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HAMILTON SEQUENCES FOR EXTREMAL QUASICONFORMAL MAPPINGS OF DOUBLY-CONNECTED DOMAINS

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Abstract

Let $T(S)$ be the Teichmüller space of a hyperbolic Riemann surface S. Suppose that μ is an extremal Beltrami differential at a given point τ of $T(S)$ and $\{\phi_n\}$ is a Hamilton sequence for μ. It is an open problem whether the sequence $\{\phi_n\}$ is always a Hamilton sequence for all extremal differentials in τ. S. Wu ['Hamilton sequences for extremal quasiconformal mappings of the unit disk', *Sci. China Ser. A* **42** (1999), 1033–1042] gave a positive answer to this problem in the case where S is the unit disc. In this paper, we show that it is also true when S is a doubly-connected domain.

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1. Introduction

Let S be a Riemann surface whose universal covering surface is the unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and let S be represented by a Fuchsian group Γ acting on Δ as $S = \Delta/\Gamma$. Let $QC(S)$ be the space of all quasiconformal mappings f from R to a variable Riemann surface $f(S)$. The Teichmüller space $T(S)$ is the space of these mappings factored by an equivalence relation. A quasiconformal mapping f in $QC(S)$ can be lifted to a quasiconformal mapping $\widetilde{f}$ from Δ onto itself. Two mappings, f and g, are equivalent (and therefore their Beltrami differentials are called equivalent) if there exist lifts $\widetilde{f}, \widetilde{g}$ of f, g such that $\widetilde{f}$ agrees with $\widetilde{g}$ on $\partial\Delta$. Let $[f]$ or $[\mu]$ denote the equivalence class of a quasiconformal mapping f in $QC(S)$, where μ is the Beltrami differential of f. Since the Beltrami differential μ uniquely determines the mapping f up to postcomposition by a conformal mapping, the Teichmüller space $T(S)$ may be represented as the space of equivalence classes of Beltrami differentials μ in the unit ball $M(S)$ of the space $L^\infty(S)$. The equivalence class of the Beltrami differential zero is the basepoint of $T(S)$.

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Given $f \in QC(S)$, let $\mu \in$ be the Beltrami differential of f. We define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

A quasiconformal mapping f of S onto $f(S)$ is said to be extremal in its class $[f]$ if its Beltrami differential μ is extremal in $[\mu]$, that is, $\|\mu\|_\infty = k_0([\mu])$. Note that $[\mu]$ may contain more than one extremal element.

Let $\mathcal{A}(\Gamma)$ denote the Banach space of holomorphic quadratic differentials ϕ on $S = \Delta/\Gamma$ with L^1-norm

$$\|\phi\|_S = \iint_S |\phi(z)|\, dx\, dy < \infty.$$

Let $\mathcal{A}_1(\Gamma)$ denote the unit sphere of $\mathcal{A}(\Gamma)$. In particular, we denote by $\mathcal{A}(1)$ the the Banach space of integrable holomorphic quadratic differentials on Δ.

The following theorem due to Hamilton, Kruškál, Reich and Strebel is a characterisation of extremal quasiconformal mappings (see [2]).

Theorem A. *A quasiconformal mapping f of S is extremal if and only if its Beltrami differential μ has a so-called Hamilton sequence $\{\phi_n : \phi_n \in \mathcal{A}_1(\Gamma)\}$ such that*

$$\lim_{n\to\infty}\left|\iint_S \mu(z)\phi_n(z)\, dx\, dy\right| = \|\mu\|_\infty.$$

It is known that there exists at least a common Hamilton sequence formed by Strebel differentials for all extremal differentials in $[\mu]$ (see [1, 3]). Suppose μ is an extremal Beltrami differential in its class $[\mu]$ and $\{\phi_n\}$ is a Hamilton sequence for μ. The following question was posed by Li in [3].

Problem. Is the sequence $\{\phi_n\}$ always a Hamilton sequence for all extremal differentials in $[\mu]$?

The problem is of interest only when $T(S)$ is infinite-dimensional. Up to now, we have an affirmative answer, given by Wu [5], only when $S = \Delta$.

Theorem B. *Let $[\mu]$ be in $T(\Delta)$ where μ is an extremal differential. Then a Hamilton sequence $\{\phi_n\}$ for μ is a Hamilton sequence for all extremal differentials in $[\mu]$.*

The aim of this paper is to show that the answer is also positive when S is a doubly-connected domain. Up to conformal mappings, we may assume that S is either $\Delta^* = \Delta\backslash\{0\}$ or a ring domain $U_r = \{z \in \mathbb{C} :\ 1 < |z| < r\}$ for some $r > 1$.

Theorem 1.1. *Let S be a doubly-connected domain in the complex plane. Suppose that μ is an extremal Beltrami differential at a point τ of $T(S)$ and $\{\phi_n\}$ is a Hamilton sequence for μ. Then the sequence $\{\phi_n\}$ is a Hamilton sequence for all extremal differentials in τ.*

2. Proof of Theorem 1.1

Let $S = \Delta/\Gamma$ be a doubly-connected domain. Suppose that $f, g \in [f]$ are two extremal quasiconformal mappings from S onto $f(S)$, $g(S)$, respectively. Let μ and ν be the Beltrami differentials of f and g, respectively. Let $\widetilde{f}$ and $\widetilde{g}$ be their lifts such that $\widetilde{f}|_{\partial\Delta} = \widetilde{g}|_{\partial\Delta}$; accordingly, let $\widetilde{\mu}$ and $\widetilde{\nu}$ be the lifts of μ, ν, that is, they are the Beltrami differentials of $\widetilde{f}, \widetilde{g}$, respectively. Since the covering transformation group Γ is an Abelian group generated by a conformal self-mapping of Δ, $\widetilde{f}$ and $\widetilde{g}$ are still extremal in the class $[\widetilde{f}]$ by [4, Theorem 1].

It is well known that the lift $\widetilde{\mu}$ of μ satisfies (as does $\widetilde{\nu}$) the Γ-invariance condition

$$(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \quad \text{for all } \gamma \in \Gamma.$$

Let ϕ be an element of $\mathcal{A}(\Gamma)$ and $\widetilde{\phi}(z)\, dz^2$ be the lift of ϕ. Then $\widetilde{\phi}$ satisfies

$$\widetilde{\phi}(\gamma(z))[\gamma'(z)]^2 = \widetilde{\phi}(z), \quad \gamma \in \Gamma, z \in \Delta.$$

On the other hand, there exists a holomorphic quadratic differential $\Phi(z)\, dz^2 \in \mathcal{A}(1)$ such that the Poincaré series of Φ,

$$\Theta_\Gamma \Phi(z) = \sum_{\gamma\in\Gamma} \Phi(\gamma(z))[\gamma'(z)]^2,$$

is equal to $\widetilde{\phi}$ (see [2, Ch. 4, Theorem 3]). For every $\phi \in \mathcal{A}_1(\Gamma)$, define

$$I(\phi) = \inf\{\|\Phi\|_\Delta : \Theta_\Gamma\Phi = \widetilde{\phi}, \Phi \in \mathcal{A}(1)\}.$$

Since Γ is also an infinite cyclic group, [4, Lemma 3] tells us that $I(\phi) \equiv 1$ for all $\phi \in \mathcal{A}_1(\Gamma)$.

Now, assuming that $\{\phi_n : \phi_n \in \mathcal{A}_1(\Gamma)\}$ is a Hamilton sequence for μ, we need to prove that

$$\lim_{n\to\infty} \left| \iint_S \nu(z)\phi_n(z)\, dx\, dy \right| = \|\nu\|_\infty = k_0([\mu]).$$

Let Ω be a fundamental region for Γ in Δ. Let $\widetilde{\phi}_n\, dz^2$ be the lift of ϕ_n. Then

$$\iint_S \mu(z)\phi_n(z)\, dx\, dy = \iint_\Omega \widetilde{\mu}(z)\widetilde{\phi}_n(z)\, dx\, dy. \tag{2.1}$$

Since $I(\phi_n) \equiv 1$, we can choose $\Phi_n(z)\, dz^2 \in \mathcal{A}(1)$ such that $\Theta_\Gamma\Phi_n = \widetilde{\phi}_n$ and

$$\|\Phi_n\|_\Delta = 1 + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{2.2}$$

We easily derive

$$\begin{aligned} \iint_\Omega \widetilde{\mu}(z)\widetilde{\phi}_n(z)\, dx\, dy &= \sum_{\gamma\in\Gamma} \iint_\Omega \widetilde{\mu}(z)\Phi_n(\gamma(z))[\gamma'(z)]^2\, dx\, dy \\ &= \sum_{\gamma\in\Gamma} \iint_{\gamma(\Omega)} \widetilde{\mu}(z)\Phi_n(z)\, dx\, dy = \iint_\Delta \widetilde{\mu}(z)\Phi_n(z)\, dx\, dy. \end{aligned} \tag{2.3}$$

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Thus, combining (2.1)–(2.3),

$$\begin{aligned}\|\mu\|_\infty &= \lim_{n\to\infty}\iint_S \mu(z)\phi_n(z)\,dx\,dy = \lim_{n\to\infty}\iint_\Delta \widetilde{\mu}(z)\Phi_n(z)\,dx\,dy\\ &= \lim_{n\to\infty}\iint_\Delta \widetilde{\mu}(z)\frac{\Phi_n(z)}{\|\Phi_n\|_\Delta}\,dx\,dy,\end{aligned}$$

which indicates that $\{\Phi_n(z)/\|\Phi_n\|_\Delta\}$ is a Hamilton sequence for $\widetilde{\mu}$. Furthermore, by Theorem B, it is also a Hamilton sequence for $\widetilde{\nu}$. Therefore, by the same reasoning as in deriving (2.3),

$$\begin{aligned}\|\nu\|_\infty &= \lim_{n\to\infty}\iint_\Delta \widetilde{\nu}(z)\frac{\Phi_n(z)}{\|\Phi_n\|_\Delta}\,dx\,dy = \lim_{n\to\infty}\iint_\Delta \widetilde{\nu}(z)\Phi_n(z)\,dx\,dy\\ &= \lim_{n\to\infty}\sum_{\gamma\in\Gamma}\iint_{\gamma(\Omega)} \widetilde{\nu}(z)\Phi_n(z)\,dx\,dy = \lim_{n\to\infty}\sum_{\gamma\in\Gamma}\iint_\Omega \widetilde{\nu}(z)\Phi_n(\gamma(z))[\gamma'(z)]^2\,dx\,dy\\ &= \lim_{n\to\infty}\iint_\Omega \widetilde{\nu}(z)\widetilde{\phi}_n(z)\,dx\,dy = \lim_{n\to\infty}\iint_S \nu(z)\phi_n(z)\,dx\,dy,\end{aligned}$$

that is, $\{\phi_n\}$ is also a Hamilton sequence for ν. This completes the proof of Theorem 1.1.

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