uf}_{A,l}(L)$ map to the same point $a_l \in \mathfrak{Eis}_l^{(m)}(L)$. Let $b, b' \in \mathfrak{Uf}_A(L)$ be lifts of b_l, b_l' respectively and let $a := \phi_\infty(b) \in \mathfrak{Eis}^{(m)}(L)$. Then, from [CLNS18, Chapter 5, Proposition 3.1.7], there exists $c \in \mathfrak{Uf}_A$ such that $\phi_\infty(c) = a$ and the images $c_{l-m}, b'_{l-m} \in \mathfrak{Uf}_{A,l-m}$ of c, b' are the same. Note that the constant c_V appearing in the cited result is taken to be 1 in our situation, since V is smooth. The condition that $\phi_\infty(c) = \phi_\infty(b) = a$ implies that c is in the H-orbit of b. Therefore $H(c_{l-m}) = H(b_{l-m}) = H(b'_{l-m})$. This shows the lemma. □

Definition 6.3. Let r be a non-negative integer. We say that a morphism $f \colon Y \to X$ of k-varieties is an $\mathbb{A}^r$*-bundle*(resp. *weak* $\mathbb{A}^r$*-bundle*) if for every geometric point $x \in X(K)$, the fiber $f^{-1}(x)$ is K-isomorphic to $\mathbb{A}^r_K$ (resp. to the quotient $\mathbb{A}^r_K/C$ of the affine space $\mathbb{A}^n_K$ by a finite cyclic group C of order coprime to p).

Definition 6.4. A variety X over k is said to be a *weak affine space* if there exists a sequence of k-varieties $X = X_0, X_1, \dots, X_l = \operatorname{Spec} k$ such that for each $0 \le i < l$, there exists a morphism between X_i and X_{i+1} in either direction which is a weak $\mathbb{A}^{r_i}$-bundle for some r_i. A weak affine space of dimension r is called also as a *weak* $\mathbb{A}^r_k$.

Lemma 6.5. *For $l \ge 2m$, the morphism $\overline{\phi_l}\colon \mathfrak{Uf}_{A,l}/H \to \psi_l^{-1}(A)$ is an $\mathbb{A}^m$-bundle.*

Proof. By base change, we may suppose that $K = k$ and they are algebraically closed. Moreover, it suffices to consider fibers over closed points. In the rest of this proof, all points that we consider are closed. For $a_l \in \psi_l^{-1}(A)$, the fiber $(\overline{\phi_l})^{-1}(a_l)$ of $\overline{\phi_l}\colon \mathfrak{Uf}_{A,l}/H \to \mathfrak{Eis}_l^{(m)}$ over a_l is contained in the fiber $(\pi^l_{l-m})^{-1}(\overline{b_{l-m}})$ of the truncation map $\pi^l_{l-m}\colon \mathfrak{Uf}_{A,l}/H \to \mathfrak{Uf}_{A,l-m}/H$ over a point $\overline{b_{l-m}} \in \mathfrak{Uf}_{A,l-m}/H$. Let $b_{l-m} \in \mathfrak{Uf}_{A,l-m}$ be a lift of $\overline{b_{l-m}}$. From Lemma 5.6, since $l - m \ge m$, the H-action on $\mathfrak{Uf}_{A,l-m}$ is free. In particular, for $h \in H \setminus \{1\}$, $h(b_{l-m}) \ne b_{l-m}$, which shows that for two distinct points of $\pi^{-1}_{l-m}(b_{l-m})$, the H-action cannot send one point to the other. From Lemma 5.4, the restriction of ϕ_∞ to $\pi^{-1}_{l-m}(b_{l-m})$ is injective. The freeness of the H-action on $\mathfrak{Uf}_{A,l-m}$ also shows that the map $(\pi^l_{l-m})^{-1}(b_{l-m}) \to (\pi^l_{l-m})^{-1}(\overline{b_{l-m}})$ is an isomorphism. From [CLNS18, Chapter 5, Theorem 3.2.2.c], $(\pi^l_{l-m})^{-1}(b_{l-m}) \to \mathfrak{Eis}_l^{(m)}$ is a piecewise $\mathbb{A}^m$-bundle over its image and so is $(\pi^l_{l-m})^{-1}(\overline{b_{l-m}}) \to \mathfrak{Eis}_l^{(m)}$. Since $(\overline{\phi_l})^{-1}(a_l) \subset (\pi^l_{l-m})^{-1}(\overline{b_{l-m}})$, we conclude that $(\overline{\phi_l})^{-1}(a_l) \cong \mathbb{A}^m_k$. □

Lemma 6.6. *Let y be a geometric point of $\psi_l^{-1}(A)$. Suppose $p \mid n$ and $l \ge \lfloor m/n \rfloor$. Then, the stabilizer* $\operatorname{Stab}(y)$ *of y with respect to the $\mathbb{G}_m$-action on $\mathfrak{Eis}_l^{(m)}$ is a tame cyclic group.*

Proof. Let L/k be the algebraically closed field such that y is an L-point. For $\lambda \in \mathbb{G}_m(L)$, if we write $y = (y_1, \dots, y_n) \in (L[\![t]\!]/(t^{l+1}))^n$, we have

$$\lambda \cdot y = (\lambda y_1, \dots, \lambda^n y_n).$$

Thus, if $y_i \ne 0$ for some i, then $\operatorname{Stab}(y)$ is contained μ_i, the group of i-th roots of 1. From Proposition 4.4, there exists $i \in \{1, \dots, n-1\}$ such that $p \nmid i$ and $y_i \ne 0$, which proves the lemma. □

From Lemma 5.3, we get a $\mathbb{G}_{m,K}$-equivariant morphism $\mathfrak{Uf}_{A,l}/H \to \psi_l^{-1}(A)$. Since $\mathfrak{Uf}_{A,l}$ is affine, so is $\mathfrak{Uf}_{A,l}/H$. The fiber $\psi_l^{-1}(A)$ is also an affine variety over K, thanks to Proposition 4.15. The $\mathbb{G}_{m,K}$-actions on $\psi_l^{-1}(A)$ and $\mathfrak{Uf}_{A,l}/H$ have finite stabilizers. Thus, the geometric quotients $\mathfrak{Uf}_{A,l}/(H\times\mathbb{G}_{m,K}) = (\mathfrak{Uf}_{A,l}/H)/\mathbb{G}_{m,K}$ and $\psi_l^{-1}(A)/\mathbb{G}_{m,K}$ exist and are again affine varieties over K.

Corollary 6.7. *For $l \ge 2m$, the morphism $\mathfrak{Uf}_{A,l}/(H \times \mathbb{G}_{m,K}) \to \psi_l^{-1}(A)/\mathbb{G}_{m,K}$ is a weak $\mathbb{A}^m$-bundle.*

Proof. Let $y \in \psi_l^{-1}(A)(L)$ be a geometric point and let $\overline{y}$ be its image in $(\psi_l^{-1}(A)/\mathbb{G}_{m,K})(L)$. Let C be the stabilizer at y for the $\mathbb{G}_{m,K}$-action on $\psi_l^{-1}(A)$, which is a finite and tame cyclic group from Lemma 6.6. From [AV02, Lemma 2.3.3], the fiber of $\mathfrak{Uf}_{A,l}/(H \times \mathbb{G}_{m,K}) \to \psi_l^{-1}(A)/\mathbb{G}_{m,K}$ over $\overline{y}$ is the quotient of the fiber of $\mathfrak{Uf}_{A,l}/H \to \psi_l^{-1}(A)$ over y by the action of C. The corollary follows from Lemma 6.5. □

Lemma 6.8. *The structure morphism $\mathfrak{Uf}_{A,l}/(H \times \mathbb{G}_{m,K}) \to \operatorname{Spec} K$ factors as*

$$\mathfrak{Uf}_{A,l}/(H \times \mathbb{G}_m) = Z_{n(l+1)-2} \to Z_{n(l+1)-3} \to \cdots \to Z_0 = \operatorname{Spec} K$$

in such a way that each morphism $Z_{i+1} \to Z_i$ is an $\mathbb{A}^1$-bundle. In particular, $\mathfrak{Uf}_{A,l}/(H \times \mathbb{G}_{m,K})$ is a weak $\mathbb{A}_K^{n(l+1)-2}$.

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Proof. Again by a base change argument, we may assume that $K = k$ and they are algebraically closed and consider only closed points. Let $\mathfrak{m}_A$ be the maximal ideal of A and let $\mathfrak{C}_{n(l+1)} \subset \mathfrak{Uf}_{A,l}$ be the subvariety corresponding to elements of

$$\mathcal{O}_A/t^{l+1}\mathcal{O}_A = \mathcal{O}_A/\mathfrak{m}_A^{n(l+1)}$$

that are equal to the chosen uniformizer ϖ modulo $\mathfrak{m}_A^2$. For $2 \leq i \leq n(l+1)$, let $\mathfrak{C}_i$ be the variety corresponding to the images of such elements in $\mathcal{O}_A/\mathfrak{m}_A^i$. As a K-variety, $\mathfrak{C}_i$ is isomorphic to $\mathbb{A}_K^{i-2}$. We define a (lower-numbering) filtration H_* of H in the way which is standard if $A/K(\!(t)\!)$ is a Galois extension: for $i \geq 0$,

$$H_i := \mathrm{Ker}(H \to \mathrm{Aut}(\mathcal{O}_A/\mathfrak{m}_A^{i+1})).$$

For a uniformizer $\varpi \in A$, the map $g \mapsto g(\varpi)/\varpi$ defines an injective homomorphism $H_i/H_{i+1} \to U^i/U^{i+1}$. From [Ser61, Section 1.6], $U^0/U^1 \cong \mathbb{G}_m$ and for $i \geq 1$, $U^i/U^{i+1} \cong \mathbb{G}_a$. Therefore H_0/H_1 is a tame cyclic group and H_i/H_{i+1}, $i \geq 1$ are elementary abelian p-groups. Similarly, for $i \geq 1$, we can have an embedding $H_i/H_{i+1} \to \mathfrak{m}_L^{i+1}/\mathfrak{m}_L^{i+2}$ by $g \mapsto g(\varpi) - \varpi$.

We claim that the composite morphism

$$\mathfrak{C}_{n(l+1)} \hookrightarrow \mathfrak{Uf}_{A,l} \to \mathfrak{Uf}_{A,l}/\mathbb{G}_m$$

is an isomorphism. Indeed, the morphism is clearly a universal homeomorphism. To show that it is an isomorphism, it is enough to show that it is étale. We have the $\mathbb{G}_m$-equivariant morphism $\mathfrak{Uf}_{A,l} \to \mathbb{G}_m$ corresponding to the map sending a uniformizer ϖ' to $\lambda \in k^*$ if $\varpi' = \lambda\varpi$ modulo $\mathfrak{m}_A^2$. This is a smooth morphism and $\mathfrak{C}_{n(l+1)}$ is the fiber over $1 \in \mathbb{G}_m$ of this morphism. This shows that $\mathfrak{C}_{n(l+1)}$ intersects with each $\mathbb{G}_m$-orbit transversally, which implies that $\mathfrak{C}_{n(l+1)} \to \mathfrak{Uf}_{A,l}/\mathbb{G}_{m,k}$ is étale. We have proved the claim.

We see that $H(\mathfrak{C}_{n(l+1)}) \subset \mathfrak{Uf}_{A,l}$ consists of $|H/H_1|$ connected components, each of which is isomorphic to $\mathfrak{C}_{n(l+1)}$. The stabilizer of the component $\mathfrak{C}_{n(l+1)} \subset H(\mathfrak{C}_{n(l+1)})$ is H_1. Therefore $\mathfrak{Uf}_{A,l}/(H\times \mathbb{G}_m) \cong \mathfrak{C}_{n(l+1)}/H_1$. We claim that for every $j \geq 1$ and every i with $2 \leq i \leq n(l+1)$, the map $\mathfrak{C}_i/H_j \to \mathfrak{C}_{i-1}/H_j$ is an $\mathbb{A}^1$-bundle. To see this, we first note that $\mathfrak{C}_i \to \mathfrak{C}_{i-1}$ is a bundle with fiber $\mathfrak{m}_L^{i-1}/\mathfrak{m}_L^i = \mathbb{G}_a$. If $j \geq i-1$, then the H_j-actions on $\mathfrak{C}_i$ and $\mathfrak{C}_{i-1}$ are trivial and the claim follows. If $j = i-2$, then the map of the claim has fibers isomorphic to

$$(\mathfrak{m}_L^{i-1}/\mathfrak{m}_L^i)/(H_{i-2}/H_{i-1}) \cong \mathbb{G}_a.$$

The last isomorphism follows from [Mil17, Corollary 14.56]. For $j < i-2$, the map of the claim is identified with

$$(\mathfrak{C}_i/H_{i-2})/(H_j/H_{i-2}) \to \mathfrak{C}_{i-1}/(H_j/H_{i-2}).$$

The claim now holds since the action of H_j/H_{i-2} on $\mathfrak{C}_{i-1}$ is free. We have proved the claim.

Thus we obtain a tower of $\mathbb{A}^1$-bundles

$$\mathfrak{Uf}_{A,l}/(H\times\mathbb{G}_m) \cong \mathfrak{C}_{n(l+1)}/H_1 \to \mathfrak{C}_{n(l+1)-1}/H_1 \to \cdots \to \mathfrak{C}_2/H_1 = \mathrm{pt}.$$

This shows the lemma. □

Lemma 6.9. *For every geometric point $A \in \Delta_n^\circ(K)$ with $\mathbf{d}_A = m$ and for $l \geq 2m$, the closed subset $\psi_l^{-1}(A)/\mathbb{G}_{m,K} \subset \mathfrak{Eis}_l^{(m)} \otimes_k K/\mathbb{G}_{m,K}$ is a weak $\mathbb{A}_K^{n(l+1)-2-m}$.*

Proof. We first note that from Proposition 4.15, $\psi_l^{-1}(A) \subset \mathfrak{Eis}_l^{(m)} \otimes_k K$ is a closed subset, which is also $\mathbb{G}_{m,K}$-invariant. Therefore, $\psi_l^{-1}(A)/\mathbb{G}_{m,K} \subset (\mathfrak{Eis}_l^{(m)} \otimes_k K)/\mathbb{G}_{m,K}$ is also a closed subset (for example, see [New09, Theorem 1.1]). From Corollary 6.7 and Lemma 6.8, $\psi_l^{-1}(A)/\mathbb{G}_{m,K}$ is a weak $\mathbb{A}_K^{n(l+1)-2-m}$. □

Lemma 6.10. *Let Y, X be k-varieties and let $f\colon Y \to X$ be a P-morphism over k with the graph $\Gamma \subset Y \times_k X$. Let pr_Y and pr_X be the projections from $Y \times_k X$ to Y and X, respectively. Suppose that for every point $w\colon \operatorname{Spec} K \to X$ with K an arbitrary field, the fiber $f^{-1}(w)$ is an irreducible closed subset of $Y \otimes_k K$.*

1. *Then, for every point $x \in X$, there exists an open dense subset U_x of $\overline{\{x\}}$ such that $\mathrm{pr}_X^{-1}(U_x) \cap \Gamma$ is a closed subset of $\mathrm{pr}_X^{-1}(U_x)$.*
2. *There exists a decomposition $X = \bigsqcup_{i=1}^{l} X_i$ of X into finitely many locally closed subsets X_i such that for each i, $\mathrm{pr}_X^{-1}(X_i) \cap \Gamma$ is a locally closed subset of $Y \times_k X$.*

Proof. (1) By assumption, $\Gamma \cap \mathrm{pr}_X^{-1}(x)$ is a closed subset of $\mathrm{pr}_X^{-1}(x)$. Let C be its closure in $Y \times X$, which is contained in $\mathrm{pr}_X^{-1}(\overline{\{x\}})$. Since $C \cap \mathrm{pr}_X^{-1}(x) = \Gamma \cap \mathrm{pr}_X^{-1}(x)$, the symmetric difference $C \triangle \Gamma$ is a constructible subset whose image in X does not contain x. Let $Z := \overline{\{x\}} \cap \mathrm{pr}_X(C \triangle \Gamma)$. This is a constructible subset of $\overline{\{x\}}$ which does not contain x. Let $U_x := \overline{\{x\}} \setminus \overline{Z}$, which is an open dense subset of $\overline{\{x\}}$. By construction, we have $C \cap \mathrm{pr}_X^{-1}(U_x) = \Gamma \cap \mathrm{pr}_X^{-1}(U_x)$, which is a closed subset of $\mathrm{pr}_X^{-1}(U_x)$.

(2) We show this in the stronger form such that for every j, $W_j := \bigsqcup_{i=j}^{l} X_i$ is a closed subset of X. We construct X_i's inductively as follows. We put X_1 to be U_η for the generic point η of an irreducible component of $W_1 = B$ having the maximal dimension. Suppose that we have constructed $X_1, \dots, X_{j-1}$ and let $W_j = X \setminus \bigcup_{i=1}^{j-1} X_i$. We take the generic point ξ of an irreducible component of W_j having the maximal dimension. We put $X_j := U_\xi \cap W_j$ and $W_{j+1} := W_j \setminus X_j$. We then have

$$(\dim W_i, \nu(W_i)) > (\dim W_{i+1}, \nu(W_{i+1})),$$

where pairs are ordered lexicographically and $\nu(-)$ means the number of irreducible components of maximal dimension. The procedure ends after finitely many steps and gives a desired decomposition of X. □

Definition 6.11. We say that a morphism $f\colon Y \to X$ of k-varieties is a *very weak $\mathbb{A}^m$-bundle* if for every geometric point $x \in X(K)$, the fiber $f^{-1}(x)$ is universally homeomorphic to a weak $\mathbb{A}_K^m$.

In particular, a universal homeomorphism of k-varieties is a very weak $\mathbb{A}^0$-bundle.

Corollary 6.12. *There exists a scheme morphism $f\colon Y \to X$ which is a very weak $\mathbb{A}^{n(l+1)-2-m}$-bundle and induces the P-morphism $\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m \to \Delta_n^{(m)}$.*

Proof. From Proposition 4.15, the P-morphism $h\colon \mathfrak{Eis}_l^{(m)}/\mathbb{G}_m \to \Delta_n^{(m)}$ satisfies the assumption of Lemma 6.10. Therefore, if Γ denotes its graph, then there exists a decomposition $\Delta_n^{(m)} = \bigsqcup_{i=1}^{l} X_i$ into locally closed subsets X_i such that for each i, $\mathrm{pr}_{\Delta_n^{(m)}}^{-1}(X_i) \cap \Gamma$ is a locally closed subset. Giving these locally closed subsets the reduced scheme structures, let us consider the coproducts $X := \coprod_{i=1}^{l} X_i$ and $Y := \coprod_{i=1}^{l} (\mathrm{pr}_{\Delta_n^{(m)}}^{-1}(X_i) \cap \Gamma)$. The natural morphism $f\colon Y \to X$ induces the P-morphism $h\colon \mathfrak{Eis}_l^{(m)}/\mathbb{G}_m \to \Delta_n^{(m)}$. From construction, the morphism

$$\mathrm{pr}_{\Delta_n^{(m)}}^{-1}(X_i) \cap \Gamma \to h^{-1}(X_i)$$

is a universal homeomorphism. From Lemma 6.9, this is a very weak $\mathbb{A}^{n(l+1)-2-m}$-bundle. □

7. Motivic mass formulas

Let $K_0(\mathbf{Var}/k)$ denote the Grothendieck ring of k-varieties and let $\mathbb{L} := [\mathbb{A}_k^1]$.

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Definition 7.1. We define $K_0^\heartsuit(\mathbf{Var}/k)$ to be the quotient of $K_0(\mathbf{Var}/k)$ by the following relation: for each very weak $\mathbb{A}^m$-bundle $Y \to X$, we have $[Y] = [X]\mathbb{L}^m$. We define $\mathcal{M}_k^\heartsuit := K_0^\heartsuit(\mathbf{Var}/k)$ to be the localization of $K_0^\heartsuit(\mathbf{Var}/k)$ by $\mathbb{L}$. For $s \in \mathbb{Z}$, let $F_s \subset \mathcal{M}_k^\heartsuit$ to be the subgroup generated by elements of the form $[X]\mathbb{L}^r$ with $\dim X + r \le -s$. We define the *dimensional completion* $\widehat{\mathcal{M}}_k^\heartsuit$ of $\mathcal{M}_k^\heartsuit$ to be the projective limit $\varprojlim \mathcal{M}_k^\heartsuit/F_s$.

There exist ring structures on $K_0^\heartsuit(\mathbf{Var}/k)$, $\mathcal{M}_k^\heartsuit$, and $\widehat{\mathcal{M}}_k^\heartsuit$ which are induced by the one on $K_0(\mathbf{Var}/k)$.

Remark 7.2. Let us suppose now that k is a finitely generated field. We denote by $\mathbf{WRep}_{G_k}(\mathbb{Q}_l)$ the abelian category of mixed l-adic representations of the absolute Galois group $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Its Grothendieck ring $K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))$ admits the completion $\widehat{K}_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))$ with respect to weights. We have a ring homomorphism $\mathcal{M}_k := K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}] \to K_0(\mathbf{WRep}_{G_k}(\mathbb{Q}_l))$ sending $[X]$ to $\sum_i (-1)^i [H_c^i(\overline{X}, \mathbb{Q}_l)]$, which extends to $\widehat{\mathcal{M}}_k^\heartsuit \to \widehat{K}_0(\mathbf{WRep}(\mathbb{Q}_l))$ by an argument similar to one in [Yas24a, Lemma 9.11] and its subsequent paragraphs. Hence, for each equality in $\widehat{\mathcal{M}}_k^\heartsuit$ that we obtain below, we have the corresponding equality in $\widehat{K}_0(\mathbf{WRep}(\mathbb{Q}_l))$, provided that k is finitely generated.

Lemma 7.3. *We have*

$$[\Delta_n^{(m)}]\mathbb{L}^{n(l+1)-2-m} = [\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m] \in K_0^\heartsuit(\mathbf{Var}/k).$$

Proof. Take a morphism $Y \to X$ as in Corollary 6.12. We have the following equalities in $K_0^\heartsuit(\mathbf{Var}/k)$:

$$[\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m] = [Y] = [X]\mathbb{L}^{n(l+1)-2-m} = [\Delta_n^{(m)}]\mathbb{L}^{n(l+1)-2-m}.$$ □

The map $\mathfrak{Eis}_{l+1}/\mathbb{G}_m \to \mathfrak{Eis}_l/\mathbb{G}_m$ is an $\mathbb{A}^n$-bundle. We define a motivic measure μ on $\mathfrak{Eis}/\mathbb{G}_m$ taking values in $\widehat{\mathcal{M}}_k^\heartsuit$ in the usual way: for a cylinder $C \subset \mathfrak{Eis}/\mathbb{G}_m$, $\mu(C) := [\pi_l(C)]\mathbb{L}^{-nl}$ for $l \gg 0$. We then extend this measure to measurable subsets (see [CLNS18, Chapter 6, Section 3] for details). In particular, $\mathfrak{Eis}/\mathbb{G}_m$, $\mathfrak{Eis}^{\mathrm{sep}}/\mathbb{G}_m$ and $\mathfrak{Eis}^{(m)}/\mathbb{G}_m$ are measurable subsets and we have

$$\begin{aligned}\mu(\mathfrak{Eis}/\mathbb{G}_m) = \mu(\mathfrak{Eis}^{\mathrm{sep}}/\mathbb{G}_m) &= \sum_m \mu(\mathfrak{Eis}^{(m)}/\mathbb{G}_m)\\ &= [\pi_1(\mathfrak{Eis})/\mathbb{G}_m]\mathbb{L}^{-n} = \mathbb{L}^{-1}.\end{aligned}$$

Theorem 7.4. *We have*

$$\int_{\Delta_n^\circ} \mathbb{L}^{-\mathbf{d}} = \mathbb{L}^{-n+1} \in \widehat{\mathcal{M}}_k^\heartsuit. \tag{7.1}$$

Proof. In $\widehat{\mathcal{M}}_k^\heartsuit$,

$$\begin{aligned}[\Delta_n^{(m)}]\mathbb{L}^{-m} &= [\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m]\mathbb{L}^{-n(l+1)+2}\\ &= \mu(\mathfrak{Eis}^{(m)}/\mathbb{G}_m)\mathbb{L}^{-n+2}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\Delta_n^\circ} \mathbb{L}^{-\mathbf{d}} &= \mathbb{L}^{-n+2}\sum_m \mu(\mathfrak{Eis}^{(m)}/\mathbb{G}_m)\\ &= \mathbb{L}^{-n+2}\mu(\mathfrak{Eis}/\mathbb{G}_m)\\ &= \mathbb{L}^{-n+1} \in \widehat{\mathcal{M}}_k^\heartsuit.\end{aligned}$$ □

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Corollary 7.5. *Let $P(n,l)$ denote the number of partitions of n into exactly l parts;*

$$P(n,l) := \{(q_1,\dots,q_l) \in (\mathbb{Z}_{>0})^l \mid q_1 \geq \cdots \geq q_l, \sum_{i=1}^{l} q_i = n\}.$$

Then, we have

$$\int_{\Delta_n} \mathbb{L}^{-\mathbf{d}} = \sum_{j=0}^{n-1} P(n,n-j)\mathbb{L}^{-j} \in \widehat{\mathcal{M}}_k^{\heartsuit}. \tag{7.2}$$

Proof. The natural P-morphism

$$\coprod_{l=1}^{n} \coprod_{(q_i)\in P(n,l)} \prod_{i=1}^{l} \Delta_{q_i}^{\circ} \to \Delta_n,\ (A_i)_{1\leq i\leq l} \mapsto \prod_{i=1}^{l} A_i$$

is geometrically bijective. Moreover, we have

$$\mathbf{d}(\prod_i A_i) = \sum_i \mathbf{d}(A_i).$$

Thus

$$\int_{\prod_{i=1}^{l} \Delta_{q_i}^{\circ}} \mathbb{L}^{-\mathbf{d}} = \prod_{i=1}^{l} \int_{\Delta_{q_i}^{\circ}} \mathbb{L}^{-\mathbf{d}} = \prod_{i=1}^{l} \mathbb{L}^{-q_i+1} = \mathbb{L}^{-n+l}$$

and

$$\begin{aligned}
\int_{\Delta_n} \mathbb{L}^{-\mathbf{d}} &= \sum_{l=1}^{n} \sum_{(q_i)\in P(n,l)} \int_{\prod_{i=1}^{l} \Delta_{q_i}^{\circ}} \mathbb{L}^{-\mathbf{d}} \\
&= \sum_{l=1}^{n} \sum_{(q_i)\in P(n,l)} \mathbb{L}^{-n+l} \\
&= \sum_{l=1}^{n} P(n,l)\mathbb{L}^{-n+l} \\
&= \sum_{j=0}^{n-1} P(n,n-j)\mathbb{L}^{-j}.
\end{aligned}$$

□

Theorem 7.6. *In what follows, we follow the convention that $0 \nmid n$ for every positive integer n.*

1. $\Delta_n^{(0)}$ *is P-isomorphic to* Spec k.
2. *We have $\Delta_n^{(m)} = \emptyset$ for $m > 0$ satisfying either of the following conditions:*
   (a) $p \nmid n,\ m \neq n-1,$
   (b) $p \mid n,\ p \nmid (m-n+1),\ m-n+1 < 0,$
   (c) $p \mid n,\ p \mid (m-n+1).$
3. *If $p \nmid n$ and $m = n-1$, then $\Delta_n^{(m)}$ is P-isomorphic to* Spec k.
4. *If $p \mid n$, $p \nmid (m-n+1)$ and $m-n+1 \geq 0$, then we have*

$$[\Delta_n^{(m)}] = (\mathbb{L}-1)\mathbb{L}^{\lfloor (m-n+1)/p \rfloor}$$

in $\mathcal{M}_k^{\heartsuit}[(\mathbb{L}-1)^{-1}]$ as well as in $\widehat{\mathcal{M}}_k^{\heartsuit}$.

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Proof. (1) For an algebraically closed field K, $\Delta_n^{(0)}(K)$ is a singleton corresponding to the trivial étale algebra $K(\!(t)\!)^n/K(\!(t)\!)$, which shows the assertion.

(2) If $A_y/K(\!(t)\!)$ corresponds to an Eisenstein polynomial $x^n + y_1 x^{n-1} + \cdots + y_n$, then from (4.1), we have

$$\mathbf{d}_{A_y} = n \min_{0 \le i \le n} \operatorname{ord}\left(\sum_{i=0}^{n} (n-i) y_i \varpi^{n-i-1}\right).$$

If $p \nmid n$, then the minimum on the right side is attained by $\operatorname{ord} n\varpi^{n-1} = (n-1)/n$ and $\mathbf{d}_{A_y} = n-1$, which shows (a). (The case (a) follows also from the proof of (3).) Suppose $p \mid n$ and $A/K(\!(t)\!)$ is a totally ramified extension of degree n and discriminant exponent m. From Proposition 4.4, for some i with $p \nmid i$ and $0 \le i \le n-1$, we have

$$m = n - i - 1 + n \operatorname{ord} y_i.$$

It follows that

$$m - n + 1 = n \operatorname{ord} y_i - i > 0$$

and

$$p \nmid (m - n + 1).$$

This shows (b) and (c).

(3) Let K be an algebraically closed field and let $A/K(\!(t)\!)$ be an extension of degree n. Let $\widetilde{A}$ be its Galois closure with Galois group G. We may identify G with a transitive subgroup of S_n. Moreover, G is isomorphic to the semidirect product $H \rtimes C$ of a p-group H and a tame cyclic subgroup C if $p > 0$ and isomorphic to a cyclic group if $p = 0$. We claim that if $p > 0$, then $H = 1$. To show this by contradiction, suppose that $H \neq 1$. Since $p \nmid n$ and H is a p-group, the H-action on $\{1, \ldots, n\}$ has at least one fixed point, say 1. Since G is transitive, there exists $g \in G$ such that $g(1)$ is not fixed by H. Then, $g^{-1}Hg \neq H$, which contradicts the fact that H is a normal subgroup of G. We have proved the claim. Then, $G = C$ is the cyclic subgroup of S_n generated by a cyclic permutation of an n-cycle. In particular, the stabilizer $\mathrm{Stab}(1)$ of $1 \in \{1, \ldots, n\}$ is trivial and $A = \widetilde{A}^G = \widetilde{A}$. We conclude that $A/K(\!(t)\!)$ is a cyclic Galois extension. As is well-known, A is isomorphic to $K(\!(t^{1/n})\!)$. Hence $\Delta_n^{(m-1)}(K)$ is a singleton, which shows the assertion.

(4) We put $c := m - n + 1$. From Lemma 7.3, we have

$$[\Delta_n^{(m)}] = [\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m]\mathbb{L}^{-nl+c+1} \in K_0^{\heartsuit}(\mathbf{Var}_k).$$

Since the $\mathbb{G}_m$-torsor $\mathfrak{Eis}_l^{(m)} \to \mathfrak{Eis}_l^{(m)}/\mathbb{G}_m$ is Zariski locally trivial thanks to Hilbert's Theorem 90, we have

$$[\mathfrak{Eis}_l^{(m)}/\mathbb{G}_m] = (\mathbb{L}-1)^{-1}[\mathfrak{Eis}_l^{(m)}]$$

in $\mathcal{M}_k^{\heartsuit}[(\mathbb{L}-1)^{-1}]$. From Corollary 4.6,

$$[\mathfrak{Eis}_l^{(m)}] = (\mathbb{L}-1)^2\mathbb{L}^{nl-c+\lfloor c/p \rfloor -1}.$$

Combining these equalities, we get

$$\begin{aligned}[\Delta_n^{(m)}] &= (\mathbb{L}-1)^{-1}\left((\mathbb{L}-1)^2\mathbb{L}^{(nl-c+\lfloor c/p\rfloor-1)}\right)\mathbb{L}^{-nl+c+1} \\ &= (\mathbb{L}-1)\mathbb{L}^{\lfloor c/p \rfloor}\end{aligned}$$

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in $\mathcal{M}_k^{\heartsuit}[(\mathbb{L}-1)^{-1}]$. Since $\mathbb{L}-1$ is invertible in $\widehat{\mathcal{M}}_k^{\heartsuit}$, we have a natural homomorphism $\mathcal{M}_k^{\heartsuit}[(\mathbb{L}-1)^{-1}] \to \widehat{\mathcal{M}}_k^{\heartsuit}$. Thus, the same equality holds also in $\widehat{\mathcal{M}}_k^{\heartsuit}$. □

The following corollary is a direct consequence of the last theorem:

Corollary 7.7. *We have*

$$\dim \Delta_n^{(m)} = \begin{cases} 0 & (p \nmid n,\ m = n-1), \\ -\infty & (p \nmid n,\ m \neq n-1), \\ \lceil (m-n+1)/p \rceil & (p \mid n,\ p \nmid (m-n+1),\ m-n+1 \geq 0), \\ -\infty & (p \mid n,\ p \nmid (m-n+1),\ m-n+1 < 0), \\ -\infty & (p \mid n,\ p \mid (m-n+1)). \end{cases}$$

Here we follow the convention that $\dim \emptyset = -\infty$ *and that if* $p = 0$*, then* $p \nmid n$*. Moreover, when* $\Delta_n^{(m)} \neq \emptyset$*, then it has only one irreducible component of the maximal dimension.*

Remark 7.8. The original mass formulas in [Kra62, Kra66, Ser78, Bha07] also hold for local fields of characteristic zero. It should be possible to similarly prove motivic mass formulas in characteristic zero, once the relevant P-moduli space is constructed.

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