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Local heights on abelian varieties with split multiplicative reduction

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Abstract. We express Néron functions and Schneider's local p-adic height pairing on an abelian variety A with split multiplicative reduction with theta functions and their automorphy factors on the rigid analytic torus uniformizing A. Moreover, we show formulas for the ρ-splittings of the Poincaré biextension corresponding to Néron's and Schneider's local height pairings.

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Introduction

The object of this paper is to 'calculate' Néron's local height pairing and Schneider's local p-adic height pairing on abelian varieties with split multiplicative reduction. Such an abelian variety admits of a rigid analytic uniformization as T/Λ, where T is a split rigid analytic torus and Λ is a lattice in T. The pullback of a divisor on A is the divisor of a theta function on T. These theta functions and their automorphy factors will be the building blocks of our formulas. Our results generalize well-known results on Tate curves to the higher dimensional case.

Concerning the Néron pairing, we actually prove a more general statement, namely a formula for the Néron map on an abelian variety with split multiplicative reduction which is similar to a formula due to Néron for an abelian variety over $\mathbb{C}$. Here we use the terminus Néron map for the association of a canonical local height function with any given divisor. These canonical local height functions were constructed by Néron in order to find quadratic global height functions on abelian varieties.

Afterwards we restrict our attention to height pairings, which means that we consider only divisors algebraically equivalent to zero. We adopt an approach to height pairings which is due to Mazur and Tate (see [Ma–Ta]). Starting with an abelian variety over a local field K and any homomorphism ρ from $K^\times$ to some abelian group Y, they define a local height pairing on A with values in Y whenever ρ can be continued to a 'bihomomorphic' map, a so-called ρ-splitting, $\sigma\colon P(K) \to Y$ on the K-rational points of the Poincaré biextension associated with A. In some

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cases there exist canonical ρ-splittings, e.g. when K is archimedean and ρ vanishes on elements of absolute value one, or when K is non-archimedean, ρ is unramified (i.e. vanishes on the group of units), and the target group Y satisfies certain divisibility conditions. These two cases suffice to describe Néron's local height pairings via canonical ρ-splittings. There is also an axiomatic characterization of the ρ-splittings leading to Schneider's local p-adic heights (as defined in [Sch]).

For an abelian variety with split multiplicative reduction we define a new ρ-splitting by a certain expression on a trivial biextension covering $P(K)$. This definition is restricted to the case where ρ is what we call Λ-invertible, i.e. where ρ maps the lattice Λ to a lattice of full rank in Y^n. If ρ is unramified, we show that ρ is Λ-invertible if and (adding an additional assumption) only if the Mazur–Tate condition for the existence of a canonical ρ-splitting is fulfilled, and that our ρ-splitting coincides with the canonical one. Furthermore, if ρ is the map corresponding to Schneider's p-adic height, then ρ is Λ-invertible if and only if Schneider's conditions are fulfilled, and in this case our ρ-splitting is the one giving rise to the p-adic height.

Moreover, we prove a formula for the height pairing defined by our ρ-splitting, using theta functions on the covering torus T and their automorphy factors, which yields the desired formulas for Schneider's and Néron's pairings.

After this paper was completed, we learnt that a generalization of our formula for the Néron map to arbitrary abelian varieties with semistable reduction was proved independently (and earlier) by M. Hindry in the unpublished preprint [Hi]. Hindry is interested in finding good representatives for the local Néron height functions associated with a divisor which are a priori only defined up to a constant, whereas our main interest lies in computing height pairings in the Mazur and Tate style and investigating their existence conditions. With this focus we only get information on divisors algebraically equivalent to zero, but we can investigate height pairings with value groups other than the real numbers. So the overlap between this paper and [Hi] concerns only our Section 3. We nevertheless decided to include this section to complete the picture.

1. Local heights

Let us first recall some facts about local height functions and height pairings. We fix an abelian variety A over a local field K, whose absolute value we will always normalize according to the product formula, i.e. if $K = \mathbb{R}$, we take the usual absolute value, if $K = \mathbb{C}$, we take the square of the usual absolute value, and if K is non-archimedean, we denote the residue class field by k and put $|x|_K = (\#k)^{-v(x)}$, where v is the valuation map (normalized so that a prime element has valuation one). By A' we denote the dual abelian variety of A. If K is non-archimedean, we denote by R its ring of integral elements, and by $\mathcal{A}$ (respectively $\mathcal{A}'$) the Néron model of A (respectively of A') over Spec R. We write Div(A) for the group of (Weil or Cartier) divisors on A, and for any nonzero f in the function field $K(A)$

of A we denote the corresponding principal divisor by $\mathrm{div}(f)$. For any Zariski open subset U of A, let $\mathcal{Q}(U)$ be the group of continuous functions $U(K) \to \mathbb{R}$, and let $\mathcal{C}(U)$ be the subgroup of constant functions on $U(K)$.

THEOREM 1.1 (Néron). *There is a unique way of associating to any $D \in \mathrm{Div}(A)$ an element*

$$\mu_{K,D} = \mu_D \in \mathcal{Q}(A \backslash \mathrm{supp}\, D)/\mathcal{C}(A \backslash \mathrm{supp}\, D)$$

with the following properties:

(i) *All representatives of μ_D have divisor D, i.e. whenever D restricted to some Zariski open subset U of A is equal to the divisor of a rational function f (restricted to U), then for every representative $\mu_D^\star$ of μ_D there exists a continuous map $\alpha\colon U(K) \to \mathbb{R}$ such that $\mu_D^\star(x) = \log|f(x)|_K + \alpha(x)$ on $(A\backslash \mathrm{supp}\, D)(K) \cap U(K)$.*
(ii) *$\mu_{D+D'} = \mu_D + \mu_{D'}$, where both sides are defined.*
(iii) *For all $f \in K(A)^\times$, we have $\mu_{\mathrm{div}\, f} \equiv \log|f|_K$ modulo $\mathcal{C}(A\backslash \mathrm{supp}\, \mathrm{div}(f))$.*
(iv) *For all $a \in A(K)$ we have $\mu_{t_a^* D} = \mu_D \circ t_a$.*

We call $D \mapsto \mu_D$ the Neron map.

Proof. Existence follows from [Né1], Theorem 1, p. 278, and [Né1], Proposition 5, p. 292. Uniqueness follows from [Né1], Lemme 7, p. 279. □

Note that the Néron map is compatible with finite base changes: For any finite extension L over K the restriction of $[L\colon K]^{-1}\mu_{L,D_{A_L}}$ to $A(K)$ is equal to $\mu_{K,D}$.

Let $Z^0(A/K)$ denote the group of all zero cycles on A with degree zero and K-rational support (i.e. the elements of degree zero in the free abelian group on $A(K)$). For any $z = \Sigma_i n_i a_i \in Z^0(A/K)$ we put $\mu_D(z) = \Sigma_i n_i \mu_D^\star(a_i)$ for an arbitrary representative $\mu_D^\star$ of μ_D. Then $\mu_D(z)$ is a well-defined real number if the support of z is disjoint from the support of D.

By $(\mathrm{Div}\ A \times Z^0(A/K))'$ we denote the set of all pairs (D, z) with disjoint supports. Then we can define a pairing $(\mathrm{Div}\ A \times Z^0(A/K))' \to \mathbb{R}$ by $(D, z) \mapsto \mu_D(z)$. We will denote the restriction of this pairing to $(\mathrm{Div}^0 A \times Z^0(A/K))'$, where $\mathrm{Div}^0\ A$ denotes the group of divisors on A algebraically equivalent to 0, by $(\ ,\ )_{N,A/K}$. An axiomatic characterization of $(\ ,\ )_{N,A/K}$ is given in [Né1], p. 294.

In [Bl], Bloch gave a description of Néron's local height pairing using a continuation of the map $\log|\ |_K$ to extensions of A by G_m. This approach was modified by Schneider in [Sch] to define an (analytic) p-adic height pairing. The conceptual background of Bloch's and Schneider's constructions becomes fully transparent in the paper [Ma–Ta] by Mazur and Tate. Let us briefly recall some of their results. Denote by P the G_m-torsor on $A \times A'$ corresponding to the Poincaré bundle expressing the duality between A and A'. P can be endowed with the structure of a biextension of A and A' by G_m (with respect to the fppf, étale or Zariski topology), see [SGA7, I, exp. VIII], p. 225. Note that $P(K)$ is a biextension of $A(K)$ and $A'(K)$ by $K^\times$ in the category of sets.

DEFINITION 1.2 ([Ma–Ta], p. 199). Let U, V, W and Y be abelian groups, let X be a biextension of U and V by W, and let ρ: $W \to Y$ be a homomorphism. A ρ-splitting of X is a map σ: $X \to Y$ such that

(i) $\sigma(wx) = \rho(w) + \sigma(x)$ for all $w \in W$ and $x \in X$.
(ii) For all $u \in U$ (respectively $v \in V$) the restriction of σ to $X \times_{U\times V} (\{u\} \times V)$ (respectively $X \times_{U\times V} (U \times \{v\})$) is a group homomorphism.

Now let ρ: $K^\times \to Y$ be a homomorphism to some abelian group Y and assume that $\sigma : P(K) \to Y$ is a ρ-splitting. Then we can define a bilinear pairing

$$(\, , \,)_{MT,\sigma}\colon \left(\mathrm{Div}^0 A \times Z^0 (A/K)\right)' \longrightarrow Y,$$

$$(D, z) \longmapsto \sigma(s_D(z)),$$

where s_D is a rational section of $P|_{A\times\{d\}} \to A$ with divisor D, and where d is the point in $A'(K)$ corresponding to D. The rational section s_D is defined only up to a constant in $K^\times$ which vanishes when we continue s_D linearly to $Z^0(A/K)$. We have $(\mathrm{div}(f), z)_{MT,\sigma} = \rho(f(z))$ and $(t_a^* D, t_a^* z)_{MT,\sigma} = (D, z)_{MT,\sigma}$ for all $a \in A(K)$ (see [Ma–Ta], 2.2, p. 212). In the following three cases, Mazur and Tate prove the existence of a canonical ρ-splitting:

(I) Let K be archimedean, i.e. $\mathbb{R}$ or $\mathbb{C}$, and assume that $\rho(c) = 0$ for all $c \in K^\times$ with $|c|_K = 1$. Put $v(c) := \log |c|_K$. Since $\rho(c)$ depends only on the value $v(c)$, there is a unique homomorphism r: $\mathbb{R} \to Y$ such that $r \circ v = \rho$. There is a unique continuous v-splitting σ_v of $P(K)$ (see [Ma–Ta], 1.8.1, p. 201), and we define the canonical ρ-splitting σ_ρ of $P(K)$ to be $\sigma_\rho = r \circ \sigma_v$. ([Ma–Ta], 1.5.1, p. 202.)

(II) Assume that K is non-archimedean, i.e. the absolute value is discrete, and that ρ vanishes on $R^\times$. If this is the case, we call ρ unramified. Assume that Y is uniquely divisible by m_A, the exponent of the group $\mathcal{A}_k(k)/\mathcal{A}_k^0(k)$, k being the residue class field of R. There exists a unique biextension P_R of $\mathcal{A}^0$ and $\mathcal{A}'$ by $G_{m,R}$ (over R and with respect to the fppf-topology) with generic fibre P, see [SGA7, I exp. VIII], 7.1 b), p. 300. The canonical ρ-splitting σ_ρ is defined as the unique ρ-splitting vanishing on $P_R(R) \subset P(K)$. ([Ma–Ta, 1.5.2, p. 202.)

(III) Assume that K is non-archimedean, $\mathcal{A}$ has ordinary reduction and Y is uniquely divisible by $m_A m_{A'} n_A n_{A'}$, where n_A is the exponent of $\mathcal{A}_k^0(k)/T_A(k)$ for the maximal torus T_A in $\mathcal{A}_k$. The formal completion P^t of P_R along the inverse image of $T_A \times T_{A'}$ in P_R is a formal biextension of the formal completion of $\mathcal{A}$ along T_A and the formal completion of $\mathcal{A}'$ along $T_{A'}$ by $G_m^\wedge$, the formal completion of $G_{m,R}$ along its special fibre. P^t is trivial and admits a unique split σ_0: $P^t \to G_m^\wedge$.

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Then the canonical ρ-splitting σ_ρ is defined as the unique ρ-splitting of $P(K)$ such that $\sigma_\rho|_{P^t(R)} = \rho \circ \sigma_0$. ([Ma–Ta], 1.5.3, p. 203.)

The canonical ρ-splitting is in all three cases compatible with base change with respect to continuous embeddings of local fields, see [Ma–Ta], 1.10.2, p. 205.

Cases (I) and (II) are sufficient to get a description of the local Néron pairings. Put $Y = \mathbb{R}$ and let $\rho\colon K^\times \to \mathbb{R}$ be the map $\rho(x) = \log|x|_K$. Then ρ vanishes on elements with absolute value 1. As the divisibility condition in (II) is fulfilled in $\mathbb{R}$, we get a canonical ρ-splitting $\sigma_\rho\colon P(K) \to \mathbb{R}$, applying case (I), if K is archimedean, and case (II), if K is non-archimedean. According to [Ma–Ta], 2.3.1, p. 212 we have $(D, z)_{MT,\sigma_\rho} = (D, z)_{N,A/K}$.

The connection to Schneider's p-adic height pairing is the following: Let K be a finite extension of $\mathbb{Q}_l$, and let $\rho\colon K^\times \to \mathbb{Q}_p$ be a non-trivial continuous homomorphism. Then ρ is continuous for the profinite topology on $K^\times$ and extends therefore uniquely to a homomorphism $\rho^\wedge$ on the profinite completion $K^{\times\wedge}$ of $K^\times$. By local class field theory, $K^{\times\wedge}$ is topologically isomorphic to $\mathrm{Gal}(K^{ab}/K)$. Then $\rho^\wedge$ determines a $\mathbb{Z}_p$-extension K_∞/K with intermediate fields K_ν which are the uniquely determined cyclic extensions of degree p^ν of K such that $\rho(N_{K_\nu/K}K_\nu^\times) = p^\nu\rho(K^\times) \subset \mathbb{Q}_p$, see [Ma–Ta], 1.11.1, p. 207. For any commutative group G over K we denote by $NG(K) \subset G(K)$ the group of universal norms with respect to K_∞/K. Furthermore, let $P(K_\nu, K)$ be the set of points in $P(K_\nu)$ which project to $A(K_\nu) \times A'(K)$. We define $NP(K) \subset P(K)$ as the intersection of all $N_{K_\nu/K}P(K_\nu, K)$, where we use the group structure of P over A' to define norms. If $NA(K)$ has finite index in $A(K)$, then $NP(K)$ carries the structure of a biextension of $NA(K)$ and $A'(K)$ by $NG_m(K)$, see [Ma–Ta], 1.11.4, p. 208.

THEOREM 1.3. *If $l = p$, assume that $NA(K)$ has finite index in $A(K)$. Then there exists a unique ρ-splitting σ_ρ: $P(K) \to \mathbb{Q}_p$ vanishing on*

$$\begin{cases} NP(K) & \text{if } l = p \\ P_R(R) & \text{if } l \neq p. \end{cases}$$

We call $(\ ,\ )_{MT,\sigma_\rho}$ Schneider's local p-adic height pairing with respect to ρ.

Proof. See [Ma–Ta], 1.11.5, p. 208 for the case $l = p$. The case $l \neq p$ follows from the existence of the canonical splitting in case (II), since the homomorphism ρ: $K^\times \to \mathbb{Q}_p$ is unramified (see the beginning of the proof of 4.13). □

If ρ is unramified (so e.g. in the case $l \neq p$), the universal norm group $NA(K)$ always has finite index in $A(K)$, and the canonical ρ-splitting vanishing on $P_R(R)$ can also be described as the unique ρ-splitting vanishing on $NP(K)$, see [Ma–Ta], 1.11.6, p. 208.

Note that if L is a finite extension of K, then ρ extends to a non-trivial continuous homomorphism $\rho_L\colon L^\times \to \mathbb{Q}_p$, and we get $\sigma_\rho = \sigma_{\rho_L}|_{P(K)}$. Hence the ρ-splittings

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leading to Schneider's p-adic height pairings can be calculated after finite base changes.

2. Abelian varieties with split multiplicative reduction

Let K be a non-archimedean local field. We will denote by Rig_K the site of rigid analytic varieties over K endowed with the strong Grothendieck topology, see [BGR], p. 357. Adopting the terminology from [BGR], we call coverings with respect to this site admissible. By Zar_K we denote the big Zariski site over Spec$K$, i.e. the category of all schemes locally of finite type over Spec K endowed with the Zariski topology. The rigid analytic GAGA-functor, as explained in [Kö], Section 1 and [BGR], p. 361f, induces a morphism of sites which we denote by

$$an\colon \mathrm{Rig}_K \to \mathrm{Zar}_K.$$

For an object X and a morphism f in the Zariski category we simply write X^{an} and f^{an} for the corresponding analytic objects. There is a natural map

$$\alpha_X\colon X^{an} \longrightarrow X,$$

of locally G-ringed spaces. (See [Kö], Section 1, and for the definition of locally G-ringed spaces see [BGR], 9.3.1, p. 353). The rigid analytic analogues of Serre's GAGA theorems hold, see [Kö]. We will call a group object in Rig_K a (rigid) analytic group. A split rigid analytic torus T over K is a rigid analytic group over K such that $T \simeq (G^n_{m,K})^{an}$ for some natural number n. Its character group is the analytic Cartier dual of T, hence a constant analytic group defined by a free abelian group H of rank n contained in $\Gamma(T, \mathcal{O}^\times)$. Any choice of a basis $\chi_1, \ldots, \chi_n$ of the free $\mathbb{Z}$-module H yields an isomorphism $T \simeq (G^n_{m,K})^{an}$ and can be used to define a map

$$\begin{aligned} \nu\colon\ T(K) &\longrightarrow \mathbb{R}^n, \\ z &\longmapsto (\log|\chi_1(z)|, \ldots, \log|\chi_n(z)|). \end{aligned}$$

We call a closed analytic subgroup $\Lambda \subset T$ a split lattice if it is a constant analytic K-group whose K-rational points are mapped bijectively via ν (for some choice of $\chi_1, \ldots \chi_n$) to a lattice of full rank in $\mathbb{R}^n$, see [Bo–Lü3], p. 656.

We say that an abelian variety A over K has split multiplicative reduction if the special fibre of the identity component of the Néron model of A is a split torus. Then there exists a split analytic torus T over K and a split lattice $\Lambda \subset T$ such that we have an isomorphism $T/\Lambda \xrightarrow{\sim} A^{an}$ in the rigid analytic category. (See [Ray] or [Bo–Lü3], p. 655f.) The rigid analytic structure on the quotient T/Λ is defined so that $T \to T/\Lambda$ is locally bianalytic, see [Ge], p. 324, or Bo–Lü3], p. 661. Via the projection $T \to A^{an}$, A^{an} is the categorical quotient of T after the Λ-operation (in the sense of [Mu1], p. 3). On an abelian variety A with split multiplicative

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reduction we have a theory of theta functions analogous to the situation over $\mathbb{C}$. We have

$$\Gamma(T, \mathcal{O}^\times) = K^\times H = \{a\chi\colon a \in K^\times, \chi \in H\},$$

where $\mathcal{O}$ denotes the rigid analytic structure sheaf. (See [Ma], p. 289.) Note that the analogous group in the complex analytic setting is more complicated. That makes the definition of theta functions even easier in the rigid analytic case. Let $\mathcal{M}$ be the sheaf of meromorphic functions on T (in the sense of [Bo], p. 6).

DEFINITION 2.1 ([Ma], p. 290). A function $\Theta \in \Gamma(T, \mathcal{M}^\times)$ with Λ-invariant Cartier divisor div Θ is called a theta function.

Let Θ be a theta function. Then we have $\mathrm{div}(\Theta(\lambda x)) = \mathrm{div}(\Theta(x))$, hence $\Theta(\lambda x)$ and $\Theta(x)$ differ by an element in $\Gamma(T, \mathcal{O}^\times)$. So for all $\lambda \in \Lambda$ there exists a constant $a_\lambda \in K^\times$ and a character $\chi_\lambda \in H$ such that

$$\Theta(\lambda x) = a_\lambda \chi_\lambda^{-1}(x)\Theta(x) \quad \text{for all} \quad x \in T.$$

a_λ and χ_λ are uniquely determined, and it is easy to see that they have the following properties:

- $\lambda \mapsto \chi_\lambda$ is a homomorphism
- $\chi_{\lambda_1}(\lambda_2) = \chi_{\lambda_2}(\lambda_1)$
- $a_{\lambda_1} a_{\lambda_2} = a_{\lambda_1\lambda_2}\chi_{\lambda_1}(\lambda_2)$

DEFINITION 2.2. We say that K contains enough roots if for every $\chi \in H$ and for every $\lambda \in \Lambda$ there exists an element $\omega(\chi, \lambda) \in K^\times$ such that $\omega(\chi, \lambda)^2 = \chi(\lambda)$.

Given Λ and H, there is always a finite extension L of K which contains enough roots. We can define L by choosing bases $\lambda_1, \ldots, \lambda_n$ of Λ and $\chi_1, \ldots, \chi_n$ of H and adjoining a fixed square root of all $\chi_i(\lambda_j)$ to K.

Note. In this and in the next section we will assume that our ground field K contains enough roots.

Let Θ, a_λ and χ_λ be as above. Since K contains enough roots, we can define a bimultiplicative and symmetric map $[\,,\,]_\Theta\colon \Lambda \times \Lambda \to K^\times$ such that $[\lambda, \mu]_\Theta^2 = \chi_\lambda(\mu)$ for all $\lambda, \mu \in \Lambda$. Furthermore, as χ_λ is a character, $[\,,\,]_\Theta^2$ has an extension to a bimultiplicative map $[\,,\,]_\Theta^2\colon \Lambda \times T(K) \longrightarrow K^\times$. We define $\psi_\Theta\colon \Lambda \to K^\times$ by $\psi_\Theta(\lambda) := a_\lambda[\lambda, \lambda]_\Theta$. An easy calculation shows that ψ_Θ is a homomorphism. Then the automorphy factor of Θ has the following shape

$$\Theta(\lambda x) = \psi_\Theta(\lambda)[\lambda, \lambda]_\Theta^{-1}[\lambda, x]_\Theta^{-2}\Theta(x).$$

(The same arrangement of the automorphy factor is used in [Ma], p. 290). $[\,,\,]_\Theta$ is uniquely determined up to a bimultiplicative and symmetric map from $\Lambda \times \Lambda$ to $\{\pm 1\}$. Hence the absolute values of $[\,,\,]_\Theta$ and ψ_Θ are independent of the choice of $[\,,\,]_\Theta$.

The following result is crucial for our investigations.

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THEOREM 2.3 (Gerritzen). $H^1(T, \mathcal{O}^\times) = 1$.

Proof. See [Ge], Theorem 1, p. 326. □

This implies that every analytic Cartier divisor on T is a principal divisor. Let us denote by $\pi: T \to A^{an}$ the uniformization map and consider an algebraic divisor D on A. D induces an analytic Cartier divisor $D^{an} = \alpha_A^* D$ on A^{an}. Then $\pi^* D^{an}$ is principal, i.e. there is a function $\Theta \in \Gamma(T, \mathcal{M}^\times)$ such that $\pi^*(D^{an}) = \operatorname{div} \Theta$. Our notation is justified, since Θ is indeed a theta function in the sense of Definition 2.1. We call Θ a theta function corresponding to D. Two theta functions corresponding to the same divisor D differ by an element in $\Gamma(T, \mathcal{O}^\times) = K^\times H$. Note that for $a \in K^\times$ and $\chi \in H$ we have $[\,,\,]^2_{a\chi\Theta} = [\,,\,]^2_\Theta$. Hence we may put $[\,,\,]^2_D = [\,,\,]^2_\Theta$ for any theta function corresponding to D.

PROPOSITION 2.4. *A divisor D on A is algebraically equivalent to zero if and only if* $[\,,\,]^2_D = 1$.

Proof. Let Θ be a theta function corresponding to D. Then $\Xi(x,y) := \Theta(xy)\Theta(x)^{-1}\Theta(y)^{-1} \in \Gamma(T \times T, \mathcal{M}^\times)$ is a theta function corresponding to the divisor $m^*D - p_1^*D - p_2^*D$ on $A \times A$ (where m is multiplication and p_1, p_2 are projections). Hence, if D is algebraically equivalent to zero, we get $\Xi = a\chi \cdot h \circ (\pi \times \pi)$, where $a\chi \in \Gamma(T \times T, \mathcal{O}^\times)$ is the product of a constant $a \in K^\times$ and a character χ on $T \times T$, and where h is a rational function on $A \times A$ with divisor $m^*D - p_1^*D - p_2^*D$. In this case, we deduce for all $\lambda \in \Lambda$

$$\frac{\Theta(x^2)}{\Theta(x)\Theta(x)}\chi^{-1}(x,x) = \frac{\Theta(\lambda x^2)}{\Theta(\lambda x)\Theta(x)}\chi^{-1}(\lambda x, x),$$

which implies $\chi(\lambda, 1) = [\lambda, x]_D^{-2}$. Hence for all λ the character $[\lambda, -]_D^{-2}$ is constant, hence equal to 1, which gives indeed $[\,,\,]^2_D = 1$.

On the other hand, suppose that $[\,,\,]^2_D = 1$. This implies that $\Xi(x,y)$ is $\Lambda \times \Lambda$-invariant, hence $\Xi = (\pi \times \pi)^* h$ for some global meromorphic function h on $A^{an} \times A^{an}$. Since $A \times A$ is projective, h is actually a rational function (see e.g. [Bo], p. 12), which implies that D is algebraically equivalent to zero. □

For any divisor $D \in \operatorname{Div}^0(A)$ and any theta function Θ corresponding to D we choose $[\,,\,]_\Theta = 1$, i.e. $\psi_\Theta: \Lambda \to K^\times$ is the well-defined homomorphism satisfying $\Theta(\lambda x) = \psi_\Theta(\lambda)\Theta(x)$.

We define a homomorphism

$$\operatorname{Div}(A) \longrightarrow H^1(\Lambda, \Gamma(T, \mathcal{O}^\times)),$$

(where we take group cohomology with respect to the natural action of $\Lambda \subset T(K)$ on $\Gamma(T, \mathcal{O}^\times)$) by $D \mapsto \Theta_D(\lambda x)/\Theta_D(x)$. It is easy to see that this morphism is well-defined (i.e. independent of the choice of Θ_D), and that its kernel consists exactly of the principal divisors. Hence we get an injection

$$\phi\colon \mathrm{CH}^1(A) \hookrightarrow H^1(\Lambda, \Gamma(T, \mathcal{O}^\times)).$$

In accordance with [Mu1], 1.6, p. 30, we call a collection of morphisms $u_\lambda\colon G_m^{an} \times T \xrightarrow{\sim} G_m^{an} \times T$ $(\lambda \in \Lambda)$ such that $u_{\lambda_1\lambda_2} = u_{\lambda_1} \circ u_{\lambda_2}$ and such that for all $\lambda \in \Lambda$ there exists an element $e_\lambda \in \Gamma(T, \mathcal{O}^\times)$ with $u_\lambda(\alpha, x) = (\alpha e_\lambda(x), \lambda x)$, a Λ-linearization on the trivial G_m^{an}-torsor $G_m^{an} \times T$ over T. For every G_m-torsor L on A there is a trivialization $\pi^* L^{an} \xrightarrow{\sim} G_m^{an} \times T$, hence the natural Λ-operation on $\pi^* L^{an}$ (given by multiplication by λ on T) induces a Λ-linearization $\{u_\lambda\}$ on $G_m^{an} \times T$. It is clear that a different trivialization of $\pi^* L^{an}$ gives rise to a Λ-linearization v_λ such that $u_\lambda = h^{-1} \circ v_\lambda \circ h$ for some fixed G_m^{an}-torsor isomorphism $h\colon G_m^{an} \times T \xrightarrow{\sim} G_m^{an} \times T$. If this is the case, we call u_λ and v_λ congruent. Let us denote the set of congruence classes of Λ-linearizations by LinCl. Then we get a map

$$\psi_1\colon \mathrm{CH}^1(A) \longrightarrow \mathrm{LinCl}.$$

As in the complex case (see [Mu2], Chapter I, Section 2) it is possible to define an inverse map by constructing the quotient of $G_m^{an} \times T$ after a Λ-linearization. Hence ψ_1 is a bijection. (See also [Bo–Lü3], Lemma 2.2, p. 662.) Furthermore, if we choose an isomorphism $i\colon G_m^{an} \times T \xrightarrow{\sim} \pi^* L^{an}$, then $(L^{an}, G_m^{an} \times T \xrightarrow{i} \pi^* L^{an} \xrightarrow{\pi} L^{an})$ is the categorical quotient (in Rig_K) of $G_m^{an} \times T$ by the operation of Λ given by the Λ-linearization u_λ derived from i. (For the definition of categorical quotients see [Mu1], p. 3)

On the other hand, we have a natural bijection $\psi_2\colon H^1(\Lambda, \Gamma(T, \mathcal{O}^\times)) \longrightarrow \mathrm{LinCl}$ induced by mapping a cocycle $\{e_\lambda\}$ to the set of maps u_λ defined by $u_\lambda(\alpha, x) = (\alpha e_\lambda(x), \lambda x)$. Following all our constructions, it turns out that $\psi_2 \circ \phi\colon \mathrm{CH}^1(A) \longrightarrow \mathrm{LinCl}$ coincides with the bijection ψ_1 defined above. In particular, we deduce that ϕ is an isomorphism.

3. The Néron map

If we assume for a moment that we are dealing with an abelian variety A over the complex numbers, a result of Néron tells us how to calculate the Néron map on A. Let $\pi\colon V \to A(\mathbb{C})$ be the complex uniformization of $A(\mathbb{C})$, so that $A(\mathbb{C}) = V/\Lambda$, where V is a complex vector space and Λ is a lattice in V. For every divisor D on A there is a normalized theta function Θ_D on V with divisor $\pi^* D$, i.e. Θ_D is a meromorphic function on V such that for all $z \in V$ and for all $\lambda \in \Lambda$

$$\Theta(z+\lambda) = \Theta(z)\ \exp\left[\pi H\left(z + \frac{\lambda}{2}, \lambda\right) + 2\pi i K(\lambda)\right],$$

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where H is a hermitian form on $V \times V$ and $K(\lambda)$ is real for all $\lambda \in \Lambda$. (For a proof see [La1] or [Ro].) Put

$$\mu_D^{\star}(x) = \log|\Theta_D(z)| - \frac{\pi}{2}H(z,z),$$

where $z \in V$ is an arbitrary preimage of $x \in A(\mathbb{C})$. Then $\mu\colon D \mapsto \mu_D^{\star} \bmod \mathcal{C}$ is the Néron map on A. (See [Né1], p. 329.)

We will now derive a similar formula in the non-archimedean case. So let us return to the case that A is an abelian variety with split multiplicative reduction over a non-archimedean local ground field K. Then A^{an} is uniformized as T/Λ, and we denote by $\pi\colon T \to A^{an}$ the projection map. We fix a basis $\chi_1, \ldots, \chi_n$ for the character group H of T and a basis $\lambda_1, \ldots, \lambda_n$ for the lattice $\Lambda \subset T(K)$. By v we denote the valuation map on $K^{\times}$. Note that $(v(\chi_j\lambda_i)_{i,j})$ is a matrix of full rank over $\mathbb{Z}$, since Λ is a split lattice. Fix a natural number N satisfying $N \det(v(\chi_j\lambda_i)_{i,j})^{-1} \in \mathbb{Z}$. In analogy to the complex situation we define

DEFINITION 3.1. We call a theta function Θ on T normalized, if $|\psi_\Theta(\lambda)| = 1$ for all $\lambda \in \Lambda$.

Then we have the following

PROPOSITION 3.2. *For any divisor D on A there is a normalized theta function Θ' corresponding to the divisor ND, which is uniquely determined up to a constant $a \in K^{\times}$.*

Proof. Consider a divisor D on A and a theta function Θ corresponding to D. We have to find a character $\chi \in H$ such that $\Theta' = \chi\Theta^N$ is a normalized theta function. Now, according to our definition, $\chi\Theta^N$ is normalized if and only if for all $\lambda \in \Lambda$ we have $1 = |\psi_{\chi\Theta^N}(\lambda)| = |\chi(\lambda)||\psi_\Theta(\lambda)|^N$. For brevity, put $\psi := \psi_\Theta$. To construct χ, we solve the linear system of equations $v(\psi(\lambda_i)) = \Sigma_{k=1}^n a_k v(\chi_k\lambda_i)$ for all $i = 1, \ldots, n$ with uniquely determined $a_k \in \mathbb{Q}$. Then $a_k \in \det(v(\chi_j\lambda_i)_{i,j})^{-1}\mathbb{Z}$ for all k, hence $Na_k \in \mathbb{Z}$. Define $\chi := \chi_1^{-Na_1} \cdots \chi_n^{-Na_n} \in H$. An easy calculation now shows that $|\chi(\lambda_i)|^{-1} = |\psi(\lambda_i)|^N$ for all elements of our chosen basis. This implies that $|\chi(\lambda)|^{-1} = |\psi(\lambda)|^N$ for all $\lambda \in \Lambda$, as both sides are multiplicative in λ. Hence $\chi\Theta^N$ is normalized. The second part of our assertion is obvious, since two theta functions corresponding to the same divisor differ by an element of $K^{\times}H$. Note that any character mapping Λ to $R^{\times}$ is trivial. □

DEFINITION 3.3. (i) For any $z \in T(K)$ we denote by $w_j(z) \in \mathbb{Q}$, $j \in \{1, \ldots, n\}$, the uniquely determined solution of the following system of linear equations

$$v(\chi_i(z)) = \sum_{j=1}^{n} w_j(z) v(\chi_i(\lambda_j)) \quad \text{for } i = 1, \ldots, n.$$

Note that $Nw_j(z)$ is an integer.

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(ii) For every divisor D on A put for $y, z \in T(K)$

$$H_D(y, z) = -\sum_{i,j}^{n} w_i(y) w_j(z) \log |[\lambda_i, \lambda_j]_D|.$$

Note that H_D is a bilinear, symmetric map $T(K) \times T(K) \longrightarrow \mathbb{R}$, which is obviously continuous in both arguments. Some other properties of H_D which we need in our calculation of the Néron map are established in the following lemma.

LEMMA 3.4.

(i) $H_{D_1+D_2} = H_{D_1} + H_{D_2}$.
(ii) *Let* χ *be a character on* T. *Then* $\log |\chi(z)| = \Sigma_j w_j(z) \log |\chi(\lambda_j)|$.
(iii) *For all* $y, z \in T(K)$ *we have* $H_D(y, z) = -\frac{1}{2} \Sigma_i w_i(y) \log |[\lambda_i, z]_D^2|$.
(iv) *For* $\lambda \in \Lambda$ *and* $z \in T(K)$ *we have* $H_D(\lambda, z) = -\frac{1}{2} \log |[\lambda, z]_D^2|$, *which gives for* $z = \mu \in \Lambda$ *the equation* $H_D(\lambda, \mu) = -\log |[\lambda, \mu]_D|$.

Proof. (i) is obvious from the definitions.

(ii) By definition, $v(\chi_i(z)) = \Sigma_j w_j(z) v(\chi_i \lambda_j)$ for all i. Hence we get for all i: $\log |\chi_i(z)| = \Sigma_j w_j(z) \log |\chi_i(\lambda_j)|$. By additivity, this holds for all $\chi \in H$.

(iii) From (ii), we get $\log |[\lambda, z]_D^2| = \Sigma_j w_j(z) \log |[\lambda, \lambda_j]_D^2|$, which implies our claim.

(iv) By (iii), we get for all i: $H_D(\lambda_i, z) = -\frac{1}{2} \Sigma_j w_j(\lambda_i) \log |[\lambda_j, z]_D^2| = -\frac{1}{2} \log |[\lambda_i, z]_D^2|$, since $w_j(\lambda_i) = \delta_{ij}$. As the λ_i are a basis of Λ, this formula holds for all $\lambda \in \Lambda$. $\square$

We can now prove a formula for the Néron map on A. The following theorem can be deduced from a result due to M. Hindry (Théorème D in [Hi]) who independently proved a formula for the Néron map on an abelian variety with semistable reduction via its Raynaud uniformization. Nevertheless, since our arguments are in a different spirit, it might be useful for the reader to give a proof here.

THEOREM 3.5. *For any divisor* D *on* A *let* Θ_{ND} *be a normalized theta function corresponding to the divisor* ND. *We define for all* $x \in (A \backslash \operatorname{supp} D)(K)$

$$\mu_{\Theta_{ND}}(x) := \frac{1}{N} \log |\Theta_{ND}(z)| - \frac{1}{N} H_{ND}(z, z),$$

where $z \in T(K)$ *is an arbitrary point with* $\pi(z) = x$. *Then*

$$\mu \colon D \longmapsto (\mu_D = \mu_{\Theta_{ND}} \bmod \mathcal{C}(A \backslash \operatorname{supp} D)),$$

is the Néron map for A, *as defined in* 1.1.

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Proof. First of all, note that a different choice of Θ_{ND} amounts to adding a constant, so that μ_D depends solely on D. Next, we will show that $\mu_{ND}(x)$ is well-defined, i.e. independent of the choice of a preimage z of x. Indeed, for $\lambda \in \Lambda$ and $z \in \pi^{-1}(\{x\})$ we have

$$\begin{aligned}\log|\Theta_{ND}(\lambda z)| - H_{ND}(\lambda z, \lambda z) \; &= \; \log|[\lambda,\lambda]_{ND}^{-1}[\lambda,z]_{ND}^{-2}\Theta_{ND}(z)| \\ &\quad -H_{ND}(\lambda,\lambda) - H_{ND}(z,z) \\ &\quad -2H_{ND}(\lambda,z) \\ &= \; \log|\Theta_{ND}(z)| - H_{ND}(z,z),\end{aligned}$$

according to 3.4, (iv). Now we have to check properties (i) to (iv) of 1.1.

(i) Assume that the restriction of D to the Zariski open subset U of A is equal to div f for some $f \in K(A)$. Then D^{an} restricted to U^{an} equals div f^{an}. Choose a normalized theta function $\Theta = \Theta_{ND}$ for ND. We have $\mathrm{div}((f^{an})^N \circ \pi)|_{\pi^{-1}U^{an}} = \mathrm{div}(\Theta)|_{\pi^{-1}U^{an}}$, hence $\Theta/(f^{an}\circ\pi)^N \in \Gamma(\pi^{-1}U^{an}, \mathcal{O}^\times)$. Thus, $\log|\Theta(z)/f^N(\pi z)| \in \mathbb{R}$ for all $z \in \pi^{-1}U^{an}$. As H_{ND} has no singularities, and $\mu_\Theta(\pi z) = \log|f(\pi z)| + (1/N)\log|\Theta(z)/f^N(\pi z)| - (1/N)H_{ND}(z,z)$, this proves our claim.

(ii) is clear since the product of two normalized theta functions is normalized.

(iii) Let $f \neq 0$ be a rational function on A. Then f^{an} is a global meromorphic function on A^{an}, and $(\pi^* f^{an})^N$ is a normalized theta function corresponding to $N\,\mathrm{div}(f)$, which implies that $\mu_{\mathrm{div}(f)} \equiv \log|f| \bmod \mathcal{C}$.

(iv) Finally, we have to check translation invariance. Let a be a point in $A(K)$. Fix some $b \in T(K)$ with $\pi(b) = a$. Denote by t_a respectively t_b the translation maps $x \mapsto ax$ on A respectively $z \mapsto bz$ on T. Then obviously $\pi \circ t_b = t_a \circ \pi$. So, if $\Theta = \Theta_{ND}$ is a normalized theta function corresponding to ND, we have $\pi^*(Nt_a^*D) = (\pi\circ t_b)^*(ND) = t_b^*(\mathrm{div}\,\Theta)$. Hence $\Theta\circ t_b$ is a theta function corresponding to the divisor Nt_a^*D. For all $\lambda \in \Lambda$ and all $z \in T(K)$ we have $\Theta(b\lambda z) = \psi_\Theta(\lambda)[\lambda,b]_{ND}^{-2}[\lambda,\lambda]_{ND}^{-1}[\lambda,z]_{ND}^{-2}\Theta(bz)$, which implies

$$|[\,,\,]_{Nt_a^*D}| = |[\,,\,]_{ND}| \quad \text{and} \quad |\psi_{\Theta\circ t_b}(\lambda)| = |\psi_\Theta(\lambda)[\lambda,b]_{ND}^{-2}|.$$

Now let Θ' be a normalized theta function corresponding to $N(t_a^*D)$. Then $\Theta\circ t_b = \Theta' a\chi$ for some $a \in K^\times$ and $\chi \in H$, as both theta functions have the same divisor. Hence $|\psi_{\Theta\circ t_b}(\lambda)| = |\psi_{\Theta'}(\lambda)\chi(\lambda)|$, which implies $|[\lambda,b]_{ND}^{-2}| = |\chi(\lambda)|$, since Θ and Θ' are normalized. Then we get, using again 3.4, $H_{ND}(z,b) = -\frac{1}{2}\Sigma_j w_j(z)\log|[\lambda_j,b]_{ND}^2| = \frac{1}{2}\Sigma_j w_j(z)\log|\chi(\lambda_j)| = \frac{1}{2}\log|\chi(z)|$. Now we calculate for all $z \in T(K)$ with $\pi z = x$

$$\mu_\Theta(ax) \; = \; \frac{1}{N}\log|\Theta(bz)| - \frac{1}{N}H_{ND}(bz,bz)$$

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$$= \frac{1}{N}\log|\Theta'(z)| + \frac{1}{N}\log|a| + \frac{1}{N}\log|\chi(z)| - \frac{1}{N}H_{ND}(b,b)$$
$$- \frac{2}{N}H_{ND}(z,b) - \frac{1}{N}H_{ND}(z,z)$$
$$\equiv \mu_{\Theta'}(x) \ \text{ modulo } \mathcal{C}.$$

Hence, $\mu_{t_a^* D} = \mu_D \circ t_a$, as claimed. □

Theorem 3.5 generalizes a well-known result on Tate curves, see [La2], Chapter III, Section 5. We used normalized theta functions in order to get a formula similar to Néron's result in the complex case. We could avoid this notion altogether, as the following corollary shows:

COROLLARY 3.6. *Let D be a divisor on A and let Θ_D be a theta function corresponding to D. Define for all $z \in T(K)$ the vector $(w_1(z), \ldots, w_n(z))$ as in* 3.3. *Then put for all $x \in (A \backslash \operatorname{supp} D)(K)$*

$$\mu^{\star}_{\Theta_D}(x) = \log|\Theta_D(z)| - \sum_{j=1}^{n} w_j(z)\log|\psi_{\Theta_D}(\lambda_j)| - H_D(z,z),$$

where $z \in T(K)$ is an arbitrary preimage of x. Substituting Θ_D by another theta function corresponding to D amounts to adding a constant on the right-hand side, hence we get a well-defined map

$$D \longmapsto \mu_D := (\mu^{\star}_{\Theta_D} \ \text{ modulo } \ \mathcal{C}(A \backslash \operatorname{supp} D)),$$

which is the Néron map.

Proof. An easy calculation, similar to the one in the proof of 3.5, shows that the expression we used to define $\mu^{\star}_{\Theta_D}(x)$ does not depend on the choice of a preimage z. Furthermore, if a is an element in $K^{\times}$ and χ is a character, we get $\mu^{\star}_{\Theta}(x) - \mu^{\star}_{a\chi\Theta}(x) = -\log|a| - \log|\chi(z)| + \Sigma_j w_j(z)\log|\chi(\lambda_j)| = -\log|a|$ according to 3.4, (ii). This proves that using a different theta function corresponding to D amounts to adding a constant. Therefore, it remains to be shown that μ_D is the Néron map, which follows easily by comparison with the expression we found in 3.5. □

We can now derive a formula for Néron's local height pairing.

COROLLARY 3.7. *For $(D, z) \in (\operatorname{Div}^0 A \times Z^0(A/K))'$ with $z = \Sigma n_i a_i$ we have*

$$(D,z)_{N,A/K} = \log\left|\prod_i \Theta_D(b_i)^{n_i}\right| - \sum_i n_i \sum_j w_j(b_i)\log|\psi_{\Theta_D}(\lambda_j)|,$$

for any theta function Θ_D corresponding to D and for arbitrary preimages b_i of a_i in $T(K)$.

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Proof. Our statement follows immediately from 3.6. Note that for a divisor D which is algebraically equivalent to zero H_D vanishes identically according to 2.4. □

For a different investigation of Néron's local height pairing on principally polarized abelian varietes with split multiplicative reduction see [Tu].

4. Local Mazur–Tate height pairings

We consider again an abelian variety A with split multiplicative reduction over a non-archimedean local ground field K such that A^{an} is uniformized as T/Λ, and we denote by $\pi\colon T \to A^{an}$ the projection map. Let T_Λ be the split torus over K with character group Λ. We put

$$T' := {T_\Lambda}^{an}.$$

Then T' is a split analytic torus. The character group H of T can be regarded as a subgroup of $T'(K) = \mathrm{Hom}\,(\Lambda, K^\times)$. Furthermore, the constant analytic group defined by H is a split lattice in T'. Hence the quotient T'/H is a rigid analytic variety over K, see [Bo–Lü3], p. 661. It is algebraic, i.e. there is an abelian variety B over K with $B^{an} \simeq T'/H$ (see [Ge], p. 341). We will denote by π' the uniformization map $\pi'\colon T' \to B^{an}$. We define now a 1-cocycle in $H^1(\Lambda \times H, \Gamma(T \times T', \mathcal{O}^\times))$ by $e_{(\lambda,\chi)} = \chi(\lambda)\chi\lambda \in \Gamma(T \times T', \mathcal{O}^\times)$. It is easy to see that the cocycle condition is satisfied. $e_{(\lambda,\chi)}$ defines a $\Lambda \times H$-linearization

$$u_{(\lambda,\chi)}(\alpha, x, y) = (e_{(\lambda,\chi)}(x,y)\alpha, \lambda x, \chi y),$$

on the trivial G_m^{an}-torsor over $T \times T'$. Let P be a line bundle on $A \times B$ corresponding to $(u_{(\lambda,\chi)})$ via the map ψ_1 from Section 2.

PROPOSITION 4.1. *(B, P) is the dual abelian variety corresponding to A.*

Proof. Our claim follows from combining [Bo–Lü1], Proposition 1.1, p. 258, and [Bo–Lü3], proof of 2.1, p. 663. (See also [Ge], Section 5.) □

Because of this Proposition, we will henceforth write A' instead of B.

Recall that we proved in 2.4 that a divisor D on A is algebraically equivalent to zero if and only if $[\ ,\ ]_D^2 = 1$. The identification of $\mathrm{Pic}^0(A)$ with $A'(K)$ is given by mapping an isomorphism class of a line bundle M in $\mathrm{Pic}^0(A)$ to the point $y \in A'(K)$ satisfying $P|_{A\times\{y\}} \simeq M$. This is exactly the point $y \in A'(K) = T'(K)/H = \mathrm{Hom}(\Lambda, K^\times)/H$ corresponding to the class of $\psi_\Theta \in \mathrm{Hom}(\Lambda, K^\times)$ for any theta function Θ for M.

P^{an} (respectively its associated G_m^{an}-torsor) is a rigid analytic biextension of A^{an} and $A'^{\,an}$ by G_m^{an}. As we explained at the end of Section 2, there is an analytic morphism

$$\phi_0\colon G_m^{an} \times T \times T' \xrightarrow{\sim} (\pi \times \pi')^* P^{an} \to P^{an},$$

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such that (P^{an}, ϕ_0) is the categorical quotient of $G_m^{an} \times T \times T'$ after the $u_{(\lambda,\chi)}$-operation. ϕ_0 is a morphism of G_m^{an}-torsors, i.e. ϕ_0 lies over the projection $T \times T' \to A^{an} \times A'^{\,an}$ and commutes with the operation of G_m^{an}. But as $G_m^{an} \times T \times T'$ is the trivial biextension of T and T' by G_m^{an}, both sides carry additional structures. The next Proposition investigates in how far ϕ_0 is already a morphism of biextensions.

PROPOSITION 4.2. *There is a unique point $c \in K^\times$ such that the map*

$$\phi := m_c \circ \phi_0\colon G_m^{an} \times T \times T' \longrightarrow P^{an},$$

is a morphism of biextension. Here m_c denotes the G_m^{an}-torsor operation by c on P^{an}.

Proof. Since for any $c \in K^\times$ the map $m_c \circ \phi_0$ is compatible with the torsor structures, the only problem is to find an element $c \in K^\times$ such that $\phi := m_c \circ \phi_0$ is a homomorphism with respect to both group structures. If c is such an element, then $\phi_0(c, 1_T, 1_{T'})$ must be equal to the unit section of P^{an} over $A'^{\,an}$ applied to the unit section of $A'^{\,an}$ over K. Let us denote this element by e. On the other hand, since both $\phi_0(1_{G_m^{an}}, 1_T, 1_{T'})$ and e project to $(1_{A^{an}}, 1_{A'^{\,an}})$, they differ by an element in $K^\times$. Hence we see that we have to define $c \in K^\times$ as the element satisfying $\phi_0(c, 1_T, 1_{T'}) = e$. Put $\phi = m_c \circ \phi_0$. It remains to be shown that ϕ is indeed a morphism of biextensions.

We begin by studying the analytic morphism

$$\begin{aligned} f\colon T \times T \times T' &\longrightarrow P^{an}, \\ (u, v, w) &\longmapsto \phi(1, uv, w)\phi(1, u, w)^{-1}\phi(1, v, w)^{-1}, \end{aligned}$$

where on the right-hand side we multiply and take inverses with respect to the group structure on P^{an} over $A'^{\,an}$, and where we use functorial points (u, v, w). We will now investigate the behaviour of f under the operation of $\Lambda \times \Lambda \times H$ on $T \times T \times T'$. By definition, ϕ is invariant under the operation of $u_{(\lambda,\chi)}$, which means that $\phi(\alpha, x, y) = \phi(\alpha e_{(\lambda,\chi)}(x, y), \lambda x, \chi y)$. Hence an easy calculation using the definition of the $e_{(\lambda,\chi)}$ shows that f is $\Lambda \times \Lambda \times H$-invariant. Thus there exists a morphism $f_0\colon A^{an} \times A^{an} \times A'^{\,an} \to P^{an}$ with $f_0 \circ (\pi \times \pi \times \pi') = f$.

Denote by ν the projection $P \to A \times A'$. Then $\nu^{an}\colon P^{an} \to A^{an} \times A'^{\,an}$ is a homomorphism of analytic groups over $A'^{\,an}$. Since $\nu^{an} \circ \phi = (\pi \times \pi') \circ p_{23}$ (where p_{23} is the projection to the last two factors) is a homomorphism, we see that $\nu^{an} \circ f$ is the map $T \times T \times T' \to A^{an} \times A'^{\,an}$ given by $(u, v, w) \mapsto (1_{A^{an}}, \pi'(w))$. Hence $\nu^{an} \circ f_0$ is the morphism given by $(a, b, c) \mapsto (1_{A^{an}}, c)$. This means that $\nu^{an} \circ f_0$ factorizes through the unit section of the group $A^{an} \times A'^{\,an}$ over $A'^{\,an}$. As P^{an} is a biextension, we know that

$$0 \longrightarrow G_m^{an} \times A'^{\,an} \longrightarrow P^{an} \xrightarrow{\nu^{an}} A^{an} \times A'^{\,an} \longrightarrow 0,$$

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is an exact sequence of analytic groups over $A'^{\,an}$. Thus there exists an $A'^{\,an}$-morphism g: $A^{an} \times A^{an} \times A'^{\,an} \longrightarrow G_m^{an} \times A'^{\,an}$ which, composed with the embedding $G_m^{an} \times A'^{\,an} \to P^{an}$, gives the morphism f_0. Furthermore, g is the product of a morphism h: $A^{an} \times A^{an} \times A'^{\,an} \longrightarrow G_m^{an}$ and the projection to $A'^{\,an}$. Now h corresponds to an element in $\Gamma(A^{an} \times A^{an} \times A'^{\,an}, \mathcal{O}^\times)$, which by GAGA theorems is isomorphic to $\Gamma(A \times A \times A', \mathcal{O}^\times) = K^\times$ (see [Kö], p. 43). Hence f is the following morphism: $T \times T \times T' \to T' \xrightarrow{d \times \pi'} G_m^{an} \times A'^{\,an} \to P^{an}$, where d is a K-rational point of G_m^{an}. But, by the definition of ϕ, we have $f(1_T, 1_T, 1_{T'}) = e$, hence we get $d = 1$. This implies that ϕ is a group homomorphism with respect to the group structures over T' respectively $A'^{\,an}$.

The same reasoning, applied to the morphism

$$\begin{aligned} T \times T' \times T' &\longrightarrow P^{an}, \\ (u, v, w) &\longmapsto \phi(1, u, vw)\phi(1, u, v)^{-1}\phi(1, u, w)^{-1}, \end{aligned}$$

where we now use the group structure on P^{an} over A^{an}, implies that ϕ is also homomorphic with respect to the second group law. □

Note that P^{an} together with the new quotient morphism ϕ: $G_m^{an} \times T \times T' \to P^{an}$ is still the categorical quotient of $G_m^{an} \times T \times T'$ for the action of $\Lambda \times H$ given by the $u_{(\lambda,\chi)}$.

Let now Y be an abelian group (noted additively), and let ρ: $K^\times \to Y$ be a homomorphism. The following result characterizes all ρ-splittings in our situation:

PROPOSITION 4.3. *There is a* (1-1)*-correspondence between*

(a) *ρ-splittings τ: $P(K) \longrightarrow Y$ and*
(b) *ρ-splittings τ^*: $(G_m^{an} \times T \times T')(K) \to Y$ satisfying for all $\lambda \in \Lambda$, $\chi \in H$, $y \in T(K)$ and $z \in T'(K)$: $\tau^*(1, \lambda, z) = -\rho(\lambda(z))$ and $\tau^*(1, y, \chi) = -\rho(\chi(y))$, induced by mapping a ρ-splitting τ: $P(K) \to Y$ to $\tau^* = \tau \circ \phi$.*

Proof. As ϕ is a morphism of biextensions, for any ρ-splitting τ of $P(K)$ the map $\tau \circ \phi$ will indeed be a ρ-splitting of $(G_m^{an} \times T \times T')(K)$. Furthermore, as ϕ is $u_{(\lambda,\chi)}$-invariant, we have $\tau\circ\phi(\lambda(z), \lambda, z) = \tau\circ\phi(u_{(\lambda,1)}(1,1,z)) = \tau\circ\phi(1,1,z) = 0$. A parallel argument shows that $\tau \circ \phi(\chi(y), y, \chi) = 0$. Hence $\tau \circ \phi$ is indeed an element of the set in (b).

On the other hand, take a ρ-splitting τ^*: $K^\times \times T(K) \times T'(K) \to Y$ satisfying $\tau^*(1, \lambda, z) = -\rho(\lambda(z))$ and $\tau^*(1, y, \chi) = -\rho(\chi(y))$. Then we can calculate for all $\alpha \in K^\times$, $\lambda \in \Lambda$, $\chi \in H$, $y \in T(K)$ and $z \in T'(K)$

$$\begin{aligned} \tau^*(\alpha, \lambda y, \chi z) &= \rho(\alpha) + \tau^*(1, \lambda, \chi) + \tau^*(1, \lambda, z) + \tau^*(1, y, \chi) \\ &\quad + \tau^*(1, y, z) \\ &= -\rho(\chi(\lambda)\lambda(z)\chi(y)) + \tau^*(\alpha, y, z). \end{aligned}$$

Hence τ^* is invariant under the operation given by the $u_{(\lambda,\chi)}$. Since $P(K)$ is the categorical quotient of $K^\times \times T(K) \times T'(K)$ after the $u_{(\lambda,\chi)}$-operation, there is a unique map τ: $P(K) \to Y$ such that $\tau \circ \phi = \tau^*$. As ϕ is a homomorphism of biextensions, τ is in fact a ρ-splitting. This proves our claim. □

We will now define a new ρ-splitting for homomorphisms ρ with a certain property. As we will see later, this result can be used to calculate the canonical Mazur–Tate height in case (II) and Schneider's local p-adic height pairing on A.

DEFINITION 4.4. Let ρ: $K^\times \to Y$ be a homomorphism to an abelian group Y. We call ρ Λ-invertible, if the following two conditions are fulfilled:

(i) Y is the additive group of a commutative ring (which we also call Y).
(ii) There is a homomorphism ρ_0: $K^\times \to Y$ and an element $a \in Y$ such that $\rho = a \cdot \rho_0$, and such that for some (and hence for any) bases $\lambda_1, \dots, \lambda_n$ of Λ and $\chi_1, \dots, \chi_n$ of H the element $\det(\rho_0(\chi_j \lambda_i)_{ij})$ is a unit in Y.

From now on, we fix a basis $\chi_1, \dots, \chi_n$ of H and a basis $\lambda_1, \dots, \lambda_n$ of Λ.

THEOREM 4.5. *Assume that* ρ: $K^\times \to Y$ *is* Λ*-invertible. Let* $M \in \mathit{Mat}_{n,n}(Y)$ *be the inverse matrix of* $(\rho_0(\chi_j \lambda_i)_{ij})$.

(i) *Define the* ρ*-splitting* τ^*: $K^\times \times T(K) \times T'(K) \to Y$ *by*

$$(\alpha, y, z) \longmapsto \rho(\alpha) - (\rho_0(\chi_1 y), \dots, \rho_0(\chi_n y))\ M^t(\rho(\lambda_1 z), \dots, \rho(\lambda_n z)).$$

Then there exists a unique ρ*-splitting* τ: $P(K) \longrightarrow Y$ *such that* $\tau \circ \phi = \tau^*$.

(ii) *Denote the map* $T(K) \to Y^n$ *given by* $y \mapsto (\rho(\chi_1 y), \dots, \rho(\chi_n y))$ *by* ρ^n. *Assume that for every element* $u \in \rho^n(T(K))$ *there is a natural number* d_u *which is a unit in* Y *and which satisfies* $d_u u \in \rho^n(\Lambda)$. *Then the* ρ*-splitting* τ *defined in* (i) *is the unique* ρ*-splitting such that* $\tau \circ \phi$ *vanishes on* $\{1\} \times (\ker \rho^n) \times T'(K)$.

Proof. (i) It is easy to see that τ^* is indeed a ρ-splitting. In order to check that it gives rise to a ρ-splitting τ of $P(K)$, according to 4.3 we have to show for all λ, χ, y and z that $\tau^*(1, \lambda, z) = -\rho(\lambda(z))$ and $\tau^*(1, y, \chi) = -\rho(\chi(y))$. Since we know that τ^* respects the group laws on the biextension $K^\times \times T(K) \times T'(K)$, we can assume that $\lambda \in \Lambda$ and $\chi \in H$ are elements of the chosen bases. Note that $(\rho_0(\chi_1 \lambda_i), \dots, \rho_0(\chi_n \lambda_i))M = {}^t e_i$, where e_i is the ith unit vector, and that $M^t(\rho(\chi_j \lambda_1), \dots \rho(\chi_j \lambda_n)) = a e_j$. Hence our claim follows.

(ii) The ρ-splitting in (i) obviously vanishes on $\{1\} \times \ker \rho^n \times T'(K)$. We have to check that under the conditions of (ii) τ^* is uniquely determined by the following properties:

- $\tau^*(1, \lambda, z) = -\rho(\lambda(z))$ for all $\lambda \in \Lambda$.
- $\tau^*(1, y, \chi) = -\rho(\chi(y))$ for all $\chi \in H$.
- $\tau^*(1, y, z) = 0$ for all $y \in \ker \rho^n \subset T(K)$.

Consider elements $y \in T(K)$ and $z \in T'(K)$. Then according to our assumption we find a natural number d which is a unit in Y such that $d\rho^n(y) \in \rho^n(\Lambda) \subset Y^n$. So there exists an element $\mu \in \Lambda$ such that $\rho^n(y^d\mu^{-1}) = 0 \in Y^n$, which implies that y^d and μ differ by an element in ker ρ^n. Hence $d\tau^*(\alpha, y, z) = d\rho(\alpha) + \tau^*(1, y^d, z) = d\rho(\alpha) + \tau^*(1, \mu, z) = d\rho(\alpha) - \rho(\mu(z))$. Since d is a unit in Y, we see that τ^* is in this case uniquely determined. □

As described in Section 1, our ρ-splitting induces a local pairing

$$(\, , \,)_{MT,\tau}\colon \left(\mathrm{Div}^0 A \times Z^0\left(A/K\right)\right)' \longrightarrow Y.$$

We define meromorphic sections of a rigid analytic line bundle (respectively G_m^{an}-torsor) as in [EGA IV], 20.1.8. The following result shows that the meromorphic section of the trivial line bundle on T given by a theta function Θ_D is just the lift of a rational section with divisor D.

LEMMA 4.6. *Let $D \in \mathrm{Div}^0(A)$, and let $d \in A'(K)$ be the corresponding point. Let Θ_D be a theta function for D, and let t be the induced meromorphic section of the trivial torsor $G_m^{an} \times T$. Then there exists a rational section s_D of $P|_{A\times\{d\}}$ with divisor D, such that the following diagram commutes*

$$\begin{array}{ccc} T\backslash \mathrm{supp}\ \pi^* D^{an} & \stackrel{t}{\longrightarrow} & G_m^{an} \times T \\ \Big\downarrow{\scriptstyle \alpha_A \circ \pi} & & \Big\downarrow{\scriptstyle \alpha_P \circ \phi(-,-,d')}, \\ A\backslash \mathrm{supp}\ D & \underset{s_D}{\longrightarrow} & P|_{A\times\{d\}} \end{array}$$

where $\alpha_A\colon A^{an} \to A$ and $\alpha_P\colon P^{an} \to P$ are the canonical maps (see Sect. 2), and where $d' \in T'(K)$ is the point corresponding to the homomorphism $\Lambda \to K^\times$ given by

$$\lambda \longmapsto \psi_{\Theta_D}(\lambda) = \frac{\Theta_D(\lambda x)}{\Theta_D(x)}.$$

Proof. By construction, $\phi(-,-,d')\colon G_m^{an} \times T \times \{d'\} \to P^{an}|_{A^{an}\times\{\pi'(d')\}}$ is equal to the composition of an isomorphism $\omega\colon G_m^{an} \times T \times \{d'\} \xrightarrow{\sim} \pi^*(P^{an}|_{A^{an}\times\{d\}})$ and the projection $\pi^*(P^{an}|_{A^{an}\times\{d\}}) \to P^{an}|_{A^{an}\times\{d\}}$. Let now s_D be any rational section of $P|_{A\times\{d\}}$ with divisor D. Then s_D induces a meromorphic section $\pi^* s_D^{an}$ of $\pi^*(P^{an}|_{A^{an}\times\{d\}})$. Via the isomorphism ω, we find a meromorphic section t of $G_m^{an} \times T \times \{d'\}$ with divisor $\pi^* D^{an}$ such that the following diagram commutes

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$$\begin{array}{ccc} T\backslash \mathrm{supp}\ \pi^* D^{an} & \xrightarrow{\ t\ } & G_m^{an} \times T \\ \Big\downarrow{\scriptstyle \alpha_A \circ \pi} & & \Big\downarrow{\scriptstyle \alpha_P \circ \phi(-,-,d')} \\ A\backslash \mathrm{supp}\ D & \xrightarrow[s_D]{} & P|_{A\times\{d\}} \end{array}.$$

Furthermore, t is given by a global meromorphic function Θ on T. We now have to connect Θ to our given theta function Θ_D. We know that Θ and Θ_D have the same divisor, hence they differ by an element in $K^\times H$. Since t is a lift of s_D, we have $\phi(\Theta(x), x, d') = \phi(\Theta(\lambda x), \lambda x, d')$. On the other hand, $\phi(\Theta(x), x, d') = \phi(u_{(\lambda,1)}(\Theta(x), x, d')) = \phi(\lambda(d')\Theta(x), \lambda x, d')$, hence

$$\frac{\Theta(\lambda x)}{\Theta(x)} = \lambda(d') = \frac{\Theta_D(\lambda x)}{\Theta_D(x)}.$$

Since the presence of a character would affect the automorphy factor, Θ/Θ_D must be constant. If $\Theta_D = c\Theta$ for some $c \in K^\times$, the rational section cs_D of $P|_{A\times\{d\}}$ has also divisor D and makes the diagram in our claim commutative. □

DEFINITION 4.7. Assume that ρ is Λ-invertible. Define for all $y \in T(K)$ the vector $(w_1(y), \dots w_n(y)) \in Y^n$ to be $(\rho_0(\chi_1 y), \dots, \rho_0(\chi_n y))M$.

Then $(w_1(y), \dots, w_n(y))$ is a solution of the linear system of equations

$$\rho(\chi_i(y)) = \sum_j w_j(y)\rho(\chi_i(\lambda_j)), \quad i = 1, \dots, n.$$

COROLLARY 4.8. *Let D be in* $\mathrm{Div}^0(A)$ *and let $z = \Sigma_{i=1}^k (a_i - b_i)$ be a zero cycle with K-rational support disjoint from the support of D. Furthermore, let τ be the ρ-splitting defined in* 4.5. *For any choice of a theta function Θ_D corresponding to D and of preimages a_i', b_i' of a_i, b_i in $T(K)$ we have*

$$(D, z)_{MT,\tau} = \rho\left(\prod_i \frac{\Theta_D(a_i')}{\Theta_D(b_i')}\right) - \sum_{j=1}^n w_j \left(\prod_i \frac{a_i'}{b_i'}\right) \rho(\psi_{\Theta_D}(\lambda_j)).$$

Proof. According to 4.6, for any $y \in T(K)\backslash \mathrm{supp}\, \pi^* D^{an}$ and for any theta function Θ_D for D we have $s_D(\pi y) = \phi(\Theta_D(y), y, d')$ for a suitable rational section s_D with divisor D and for $\lambda(d') = \psi_{\Theta_D}(\lambda)$. Hence

$$\begin{aligned} \tau(s_D(\pi y)) &= \tau(\phi(\Theta_D(y), y, d')) \\ &= \tau^*(\Theta_D(y), y, d') \end{aligned}$$

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$$\begin{aligned} &= \rho(\Theta_D(y)) - (\rho_0(\chi_1 y), \ldots, \rho_0(\chi_n y)) \\ &\quad \cdot M^t(\rho(\lambda_1(d')), \ldots, \rho(\lambda_n(d'))) \\ &= \rho(\Theta_D(y)) - \sum_{j=1}^{n} w_j(y)\rho(\psi_{\Theta_D}(\lambda_j)), \end{aligned}$$

by the definition of the w_j's. This implies our claim. □

We will now investigate the ρ-splitting τ from 4.5 in the case that ρ is unramified, i.e. that ρ vanishes on units in R. Recall that if ρ is unramified and if Y is uniquely divisible by m_A, the exponent of $\mathcal{A}_k(k)/\mathcal{A}_k^0(k)$, Mazur and Tate have shown that there exists a unique ρ-splitting σ_ρ of $P(K)$ vanishing on $P_R(R)$. Here P_R is the unique biextension of $\mathcal{A}^0$ and $\mathcal{A}'$ by $G_{m,R}$ with generic fibre P. The next lemma will be needed to compare the conditions for the existence of σ_ρ and our splitting τ.

LEMMA 4.9. *Denote by* $v: K^\times \to \mathbb{Z}$ *the valuation map. Then* m_A *divides* $\det(v(\chi_j\lambda_i)_{i,j})$, *and* $\det(v(\chi_j\lambda_i)_{i,j})$ *divides* m_A^n *in* $\mathbb{Z}$.

Proof. For Tate curves, this means that $m_A = \pm v(\chi_1\lambda_1)$, which is a well-known result, see [Si], p. 358f. Recall that $\det(v(\chi_j\lambda_i)_{i,j}) \neq 0$, as Λ is a split lattice in T. Let $\mathcal{A}^{0\wedge}$ be the formal completion of $\mathcal{A}^0$ along the special fibre. We can associate to $\mathcal{A}^{0\wedge}$ its rigid analytic generic fibre $\overline{A}$ which is an open analytic subgroup variety of A^{an} (see [Bo–Lü2], Sect. 1). The uniformization map π induces an isomorphism

$$T(R) := \{y \in T(K) : |\chi_i(y)| = 1 \quad \text{for all } i\} \xrightarrow{\sim} \overline{A}(K),$$

see [Bo–Lü3], p. 655. We have furthermore a natural identification $A(K) \xrightarrow{\sim} \mathcal{A}(R)$. The preimage of $\mathcal{A}^0(R)$ under this identification is just $\overline{A}(K)$, see [Bo–Lü2], Proposition 1.3, p. 72. The restriction to the special fibre induces a homomorphism $\mathcal{A}(R) \to \mathcal{A}_k(k)$, which is surjective, since $\mathcal{A}$ is smooth over the henselian ring R ([EGA IV], 18.5.17). The preimage of $\mathcal{A}_k^0(k)$ under this map is just $\mathcal{A}^0(R)$.

So we find that the preimage of $\mathcal{A}_k^0(k)$ under the surjective reduction map $A(K) \to \mathcal{A}_k(k)$ is $\overline{A}(K)$. Hence we get an isomorphism $A(K)/\overline{A}(K) \xrightarrow{\sim} \mathcal{A}_k(k)/\mathcal{A}_k^0(k)$. Via the uniformization $\pi: T \to A^{an}$ we get an isomorphism $T(K)/T(R)\Lambda \xrightarrow{\sim} \mathcal{A}_k(k)/\mathcal{A}_k^0(k)$. Hence m_A is equal to the exponent of $T(K)/T(R)\Lambda$.

For any point $y \in T(K)$ and any natural number m the point y^m is in $T(R)\Lambda$ if and only if there are natural numbers $m_1, \ldots, m_n$ such that

$$mv(\chi_i(y)) = \sum_j m_j v(\chi_i(\lambda_j)) \quad \text{for all } i = 1 \ldots n.$$

It is easy to see that this is always the case if we choose $m = \det(v(\chi_i\lambda_j)_{i,j}) = \det(v(\chi_j\lambda_i)_{i,j})$. Hence m_A divides $\det(v(\chi_j\lambda_i)_{i,j})$. On the other hand, choose for all k a point $y_k \in T(K)$ such that $v(\chi_i(y_k)) = \delta_{ik}$. This is possible, since $(\chi_1, \ldots, \chi_n)\colon T(K) \to K^{\times n}$ is an isomorphism. As m_A is the exponent of $T(K)/T(R)\Lambda$, we find for every $k \in \{1, \ldots, n\}$ integers $m_1, \ldots, m_n$ such that

$$m_A \delta_{ik} = m_A v(\chi_i(y_k)) = \sum_j m_j v(\chi_i(\lambda_j)) \quad \text{for all } i.$$

If $M = (M_{i,j})_{i,j}$ is the inverse matrix of $(v(\chi_i \lambda_j)_{i,j})$, this is equivalent to $M_{j,k} = m_j/m_A$ for all $j = 1, \ldots, n$. Hence all coefficients of M lie in ${m_A}^{-1}\mathbb{Z}$, which implies that $\det(v(\chi_j \lambda_i)_{i,j})^{-1} = \det(M) \in m_A^{-n}\mathbb{Z}$, hence $\det(v(\chi_j \lambda_i)_{i,j})$ divides m_A^n in $\mathbb{Z}$. □

In order to compare our ρ-splitting to the canonical Mazur–Tate splitting in the unramified case, we need another lemma:

LEMMA 4.10. *Denote again by $T(R) \subset T(K)$ the set of K-rational points of T which are mapped to $R^\times$ by all characters $\chi \in H$. The morphism of biextensions*

$$\phi\colon G_m^{an} \times T \times T' \longrightarrow P^{an}$$

maps $R^\times \times T(R) \times T'(K)$ to $P_R(R) \subset P(K)$. Furthermore, every point in $P_R(R)$ has a preimage in $R^\times \times T(R) \times T'(K)$.

Proof. We fix a point $z \in T'(K)$ and put $b = \pi'(z) \in A'(K)$. We denote by b also the corresponding point in $\mathcal{A}'(R)$, and we denote the projection $P_R \to \mathcal{A}^0 \times \mathcal{A}'$ by ν. For brevity, we put $Z = P_R|_{\mathcal{A}^0 \times \{b\}}$, which is an extension of $\mathcal{A}^0$ by $G_{m,R}$. As $H^1(\operatorname{Spec} R, G_m)$ is trivial, $Z(R)$ is an extension of $\mathcal{A}^0(R)$ by $R^\times$. Now let $S = \operatorname{Spec} R$, and for all integers $n \geqslant 0$ put $S_n = \operatorname{Spec} R/\mathcal{M}^{n+1}$, where $\mathcal{M}$ is the maximal ideal in R. Note that $S_0 = \operatorname{Spec} k$. For S-schemes and S-morphisms we use subscripts n to indicate base changes by S_n, and for any S-scheme Y we denote the formal completion after its special fibre by $\widehat{Y}$. Hence we write $\widehat{S}$ for the formal spectrum of R. Then $\widehat{Z}$ is a formal extension of $\widehat{\mathcal{A}}^0$ by $\widehat{G}_{m,R}$. Since A has split multiplicative reduction, all $\mathcal{A}_n^0$ are split tori. Hence, by [SGA7, I, exp. VIII], 3.3.1, all extensions Z_n split. Choose a section $\zeta_0\colon \mathcal{A}_0^0 \to Z_0$ of the projection $\nu_0\colon Z_0 \to \mathcal{A}_0^0$. By [SGA3, II, exp. IX], 3.6, we find for all n uniquely determined S_n-homomorphisms $\zeta_n\colon \mathcal{A}_n^0 \to Z_n$ such that $\zeta_n \times_{S_n} S_0 = \zeta_0$. Furthermore, from $\nu_0 \circ \zeta_0 = \mathrm{id}_{\mathcal{A}_0^0}$, we deduce by [SGA3, II, exp. IX] 3.4, that $\nu_n \circ \zeta_n$ is the identity map on $\mathcal{A}_n^0$. Hence we found a compatible system of sections $(\zeta_n)_n$, which induces a section of the homomorphism $\widehat{Z} \to \widehat{\mathcal{A}}^0$, and hence a section of $\widehat{Z}(\widehat{S}) \to \widehat{\mathcal{A}}^0(\widehat{S})$. Since $\widehat{Z}(\widehat{S}) = Z(R)$ and $\widehat{\mathcal{A}}^0(\widehat{S}) = \mathcal{A}^0(R)$ (which follows e.g. from Grothendieck's existence theorem [EGA III], 5.4.1), we find a section ζ of the homomorphism $P_R(R)|_{\mathcal{A}^0(R) \times \{b\}} = Z(R) \to \mathcal{A}^0(R)$.

For all $y \in T(R)$ we have $\nu \circ \phi(1, y, z) = (\pi(y), \pi'(z)) \in \mathcal{A}^0(R) \times \{b\}$. Composing this map with our section ζ, we get a morphism

$$\zeta \circ \nu \circ \phi(1, y, z) : \{1\} \times T(R) \times \{z\} \longrightarrow P_R(R)|_{\mathcal{A}^0(R) \times \{b\}}.$$

As $\nu \circ \zeta \circ (\nu \circ \phi(1,y,z)) = \nu \circ \phi(1,y,z) = (\pi(y),\pi'(z))$, the maps $y \mapsto \zeta \circ \nu \circ \phi(1,y,z)$ and $y \mapsto \phi(1,y,z)$ differ by a homomorphism h: $T(R) \to K^\times$. Identifying $T(R)$ with $R^{\times n}$, we get a homomorphism h: $R^{\times n} \to K^\times$. Composing h with the natural projection $K^\times \to K^\times/R^\times \simeq \mathbb{Z}$, we get a homomorphism g: $R^{\times n} \to \mathbb{Z}$.

Assume now that $\mathrm{Im}\ g \neq \{0\}$. Then there exists an integer $\alpha > 0$ such that α is minimal among all integers $\beta > 0$ such that $\beta \in \mathrm{Im}\ g$. Let y be an element in $R^{\times n}$ with $g(y) = \alpha$. Now choose a natural number $\gamma > 1$ such that the characteristic of k does not divide γ. Let $U^{(1)} \subset R^\times$ be the units congruent to 1 modulo the valuation ideal. Then, by Hensel's Lemma, $x \mapsto x^\gamma$ induces a surjection $U^{(1)} \to U^{(1)}$. Now $R^\times \simeq W \times U^{(1)}$, where W is torsion. Hence g factorizes through the projection of $R^{\times n}$ to the direct factor $U^{(1)n}$. Let y_1 be the projection of y to $U^{(1)n}$. Then there is a $z \in U^{(1)n}$ such that $z^\gamma = y_1$. Hence $\gamma g(z) = g(y_1) = g(y) = \alpha$. As γ is bigger than 1, we get a contradiction to our choice of α.

Thus $\mathrm{Im}\ g = \{0\}$, which implies that the image of h is contained in $R^\times$. Hence we find that $\zeta \circ \nu \circ \phi(1,y,z) \cdot \phi(1,y,z)^{-1}$ is an element of $R^\times$. As $\zeta \circ \nu \circ \phi(1,y,z)$ is in $P_R(R)$, we deduce that $\phi(1,y,z) \in P_R(R)$. Let now (α,y,z) be an element of $R^\times \times T(R) \times T'(K)$. Then $\phi(\alpha,y,z) = \alpha\phi(1,y,z) \in P_R(R)$, which proves our first claim.

Finally let x be a point in $P_R(R)$, and let $(a,b) \in \mathcal{A}^0(R) \times \mathcal{A}'(R)$ be the projection to $\mathcal{A}^0 \times \mathcal{A}'$. Recall from the proof of 4.9 that the preimage of $\mathcal{A}^0(R)$ under the map $T(K) \to A(K) \overset{\sim}{\to} \mathcal{A}(R)$ is just $T(R)\Lambda$. Hence a has a preimage $y \in T(R)$. We choose any preimage $z \in T'(K)$ of b. Now $\phi(1,y,z)$ is an element of $P_R(R)$ projecting also to $(a,b) \in \mathcal{A}^0(R) \times \mathcal{A}'(R)$, hence there is an $\alpha \in R^\times$ such that $x = \alpha\phi(1,y,z) = \phi(\alpha,y,z)$, which proves our claim. □

Now we can compare our ρ-splitting to the canonical splitting of Mazur and Tate.

THEOREM 4.11. *Let ρ: $K^\times \to Y$ be unramified, and let Y be a (commutative) ring. If Y is uniquely divisible by m_A, then ρ is Λ-invertible. If ρ is Λ-invertible in such a way that $\rho = a\rho_1$ with $\det(\rho_1(\chi_j\lambda_i)_{i,j})$ a unit and a not a zero divisor in Y, then Y is uniquely divisible by m_A.*

Let us assume that Y is uniquely divisible by m_A, and put $\rho_0(x) = v(x) \cdot 1_Y$. Then $(\rho_0(\chi_j\lambda_i)_{i,j})$ has an inverse matrix $M \in \mathrm{Mat}_{n,n}(Y)$. Let τ: $P(K) \to Y$ be the ρ-splitting defined by

$$\tau(x) = \rho(\alpha) - (\rho_0(\chi_1 y), \ldots, \rho_0(\chi_n y))M^t(\rho(\lambda_1 z), \ldots, \rho(\lambda_n z)),$$

for an arbitrary preimage $(\alpha,y,z) \in K^\times \times T(K) \times T'(K)$ of $x \in P(K)$ under ϕ. Then τ is uniquely determined by the property that $\tau \circ \phi(1,y,z) = 0$ for all $y \in \ker \rho^n \subset T(K)$ and all $z \in T'(K)$.

Furthermore, the canonical Mazur–Tate splitting σ_ρ in case (II) *is equal to τ.*

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Proof. Let r be a prime element in R. The fact that ρ is unramified implies that ρ satisfies $\rho(x) = \rho(r)\rho_0(x)$. Assume now that Y is uniquely divisible by m_A. Then m_A is a unit in Y, and Lemma 4.9 implies that $\det(\rho_0(\chi_j\lambda_i)_{i,j}) = \det(v(\chi_j\lambda_i)_{i,j}) \cdot 1_Y$ is a unit in Y. Hence ρ is Λ-invertible. On the other hand, assume that there exists a homomorphism ρ_1: $K^\times \to Y$ and an element $a \in Y$ not dividing zero such that $\rho(x) = a\rho_1(x)$ and such that $\det(\rho_1(\chi_j\lambda_i)_{i,j})$ is a unit in Y. Then $\rho_1(x) = \rho_0(x)\rho_1(r)$, hence $\det(\rho_1(\chi_j\lambda_i)_{i,j}) = \rho_1(r)^n \det(\rho_0(\chi_j\lambda_i)_{i,j})$, which implies that $\det(\rho_0(\chi_j\lambda_i)_{i,j})$ is a unit in Y. From Lemma 4.9 we can now deduce that m_A is a unit in Y.

In order to show that τ is uniquely determined by the property above, by Theorem 4.5, (ii) it suffices to show that for all $y \in T(K)$ there exists a natural number $d_y \in Y^\times$ such that $d_y\rho^n(y) \in \rho^n(\Lambda)$. Put $d = \det(v(\chi_j\lambda_i)_{i,j})$. Then d is a unit in Y, as we have just seen. For any $y \in T(K)$ we can solve the system of linear equations $\Sigma_j q_j v(\chi_i(\lambda_j)) = v(\chi_i(y))$ for $i = 1, \dots, n$ with $q_1, \dots, q_n \in d^{-1}\mathbb{Z}$. Now it is easy to see that $d\rho^n(y) \in \rho^n(\Lambda)$. It remains to be shown that $\sigma_\rho = \tau$. As σ_ρ is the unique ρ-splitting vanishing on $P_R(R)$, it suffices to show that our ρ-splitting τ fulfills that condition. Let $x \in P_R(R)$. According to Lemma 4.10, we find a preimage (α, y, z) of x under ϕ in $R^\times \times T(R) \times T'(K)$. Then we have $\tau(x) = \tau^*(\alpha, y, z) = 0$, as $T(R) \subset \ker \rho^n$. □

Using Corollary 4.8, we get a formula for the canonical local height pairing in case (II) in terms of theta functions. As the homomorphism $\log|\ |_K$ which leads to Néron's local height pairing is unramified and the necessary divisibility conditions are satisfied in $Y = \mathbb{R}$, we get a formula for the local Néron height pairing on A, which coincides with the formula we derived from our description of the Néron map in 3.7. (This does not make our results in Section 3 superfluous because there we computed the whole Néron map, not only its restriction to $\mathrm{Div}^0(A)$.)

For a construction of a $K^\times$-valued pairing with different theta functions on an abelian variety from which one can deduce formulas for Néron's local height pairing and for a p-adic height pairing see [Né2], [Né3], and [Né4].

We will now derive a formula for Schneider's local p-adic height pairing on A. From now on we will assume that K is a finite extension of $\mathbb{Q}_l$ for some prime number l, and that ρ: $K^\times \to \mathbb{Q}_p$ is a non-trivial continuous homomorphism for some fixed prime number p. As we have seen in Section 1, ρ determines a $\mathbb{Z}_p$-extension K_∞/K with intermediate fields K_ν of degree p^ν over K such that $\rho(N_{K_\nu/K}K_\nu^\times) = p^\nu\rho(K^\times) \subset \mathbb{Q}_p$. Recall that if $l = p$, Schneider's local p-adic height pairing with respect to ρ is defined only under the condition that the group of universal norms $NA(K)$ has finite index in $A(K)$. We will first investigate this condition.

PROPOSITION 4.12. *If $l = p$, then ρ is Λ-invertible if and only if $NA(K)$ has finite index in $A(K)$.*

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Proof. (For Tate curves, this is also proved in [Na].) We may assume that ρ is not unramified: First of all note that ρ is unramified iff K_∞ is unramified over K. If this is the case, we find by 4.11 that ρ is Λ-invertible, and by [Ma–Ta], 1.11.6, p. 208, that the universal norm group has finite index.

Hence let us assume that ρ is not unramified. Let r is a prime element in R, so that $K^\times \simeq \langle r\rangle \times R^\times$. Since $R^\times$ is a compact subgroup of $K^\times$, it is mapped by ρ to a compact subgroup. Therefore $\rho(R^\times)$ is contained in $p^t\mathbb{Z}_p$ for some integer t, and since $\rho(R^\times)$ is closed and not zero, t can be chosen so that $\rho(R^\times) = p^t\mathbb{Z}_p$. Hence $\rho(K^\times) = \rho(r)\mathbb{Z} + p^t\mathbb{Z}_p = p^s\mathbb{Z}_p$ for some integer s.

For each intermediate field K_ν we write N_ν for the norm map $N_{K_\nu/K}$. The homomorphism ρ induces surjections ρ: $K^\times/N_\nu K_\nu^\times \longrightarrow \rho(K^\times)/p^\nu\rho(K^\times)$, since $\rho(N_\nu K_\nu^\times) = p^\nu\rho(K^\times)$. As both groups have the same cardinality, these maps are isomorphisms for all ν. Furthermore, we see that the kernel of ρ is equal to $\cap_\nu N_\nu K_\nu^\times = NG_m(K)$.

Now the preimage of $NA(K) = \cap_\nu N_\nu A(K_\nu)$ under the covering map π: $T(K) \to A(K)$ is $\cap_\nu(\Lambda N_\nu T(K_\nu))$. Hence π induces an isomorphism $\pi: T(K)/\cap_\nu(\Lambda N_\nu T(K_\nu)) \xrightarrow{\sim} A(K)/NA(K)$. Choose a basis $\chi_1,\ldots,\chi_n$ of the character group H. Via $(\chi_1,\ldots,\chi_n)$, $T(K)$ is isomorphic to $K^{\times n}$. Denote the induced lattice in $K^{\times n}$ by Λ_1. Then by definition, ρ is Λ-invertible, if and only if $\rho^n(\Lambda_1)$ contains a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p^n$, where ρ^n: $K^{\times n} \to \mathbb{Q}_p^n$ is the induced map. We get an isomorphism $K^{\times n}/\cap_\nu(\Lambda_1(N_\nu K_\nu^\times)^n) \xrightarrow{\sim} A(K)/NA(K)$. As the kernel of ρ^n is equal to $(\cap_\nu N_\nu K_\nu^\times)^n \subset \cap_\nu(\Lambda_1(N_\nu K_\nu^\times)^n)$, this induces an isomorphism

$$\frac{\rho^n(K^{\times n})}{\rho^n(\cap_\nu\Lambda_1(N_\nu K_\nu^\times)^n)} \xrightarrow{\sim} \frac{A(K)}{NA(K)}.$$

Note that $\rho^n(\cap_\nu\Lambda_1(N_\nu K_\nu^\times)^n) = \overline{\rho^n(\Lambda_1)}$, where $\overline{\rho^n(\Lambda_1)}$ denotes the p-adic closure of $\rho^n(\Lambda_1)$, so that

$$\rho^n(K^{\times n})/\overline{\rho^n(\Lambda_1)} \xrightarrow{\sim} A(K)/NA(K).$$

Now let us assume that $NA(K)$ has finite index in $A(K)$. Then, since $\rho^n(K^{\times n})$ contains a basis of $\mathbb{Q}_p^n$, the same holds for $\overline{\rho^n(\Lambda_1)}$ and hence also for $\rho^n(\Lambda_1)$. So ρ is Λ-invertible.

On the other hand, suppose that ρ is Λ-invertible. Then $\overline{\rho^n(\Lambda_1)}$ contains a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p^n$. Hence $\rho^n(K^{\times n})/\overline{\rho^n(\Lambda_1)}$ is a finitely generated torsion $\mathbb{Z}_p$-module, hence finite, which implies that $NA(K)$ has finite index in $A(K)$. □

Now we can calculate Schneider's p-adic height pairing.

THEOREM 4.13. *Let* ρ: $K^\times \to \mathbb{Q}_p$ *be a non-trivial continuous homomorphism and assume that in the case* $l = p$ *the group of universal norms* $NA(K)$ *has*

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finite index in $A(K)$. *Then the matrix* $(\rho(\chi_j\lambda_i)_{ij})$ *has an inverse matrix* $M \in \mathrm{Mat}_{n,n}(\mathbb{Q}_p)$. *Let* $\tau\colon P(K) \to \mathbb{Q}_p$ *be the* ρ*-splitting defined by*

$$\tau(x) = \rho(\alpha) - (\rho(\chi_1 y), \ldots, \rho(\chi_n y))M^t(\rho(\lambda_1 z), \ldots, \rho(\lambda_n z)),$$

for an arbitrary preimage $(\alpha, y, z) \in K^\times \times T(K) \times T'(K)$ *of* $x \in P(K)$. *Then* $(\ ,\ )_{MT,\tau}$ *is equal to Schneider's* p*-adic height pairing corresponding to* ρ.

Proof. First assume that $l \neq p$. As we have seen at the beginning of the proof of 4.12, $\rho(R^\times) = p^t\mathbb{Z}_p$ for some integer t, if $\rho(R^\times) \neq 0$. But $R^\times$ has no infinite pro-p-quotient, which implies that ρ is unramified. By Theorem 4.11, we find that τ is equal to the canonical Mazur–Tate splitting σ_ρ, which proves our claim.

We now treat the case $l = p$. The fact that $NA(K)$ has finite index in $A(K)$ implies that ρ in Λ-invertible, as we have seen in 4.12. Hence M exists. Schneider's local p-adic height pairing is equal to $(\ ,\ )_{MT,\sigma_\rho}$, where σ_ρ is the unique ρ-splitting vanishing on $NP(K)$. Hence it suffices to show that τ vanishes on $NP(K)$.

For all $(\alpha, y, z) \in K^\times \times T(K) \times T'(K)$ denote by $\tau^*(\alpha, y, z)$ again the right-hand side of the equation defining τ. Fix a point $(\alpha, y, z) \in K^\times \times T(K) \times T'(K)$ such that $\phi(\alpha, y, z)$ is in $NP(K)$. Furthermore, fix a natural number m such that the vector $M^{\,t}(\rho(\lambda_1 z), \ldots, \rho(\lambda_n z)) \in \mathbb{Q}_p^n$ is already contained in $m^{-1}\mathbb{Z}_p^n$. Bear in mind that m does not depend on α or y. We write N_ν for the norm map $N_{K_\nu/K}$. For all ν, the point $\phi(\alpha, y, z)$ is in $N_\nu P(K_\nu, K)$, i.e. there exists a point $x_\nu \in P(K_\nu)$, projecting to $A(K_\nu) \times A'(K)$, such that $N_\nu x_\nu = \phi(\alpha, y, z)$. Choose a preimage (α_ν, y_ν, z) of x_ν in $K_\nu^\times \times T(K_\nu) \times \{z\}$. Then $\phi N_\nu(\alpha_\nu, y_\nu, z) = N_\nu(x_\nu) = \phi(\alpha, y, z)$. Recall that $\rho(N_\nu K_\nu^\times) = p^\nu \rho(K^\times)$. Furthermore, we have seen in the proof of 4.12 that there is an integer s such that $\rho(K^\times) \subset p^s\mathbb{Z}_p$. Hence $\rho(N_\nu K_\nu^\times) \subset p^{\nu+s}\mathbb{Z}_p$. Then we derive for all $\nu \geqslant 1$

$$\begin{aligned} m\tau^*(\alpha, y, z) &= m\tau^*(N_\nu\alpha_\nu, N_\nu y_\nu, z) \\ &= m\rho(N_\nu\alpha_\nu) - m(\rho(N_\nu(\chi_1 y_\nu)), \ldots, \rho(N_\nu(\chi_n y_\nu))) \\ &\qquad \cdot M^t(\rho(\lambda_1 z), \ldots, \rho(\lambda_n z)) \\ &\in p^{\nu+s}\mathbb{Z}_p, \end{aligned}$$

since $\rho(N_\nu K_\nu^\times) \subset p^{\nu+s}\mathbb{Z}_p$ and $mM^t(\rho(\lambda_1 z), \ldots, \rho(\lambda_n z)) \subset \mathbb{Z}_p^n$. Therefore $\tau^*(\alpha, y, z) = 0$, which implies that τ vanishes on $NP(K)$. □

COROLLARY 4.14. *If* $l = p$, *assume that* $NA(K)$ *has finite index in* $A(K)$ (*or, equivalently, that* ρ *is* Λ*-invertible*). *Let* D *be in* $\mathrm{Div}^0(A)$ *and let* $z = \Sigma_{i=1}^k(a_i - b_i)$ *be a zero cycle with* K*-rational support which is disjoint from the support of* D. *Choose a theta function* Θ_D *corresponding to* D *and preimages* a_i', b_i' *of* a_i, b_i *in* $T(K)$. *Furthermore, define for all* $y \in T(K)$ *the vector* $(w_1(y), \ldots, w_n(y)) \in \mathbb{Q}_p^n$ *as the unique solution of the linear system of equations* $\Sigma_j w_j(y)\rho(\chi_i(\lambda_j)) =$

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$\rho(\chi_i(y))$ for $i = 1,\dots n$. Then we have the following formula for Schneider's p-adic height pairing

$$(D, z)_{MT,\sigma_\rho} = \rho\left(\prod_i \frac{\Theta_D(a_i')}{\Theta_D(b_i')}\right) - \sum_{j=1}^{n} w_j \left(\prod_i \frac{a_i'}{b_i'}\right) \rho(\psi_{\Theta_D}(\lambda_j)).$$

Proof. Immediate consequence of Corollary 4.8 and Theorem 4.13. □

If A is a Tate curve, and if

$$\rho = \begin{cases} \log_p \circ N_{K/\mathbb{Q}_l} & \text{if } l = p, \\ \log_p \circ |\ \ |_l \circ N_{K/\mathbb{Q}_l} & \text{if } l \neq p, \end{cases}$$

we get the formula proven in [Sch], p. 408.

Finally, let us briefly compare our ρ-splitting τ to the canonical ρ-splitting σ_ρ in case (III). First of all, the condition that ρ is Λ-invertible is not equivalent to the divisibility condition in case (III). But even if τ and σ_ρ exist, we should not expect them to coincide: As we have just seen, for continuous $\rho\colon K^\times \to \mathbb{Q}_p$ our τ gives rise to Schneider's p-adic height pairing, hence the corresponding height pairings do not even necessarily coincide on Tate curves, see [MTT], p. 34. The relation between τ and σ_ρ is the following: For any $x \in P(K)$ there exists a certain preimage $(\alpha, y, z) \in K^\times \times T(K) \times T'(K)$ of $x^{(m_A, m_{A'})}$ such that $\sigma_\rho(x) = (m_A m_{A'})^{-1}\rho(\alpha)$. Hence, by Theorem 4.13, the difference between σ_ρ and τ can be calculated via the bilinear term involving M.

5. The canonical Mazur–Tate splitting in the Archimedean Case

Using the result of Néron which we recalled at the beginning of Section 3, one can calculate Néron's local height pairing over an archimedean ground field via theta functions. By transcribing our arguments in Section 4 from the rigid analytic to the complex setting, we can do a bit more, namely prove a formula for Mazur and Tate's canonical ρ-splitting in case (I). From this we could reprove Néron's expression for his local height pairing with arguments analogous to those we used in Section 4.

So let A be an abelian variety over $\mathbb{C}$ of dimension n, such that $A(\mathbb{C}) = V/\Lambda$ for some n-dimensional vector space V and a lattice Λ in V. Let $\pi\colon V \to A(\mathbb{C})$ be the projection. Absolute values will from now on always be complex ones. Let $\rho\colon \mathbb{C}^\times \to Y$ be a homomorphism to an abelian group Y with $\rho(c) = 0$ whenever $|c| = 1$. We define $v(c) := \log|c|$. Recall from Section 1 that there is a unique homomorphism $r\colon \mathbb{R} \to Y$ such that $r \circ v = \rho$. Furthermore, there is a unique continuous v-splitting σ_v of $P(\mathbb{C})$, and the canonical ρ-splitting of $P(\mathbb{C})$ equals $\sigma_\rho = r \circ \sigma_v$. Hence, in order to derive a formula for σ_ρ, it suffices to treat the case $\rho = v$. The canonical ρ-splitting in case I) does also exist if the ground field

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is $\mathbb{R}$, not $\mathbb{C}$, but as canonical ρ-splittings behave well under finite base changes, it suffices to treat the complex case.

According to [Mu2], p. 86, the uniformization of the dual abelian variety A' can be described as follows: $A'(\mathbb{C}) = V'/\Lambda'$, where $V' = \mathrm{Hom}_{\mathbb{C}\text{-antilin}}(V, \mathbb{C})$, and $\Lambda' = \{l \in V'\colon \mathrm{Im}\, l(\lambda) \in \mathbb{Z} \quad \text{for all} \quad \lambda \in \Lambda\}$. Let π' be the corresponding map $V' \to A'(\mathbb{C})$, and let P be the canonical biextension of A and A' by $G_{m,\mathbb{C}}$. The pullback of $P(\mathbb{C})$ via $\pi \times \pi'\colon V \times V' \to A(\mathbb{C}) \times A'(\mathbb{C})$ is a trivial $\mathbb{C}^\times \times V \times V'$-torsor, since $H^1(V \times V', \mathcal{O}^\times) = 1$ (see [Mu2], p. 13). According to [Mu2], p. 86, $P(\mathbb{C})$ is the quotient of $\mathbb{C}^\times \times V \times V'$ for the action of $\Lambda \times \Lambda'$ given by

$$\begin{aligned} u_{(\lambda,\lambda')}(\alpha, z, z') \ = \ & (\alpha \exp(\pi[\overline{\lambda'(z)} + z'(\lambda) + \mathrm{Re}(\lambda'(\lambda))] \\ & -\pi i \ \mathrm{Im}\, \lambda'(\lambda)), \lambda + z, \lambda' + z'), \end{aligned}$$

for all $(\alpha, z, z') \in \mathbb{C}^\times \times V \times V'$ and $(\lambda, \lambda') \in \Lambda \times \Lambda'$. After multiplying by an element of $\mathbb{C}^\times$, we can assume that the quotient map $\phi\colon \mathbb{C}^\times \times V \times V' \to P(\mathbb{C})$ maps $(1, 0_V, 0_{V'})$ to $1_{P/A}(1_{A/\mathbb{C}})$. As in 4.2 one can show that ϕ is a morphism of biextensions. Define now $\tau^*\colon \mathbb{C}^\times \times V \times V' \longrightarrow \mathbb{R}$ by

$$\tau^*(\alpha, z, z') = v(\alpha) - v(\exp(\pi z'(z))).$$

Then we have

LEMMA 5.1. (i) τ^* *is a* v*-splitting of the trivial biextension* $\mathbb{C}^\times \times V \times V'$.

(ii) τ^* *is continuous.*

(iii) τ^* *is invariant under the action of* $\Lambda \times \Lambda'$ *given by* $u_{(\lambda,\lambda')}$.

Proof. (i) and (ii) are obvious.

(iii) We have

$$\begin{aligned} \tau^*(u_{(\lambda,\lambda')}(\alpha, z, z')) \ = \ & v(\alpha) + v(\exp[\pi\overline{\lambda'(z)} + \pi z'(\lambda) + \pi\, \mathrm{Re}\, \lambda'(\lambda) \\ & -\pi i \ \mathrm{Im}\, \lambda'(\lambda)]) - v(\exp[\pi(\lambda' + z')(\lambda + z)]) \\ = \ & \tau^*(\alpha, z, z') + v(\exp[-2\pi i \ \mathrm{Im}\, \lambda'(z)]) \\ & +v(\exp[-2\pi i \ \mathrm{Im}\, \lambda'(\lambda)]) \\ = \ & \tau^*(\alpha, z, z'), \end{aligned}$$

since $\mathrm{Im}\, \lambda'(\lambda) \in \mathbb{Z}$, which leads to the vanishing of the third term. The second term vanishes as $|\exp(-2\pi i \ \mathrm{Im}\, \lambda'(z))| = 1$. □

From this we deduce immediately

THEOREM 5.2. *The map* $\tau\colon P(\mathbb{C}) \longrightarrow \mathbb{R}$ *given by*

$$x \longmapsto \tau^*(\alpha, z, z') = v(\alpha) - v(\exp[\pi z'(z)]),$$

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where $(\alpha, z, z') \in \mathbb{C}^\times \times V \times V'$ *is an arbitrary preimage of* $x \in P(\mathbb{C})$ *under* ϕ*, is a continuous* v*-splitting of* $P(\mathbb{C})$*. Hence* τ *is equal to the canonical* v*-splitting of Mazur and Tate.*

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References

[Bl] Bloch, S.: A note on height pairings, Tamagawa numbers, and the Birch and Swinnerton-Dyer conjecture *Invent. Math.* 58 (1980) 65–76.

[Bo] Bosch, S.: Meromorphic functions on proper rigid analytic varieties, Sém. Théorie des Nombres de Bordeaux, Année 1982–1983, exp. 34.

[BGR] Bosch, S., Güntzer, U., Remmert, R.: *Non-Archimedean analysis*, Springer 1984.

[Bo-Lü1] Bosch, S., Lütkebohmert, W.: Stable reduction and uniformization of abelian varieties II, *Invent. Math.* 78 (1984) 257–297.

[Bo-Lü2] Bosch, S. and Lütkebohmert, W.: Néron models from the rigid analytic viewpoint, *J. Reine Angew. Math.* 364 (1986) 69–84.

[Bo-Lü3] Bosch, S. and Lütkebohmert, W.: Degenerating abelian varieties, Topology 30, No.4 (1991) 653–698.

[EGA III] Grothendieck, A.: Éléments de géométrie algébrique III, *Publ. Math. IHES* 11 (1961), 17 (1963).

[EGA IV] Grothendieck, A.: Éléments de géométrie algébrique IV, *Publ. Math. IHES* 20 (1964), 24 (1965), 28 (1966), 32 (1967).

[Ge] Gerritzen, L.: On non-archimedean representations of abelian varieties, *Math. Ann.* 196 (1972) 323–346.

[Hi] Hindry, M.: Sur les hauteurs locales de Néron sur les variétés abéliennes, Prépublications Mathématiques de l'U.R.A. 212 '*Théories Géométriques*' no. 51, Université Paris 7 1993.

[Kö] Köpf, U.: Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen, Schriftenreihe Math. Inst. Univ. Münster, 2. Serie, Heft 7.

[La1] Lang, S.: *Introduction to Algebraic and Abelian Functions*, Addison-Wesley 1972.

[La2] Lang, S.: Elliptic Curves: *Diophantine Analysis*, Springer 1978.

[Ma] Manin, Y. I.: P-adic automorphic functions, *J. Soviet Math.* 5 (1976) 279–333.

[Ma-Ta] Mazur, B., Tate, J.: Canonical height pairings via biextensions, In: Arithmetic and Geometry, Papers dedicated to I.R. Shafarevich, *Progress in Mathematics* 35, Birkhäuser 1983 195–235.

[MTT] Mazur, B., Tate, J. and Teitelbaum, J.: On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer, *Invent. Math.* 84 (1986) 1–48.

[Mu1] Mumford, D.: *Geometric Invariant Theory*, Springer 1965.

[Mu2] Mumford, D.: *Abelian Varieties*, Oxford University Press 1970.

[Na] Nasybullin, A. G.: Elliptic Tate curves over local Γ-extensions, *Math. Notes USSR* 13 (1973) 322–327.

[Né1] Néron, A.: Quasi-fonctions et hauteurs sur les variétés abéliennes, *Ann. Math.* 82 (1965) 249–331.

[Né2] Néron, A.: Hauteurs et fonctions thêta, *Rendiconti del Sem. Math. Milano* 46 (1976) 111–135.

[Né3] Néron, A.: Fonctions thêta p-adiques et hauteurs p-adiques, Sém. Théorie des Nombres Paris 1980–81, *Progress in Mathematics* 22, Birkhäuser 1982 149–174.

[Né4] Néron, A.: Fonctions thêta p-adiques, *Symposia Mathematica* 24 (1981) 315–345.

[Ray] Raynaud, M.: Variétés abéliennes et géométrie rigide, *Actes, Congrès intern. math.* 1970, Tome 1 473–477.

[Ro] Rosen, M.: Abelian varieties over $\mathbb{C}$. In: Cornell, G., Silverman, J. H. (eds.): *Arithmetic Geometry*, Springer 1986 79–101.

[Sch] Schneider, P.: P-adic height pairings I, *Invent. Math.* 69 (1982) 401–409.

[SGA3] Grothendieck, A. and Demazure, M. et al.: Séminaire de géométrie algébrique du Bois-Marie 1962/64, Schémas on groupes, *Lecture Notes in Mathematics* 151, 152, 153, Springer 1970.

[SGA7, I] Grothendieck, A. et al.: Séminaire de géométrie algébrique du Bois-Marie 1967/69, Groupes de monodromie en géométrie algébrique I, *Lecture Notes in Mathematics* 288, Springer 1972.

[Si] Silverman, J.H.: *The Arithmetic of Elliptic Curves*, Springer 1986.

[Tu] Turner, S.: Nonarchimedean theta functions and the Néron symbol, *J. Number Th.* 21 (1985) 1–16.