A COUNTEREXAMPLE IN THE THEORY OF DERIVATIONS

by FENG WENYING and JI GUOXING

(Received 18 November, 1987)

Let $B(H)$ be the algebra of all bounded linear operators on a separable, infinite dimensional complex Hilbert space H. Let C_2 and C_1 denote respectively, the Hilbert–Schmidt class and the trace class operators in $B(H)$. It is known that C_2 and C_1 are two-sided*-ideals in $B(H)$ and C_2 is a Hilbert space with respect to the inner product

$$(X, Y) = \operatorname{tr}(Y^*X) \qquad (X, Y \in C_2),$$

(where tr denotes the trace). For any Hilbert–Schmidt operator X let $\|X\|_2 = (X, X)^{1/2}$ be the Hilbert–Schmidt norm of X.

For fixed $A \in B(H)$ let δ_A be the operator on $B(H)$ defined by

$$\delta_A(X) = AX - XA \qquad (X \in B(H)). \tag{1}$$

Operators of the form (1) are called *inner derivations* and they (as well as their restrictions $\delta_A|_{C_2}$) have been extensively studied (for example [**1–3**]). In [**1**], Fuad Kittaneh proved the following result.

THEOREM K. *If A is a cyclic subnormal operator and $S \in C_2$ is an operator such that $AS = SA$, then for all $X \in B(H)$ we have*

$$\|AX - XA + S\|_2^2 = \|AX - XA\|_2^2 + \|S\|_2^2.$$

Hence the range of $\delta_A|_{C_2}$ is orthogonal to the null space of $\delta_A|_{C_2}$ in the usual Hilbert space sense.

In this paper, we give an example which provides an affirmative answer to the following question in [**1**].

QUESTION 1. Is it necessary to assume A cyclic in Theorem K?

First we prove a lemma.

LEMMA. *If A is an operator in $B(H)$ and $S \in C_2$ with $AS = SA$, and for all $X \in B(H)$.*

$$\|AX - XA + S\|_2^2 = \|AX - XA\|_2^2 + \|S\|_2^2,$$

then $AS^* = S^*A$.

Proof. $\delta_A|_{C_2}$ is a bounded linear operator acting on the Hilbert space C_2 and $(\delta_A|_{C_2})^* = \delta_{A^*}|_{C_2}$. Noting that $\overline{R(\delta_A|_{C_2})}^{\perp} = N(\delta_{A^*}|_{C_2})$, we have $N(\delta_A|_{C_2}) \subset N(\delta_{A^*}|_{C_2})$; therefore $AS^* = S^*A$. (Here $R(\delta_A|_{C_2})$ and $N(\delta_A|_{C_2})$ denote respectively the range of $\delta_A|_{C_2}$ and the null space of $\delta_A|_{C_2}$).

We give an example showing that A is necessarily cyclic in Theorem K.

EXAMPLE. Let $\{e_n\}_{n=-\infty}^{+\infty}$ be an orthonormal basis for a Hilbert space H and let $\{c_n\}_{n=-\infty}^{+\infty}$ and $\{d_n\}_{n=-\infty}^{+\infty}$ be bounded sequences of positive numbers as follows:

$$c_n = \begin{cases} \dfrac{n+1}{n+2} & (n \geqslant 0), \\ \dfrac{1}{2-n} & (n < 0); \end{cases}$$

$$d_n = \begin{cases} c_{n-1} + \dfrac{1}{2(n+2)(n+1)} = \dfrac{2n^2+4n+1}{2(n+2)(n+1)} & (n \geqslant 1), \\ c_{n-1} + \dfrac{1}{2(3-n)(2-n)} = \dfrac{5-2n}{2(3-n)(2-n)} & (n \leqslant 0); \end{cases}$$

then for each integer n, $c_{n+1} < d_n < c_n$. Let P_1 and P_2 be operators in $B(H)$ defined by $P_1 e_n = c_n e_n$ and $P_2 e_n = d_n e_n$, for each integer n. Let $\hat{H} = \sum_{n=1}^{+\infty} \oplus H_n$, $\hat{P}_1 = \sum_{n=1}^{+\infty} \oplus P_n$ and $\hat{P}_2 = \sum_{n=1}^{+\infty} \oplus P'_n$, where for each n, $H_n = H$, $P_n = P_1$ and $P'_n = P_2$. Let V_H denote the unilateral shift on $\hat{H}$, i.e., $V_H(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for each $(x_1, x_2, \ldots)$ in $\hat{H}$. Let $T_1 = V_H \hat{P}_1$ and $T_2 = V_H \hat{P}_2$ on $\hat{H}$. The operators T_1 and T_2 are pure quasinormal operators by **[4]**, and so they are pure subnormal operators.

For each positive integer m define X_m and Y_m in $B(H)$ by

$$X_m e_n = \frac{1}{2^{|n|}} \left(\frac{d_n}{c_n} \right)^m e_n$$

and

$$Y_m e_n = \frac{1}{2^{|n|}} \left(\frac{c_{n-1}}{d_n} \right)^m e_{n-1},$$

for $n = \ldots -2, -1, 0, 1, 2, \ldots$. Observe that X_m and Y_m are compact $(m = 1, 2, \ldots)$, that $\|X_m\| \to 0$ and $\|Y_m\| \to 0$. Also $X_m P_1 = P_2 X_{m-1}$, $Y_m P_2 = P_1 Y_{m-1}$, $(m = 2, 3, \ldots)$.

Let $X = \sum_{m=1}^{+\infty} \oplus X_m$ and $Y = \sum_{m=1}^{+\infty} \oplus Y_m$ on $\hat{H}$. Thus X and Y are compact, $XT_1 = T_2 X$, $T_1 Y = Y T_2$; hence $(YX)T_1 = YT_2 X = T_1(YX)$.

Next, we shall prove that X is a Hilbert–Schmidt operator.

Let v_{ij} have e_i in the jth position, zeros elsewhere for $j = 1, 2, \ldots$ and $i = \ldots, -2, -1, 0, 1, 2, \ldots$; thus $\{v_{ij}\}$ is an orthonormal basis for the Hilbert space $\hat{H}$. From the definition of X, we have

$$Xv_{ij} = \left(\ldots, 0, \ldots, 0, \frac{1}{2^{|i|}} \left(\frac{d_i}{c_i} \right)^j e_i, 0, \ldots \right).$$

$$\|Xv_{ij}\| = \frac{1}{2^{|i|}}\left(\frac{d_i}{c_i}\right)^j,$$

$$\begin{aligned}\sum_{i,j}\|Xv_{ij}\| &= \sum_{i=-\infty}^{+\infty}\sum_{j=1}^{+\infty}\frac{1}{2^{|i|}}\left(\frac{d_i}{c_i}\right)^j \\ &= \sum_{i=-\infty}^{+\infty}\frac{1}{2^{|i|}}\frac{d_i}{c_i-d_i} \\ &= \sum_{i=-\infty}^{0}\frac{1}{2^{|i|}}\frac{d_i}{c_i-d_i}+\sum_{i=1}^{+\infty}\frac{1}{2^i}\frac{d_i}{c_i-d_i} \\ &= \sum_{i=0}^{+\infty}\frac{1}{2^i}\frac{d_{-i}}{c_{-i}-d_{-i}}+\sum_{i=1}^{+\infty}\frac{1}{2^i}\frac{d_i}{c_i-d_i} \\ &= \sum_{i=0}^{+\infty}\frac{2i+5}{2^i}+\sum_{i=1}^{+\infty}\frac{2i^2+4i+1}{2^i}<+\infty.\end{aligned}$$

Thus X is a Hilbert–Schmidt operator, and so is YX. Note that the operator YX is compact and $(YX)T_1 = T_1(YX)$, but T_1 is a pure subnormal operator; therefore $(YX)^*T_1 \neq T_1(YX)^*$. From the lemma, we know that Theorem K does not hold for T_1; thus if Theorem K holds, then A must by cyclic.

The example above gives an affirmative answer to Question 1.

REFERENCES

1. Fuad Kittaneh, On normal derivations of Hilbert–Schmidt type, *Glasgow Math. J.* **29** (1987), 245–248.

2. J. H. Anderson, On normal derivations, *Proc. Amer. Math. Soc.* **38** (1973), 135–140.

3. L. A. Fialkow, A note on norm ideals and the operator $X \rightarrow AX\text{-}XB$, *Israel J. Math.* **32,** (1979), 331–348.

4. L. R. Williams, Quasisimilarity and hyponormal operators, *J. Operator Theory* **5** (1981), 127–139.

Department of Mathematics
Shaanxi Normal University
Xi'an
People's Republic of China