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DUALIZING MODULES AND n-PERFECT RINGS

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Abstract In this article we extend the results about Gorenstein modules and Foxby duality to a non-commutative setting. This is done in § 3 of the paper, where we characterize the Auslander and Bass classes which arise whenever we have a dualizing module associated with a pair of rings. In this situation it is known that flat modules have finite projective dimension. Since this property of a ring is of interest in its own right, we devote § 2 of the paper to a consideration of such rings. Finally, in the paper's final section, we consider a natural generalization of the notions of Gorenstein modules which arises when we are in the situation of § 3, i.e. when we have a dualizing module.

Keywords: n-perfect ring; dualizing module; Gorenstein modules

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1. Introduction

Grothendieck in [**11**] introduced the notion of a dualizing complex for a commutative Noetherian ring R. A dualizing module (or canonical module) for R is a module whose deleted injective resolution is a dualizing complex for R. Gorenstein local rings are precisely those local rings for which R itself is a dualizing module for R. All these notions have been extensively developed and applied in commutative algebra and algebraic geometry. In recent years, it has become clear that all these notions have useful non-commutative versions which can be applied in various areas of algebra such as the theory of modular group representations. Bass had noted that Gorenstein local rings are precisely those whose self injective dimension is finite (see [**2**]). But integral group rings of finite groups (and, in fact, of many infinite groups) have the same property. So these rings can also be said to serve as their own dualizing modules.

In this article we consider a general notion of a dualizing bimodule for a pair of rings S and R. We use this terminology since we want to generalize the main result of [**9**] to the non-commutative setting, that is, we want to characterize the Auslander and Bass classes which arise in this situation in terms of the Gorenstein injective and projective dimensions. We note that the language of cotilting theory is also appropriate in this setting. An important property of modules over rings admitting a dualizing module is that flat modules have finite projective dimensions (cf. [**14**, Corollary 3.2.7] or [**15**, Theorem 4.2.8]). In § 2 we introduce a class of rings with this property.

2. n-perfect rings

Left perfect rings R are characterized in Bass's Theorem P as those rings such that every left flat R-module is projective (cf. [**2**]). In this sense the following definition is an extension of that of left perfect rings.

Definition 2.1. A ring R is said to be left (right) n-perfect if every left (right) flat R-module has projective dimension less or equal than n.

Example 2.2. As we have noted before, it is immediate that left perfect rings are left 0-perfect, and so a ring may be left n-perfect and not right n-perfect (see [**1**, Exercise 2, p. 322]).

Example 2.3. If R is a local commutative Noetherian ring of Krull dimension d, then R is a left (and right) d-perfect ring (cf. [**15**, Theorem 4.2.8]).

Example 2.4. Let R be a left Noetherian ring such that $\mathrm{id}({}_RR) \leqslant n$. Then, by [**6**, Proposition 9.1.2], R is left n-perfect. In particular, if R is n-Gorenstein, i.e. R is left and right Noetherian and $\mathrm{id}({}_RR), \mathrm{id}(R_R) \leqslant n$, then R is left and right n-perfect.

Example 2.5. If R is a left n-perfect ring, then $R[x]$ is $(n+1)$-perfect. To show this, let F be a flat left $R[x]$-module. Then $F = \varinjlim P_i$, where $P_i \in R[x]$-Mod are projective and so are direct summands of a free $R[x]$-module. Now since $R[x]$ is a free left R-module, it follows that $F = \varinjlim P_i$, with P_i projective in R-Mod. But then $F[x] = \varinjlim P_i[x]$, where $P_i[x]$ are projective in $R[x]$-Mod, and so $F[x]$ is flat in $R[x]$-Mod.

Now suppose that $\mathrm{pd}(F) \leqslant n$ and let

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0,$$

with each P_i projective. Then

$$0 \to P_n[x] \to \cdots \to P_1[x] \to P_0[x] \to F[x] \to 0$$

shows that $\mathrm{pd}(F[x]) \leqslant n$ in $R[x]$-Mod, and since we have an exact sequence

$$0 \to F[x] \to F[x] \to F \to 0,$$

it follows that $\mathrm{pd}(F) \leqslant n+1$ in $R[x]$-Mod.

Example 2.6. Let R be a filtered ring and $G(R)$ be its associated graded ring. If $G(R)$ is left n-perfect, then R is n-perfect. Let $F \in R$-Mod be a flat module. Then $F = \varinjlim P_i$, with P_i finitely generated projective R-modules. Now, by [**12**, Proposition I.4.2.2], if we take a good filtration on each P_i, we get that $G(F) = G(\varinjlim P_i) = \varinjlim G(P_i)$, and so, using Proposition I.6.5 and Lemma I.5.4 of [**12**], $G(F)$ is flat in $G(R)$-gr. Therefore, there exists an exact sequence in $G(R)$-gr

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to G(F) \to 0,$$

where each P_i is projective in R-Mod. Now we use Eilenberg's trick, i.e. if P is projective, then there is a free module X such that $P \oplus X$ is free (if $P \oplus P'$ is free, take $X = P' \oplus P \oplus P' \oplus \cdots$), and so we can take the direct sum of the preceding complex with complexes of the form

$$0 \to \cdots \to 0 \to X \xrightarrow{\text{id}} X \to 0 \to \cdots \to 0,$$

where X is a gr-free graded $G(R)$-module, to get a projective resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to G(F) \to 0,$$

where every F_i is gr-free. Now, by [**12**, Lemma 6.2.4], there exist filtered free R-modules H_i such that $G(H_i) = F_i$ for all $i = 1, \dots, n$, and so, by [**12**, Lemma 6.2.6], we get an exact sequence in R-Mod

$$0 \to H_n \to \cdots \to H_1 \to H_0 \to F \to 0,$$

which shows that $\mathrm{pd}(F) \leqslant n$.

Now, from Examples 2.5 and 2.6, it is possible to get numerous examples of classical rings that are left n-perfect.

3. Gorenstein modules

In this section we characterize Gorenstein modules in terms of the so-called Auslander and Bass classes and generalize results obtained in [**9**]. We point out that similar results have been recently obtained for the commutative case in [**4**].

We recall from [**5**] that a left R-module N is said to be Gorenstein injective if there exists an exact sequence of injective left R-modules $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ such that $N = \mathrm{Ker}(E^0 \to E^1)$ and that remains exact when $\mathrm{Hom}_R(E, \cdot)$ is applied for every injective left R-module E. Gorenstein projective modules are defined dually.

Definition 3.1. Let R and S be right and left Noetherian rings, respectively, and let ${}_SV_R$ be an S–R-bimodule such that $\mathrm{End}_S(V) = R$ and $\mathrm{End}_R(V) = S$. Then V is said to be a dualizing module if it satisfies the following three conditions.

(i) $\mathrm{id}({}_SV) \leqslant r$ and $\mathrm{id}(V_R) \leqslant r$.

(ii) $\mathrm{Ext}^i_R(V, V) = \mathrm{Ext}^i_S(V, V) = 0$ for all $i \geqslant 1$.

(iii) ${}_SV$ and V_R are finitely generated.

Example 3.2. If R is a Cohen–Macaulay local ring of Krull dimension d admitting a dualizing module Ω (see [**9**]), then Ω is a dualizing module in this sense.

Example 3.3. If R is an n-Gorenstein (not necessarily commutative), then ${}_RR_R$ is a dualizing module.

Example 3.4. Let R and S be strongly graded rings over finite groups, right and left Noetherian, respectively, and let ${}_{S_e}V_{R_e}$ be a dualizing module (for R_e and S_e). Then $W = S \otimes_{S_e} V \otimes_{R_e} R$ is a dualizing module (for R and S). Let us show this:

$$\begin{aligned} \mathrm{End}_S(W) &= \mathrm{Hom}_S(W, W) \\ &\cong \mathrm{Hom}_{S\text{-gr}}(W, W) \otimes_{R_e} R \\ &\cong \mathrm{Hom}_{S_e}(V, V) \otimes_{R_e} R \\ &\cong R_e \otimes_{R_e} R \\ &\cong R. \end{aligned}$$

Analogously, $\mathrm{End}_R(W) = S$. If $0 \to V \to E^0 \to \cdots \to E^r \to 0$ is an injective resolution of V, then $0 \to V \otimes_{R_e} R \to E^0 \otimes_{R_e} R \to \cdots \to E^r \otimes_{R_e} R \to 0$ is an injective resolution of W_R and, analogously, $\mathrm{id}({}_SW) \leqslant n$.

$$\mathrm{Ext}^i_S(W, W) \cong \mathrm{Ext}^i_{S\text{-gr}}(W, W) \otimes_{R_e} R \cong \mathrm{Ext}^i_{S_e}(V, V) = 0$$

for all $i \geqslant 1$ and, analogously, $\mathrm{Ext}^i_R(W, W) = 0$ for all $i \geqslant 1$. Finally, it is immediate that W_R and ${}_SW$ are finitely generated.

Example 3.5. Let R and S be right and left Noetherian rings and let ${}_SV_R$ be a dualizing module. Then ${}_{S[\![x]\!]}V[\![x]\!]_{R[\![x]\!]}$ is a dualizing module. Let us show this. First, it is immediate that ${}_{S[\![x]\!]}V[\![x]\!]$ and $V[\![x]\!]_{R[\![x]\!]}$ are finitely generated and, if $\mathrm{id}({}_SV), \mathrm{id}(V_R) \leqslant r$, then, by [**10**, Proposition 2.7], we have that $\mathrm{id}({}_{S[\![x]\!]}V[\![x]\!]), \mathrm{id}(V[\![x]\!]_{R[\![x]\!]}) \leqslant r+1$. Moreover,

$$\mathrm{Hom}_{R[\![x]\!]}(V[\![x]\!], V[\![x]\!]) \cong \mathrm{Hom}_R(V, V)[\![x]\!]$$

(cf. [**13**]), and so $\mathrm{End}_{R[\![x]\!]}(V[\![x]\!]) \cong S[\![x]\!]$ and, analogously, $\mathrm{End}_{S[\![x]\!]}(V[\![x]\!]) \cong R[\![x]\!]$. We also have that, for every left $R[\![x]\!]$-module L,

$$R[\![x]\!] \otimes_{R[\![x]\!]} L \cong R \otimes_R L \cong L,$$

and so, if $P \in \text{Mod-}R$ is projective, then it is easy to see that

$$P[\![x]\!] \otimes_{R[\![x]\!]} L \cong P \otimes_R L.$$

Hence, if $M \in \text{Mod-}R$ and $\cdots \to P_1 \to P_0 \to M \to 0$ is projective resolution of M, then we have a commutative diagram

$$\begin{array}{ccccccc} P_1 \otimes_R L & \longrightarrow & P_0 \otimes_R L & \longrightarrow & M \otimes_R L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ P_1[\![x]\!] \otimes_{R[\![x]\!]} L & \longrightarrow & P_0[\![x]\!] \otimes_{R[\![x]\!]} L & \longrightarrow & M[\![x]\!] \otimes_{R[\![x]\!]} L & \longrightarrow & 0 \end{array}$$

and so $M \otimes_R L \cong M[\![x]\!] \otimes_{R[\![x]\!]} L$. From this, we have that

$$\operatorname{Ext}^n_{R[\![x]\!]}(V[\![x]\!], V[\![x]\!]) \cong \operatorname{Ext}^n_R(V, V)[\![x]\!] = 0 \quad \forall n \geqslant 1$$

and, analogously, $\operatorname{Ext}^n_{R[\![x]\!]}(V[\![x]\!], V[\![x]\!]) = 0$ for all $n \geqslant 1$. Therefore, ${}_{S[\![x]\!]}V_{R[\![x]\!]}$ is a dualizing module.

Now we use the bimodule ${}_SV_R$ to define the two following classes.

Definition 3.6. Let R and S be right and left Noetherian rings, respectively, and let ${}_SV_R$ be a dualizing module. We define the Auslander class $\mathcal{A}(R)$ (relative to V) as those left R-modules M such that $\operatorname{Tor}^R_i(V, M) = 0$ and $\operatorname{Ext}^i_S(V, V \otimes_R M) = 0$ for all $i \geqslant 1$ and such that the natural morphism $M \to \operatorname{Hom}_S(V, V \otimes_R M)$ is an isomorphism.

The Bass class $\mathcal{B}(S)$ (relative to V) is defined as those left S-modules N such that $\operatorname{Ext}^i_S(V, N) = 0$ and $\operatorname{Tor}^R_i(V, \operatorname{Hom}_S(V, N)) = 0$ for all $i \geqslant 1$ and such that the natural morphism $V \otimes_R \operatorname{Hom}_S(V, N) \to N$ is an isomorphism.

It is an important property of Auslander and Bass classes that they are equivalent under the pair of functors

$$\mathcal{A}(R) \underset{\operatorname{Hom}_S(V,\cdot)}{\overset{V\otimes_R \cdot}{\rightleftarrows}} \mathcal{B}(S)$$

This can be shown as in [**3**, Theorem 3.3.2].

Lemma 3.7. *Let $E \in S$-*Mod *be injective. Then $E \in \mathcal{B}(S)$.*

Proof. It is immediate that $\operatorname{Ext}^i_S(V, E) = 0$ for all $i \geqslant 1$. On the other hand, since ${}_SV$ is finitely presented and E is injective, we have the isomorphism

$$V \otimes_R \operatorname{Hom}_S(V, E) \cong \operatorname{Hom}_S(\operatorname{Hom}_R(V, V), E).$$

But $\operatorname{Hom}_R(V, V) = S$ and $\operatorname{Hom}_S(S, E) \cong E$, and so $V \otimes_R \operatorname{Hom}_S(V, E) \cong E$.

Finally, since R is right Noetherian, V_R is finitely generated and E is injective, then we have the isomorphism

$$\operatorname{Tor}^R_i(V, \operatorname{Hom}_S(V, E)) \cong \operatorname{Hom}_S(\operatorname{Ext}^i_R(V, V), E),$$

and since $\operatorname{Ext}^i_R(V, V) = 0$ for all $i \geqslant 1$, it follows that $\operatorname{Tor}^R_i(V, \operatorname{Hom}_S(V, E)) = 0$ for all $i \geqslant 1$. □

Proposition 3.8. *If $N \in S$-*Mod *is Gorenstein injective, then $E \in \mathcal{B}(S)$.*

Proof. Let $\cdots \to E_{-1} \to E_0 \to E_1 \to \cdots$ exact in S-Mod with every E_i injective, $N = \operatorname{Ker}(E_0 \to E_1)$ and such that $\operatorname{Hom}_S(E, \cdot)$ leaves it exact for every $E \in S$-Mod injective. Since ${}_SV$ has finite injective dimension, then $\operatorname{Hom}_S(V, \cdot)$ leaves the preceding sequence exact, which gives that $\operatorname{Ext}^i_S(V, N) = 0$ for all $i \geqslant 1$. Moreover, we have an exact sequence

$$\operatorname{Hom}_S(V, E_{-1}) \to \operatorname{Hom}_S(V, E_0) \to \operatorname{Hom}_S(V, N) \to 0,$$

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and so we get a commutative diagram

$$\begin{array}{ccccccc} V \otimes_R \mathrm{Hom}_S(V, E_{-1}) & \longrightarrow & V \otimes_R \mathrm{Hom}_S(V, E_0) & \longrightarrow & V \otimes_R \mathrm{Hom}_S(V, N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ E_{-1} & \longrightarrow & E_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Now, as the two vertical maps on the left are isomorphisms, since $E_i \in \mathcal{B}(S)$, it follows that $N \cong V \otimes_R \mathrm{Hom}_S(V, V)$. Let us now consider the exact sequence $0 \to K \to E_0 \to N \to 0$. Since K is Gorenstein injective by [**6**, Remark 10.1.5], then, by the preceding $K \cong V \otimes_R \mathrm{Hom}_S(V, K)$. The fact that $\mathrm{Hom}_S(V, \cdot)$ leaves the first sequence exact gives that

$$0 \to \mathrm{Hom}_S(V, K) \to \mathrm{Hom}_S(V, E_0) \to \mathrm{Hom}_S(V, N) \to 0$$

is exact and so, applying $V \otimes_R \cdot$, we get a long exact sequence

$$0 = \mathrm{Tor}_1^R(V, \mathrm{Hom}_S(V, E_0)) \to \mathrm{Tor}_1^R(V, \mathrm{Hom}_S(V, N)) \to K \to E_0 \to N \to 0,$$

which implies that $\mathrm{Tor}_1^R(V, \mathrm{Hom}_S(V, N)) = 0$. By the same reasoning, we get that $\mathrm{Tor}_1(V, \mathrm{Hom}_S(V, K)) = 0$, which gives that $\mathrm{Tor}_2^R(V, \mathrm{Hom}_S(V, N)) = 0$, since $E_0 \in \mathcal{B}(S)$. An easy induction terminates the proof. □

A dual argument gives the following result.

Proposition 3.9. *If $M \in R$-Mod is Gorenstein projective, then $M \in \mathcal{A}(R)$.*

We have shown that $\mathcal{A}(R)$ and $\mathcal{B}(S)$ are not empty classes, since they contain the projective and injective modules, respectively. Now we are going to give a characterization of these classes.

Let $\mathcal{W}$ be the class of all left S-modules M such that $M \cong V \otimes_R P$ for some $P \in R$-Mod projective and let $\mathcal{U}$ be the class of all left R-modules N such that $N \cong \mathrm{Hom}_S(V, E)$ for some $E \in S$-Mod injective.

Proposition 3.10. *Every left S-module has a $\mathcal{W}$-precover and every left R-module has an $\mathcal{U}$-pre-envelope.*

Proof. It is clear that $V \in \mathcal{W}$, since $V \cong V \otimes_R R$, and that $V^{(I)} \in W$. In this way, if $M \in S$-Mod, then a $\mathcal{W}$-precover of M is $V^{(\mathrm{Hom}_S(V,M))} \to M$, taking into account that $W \cong V \otimes_R P \xrightarrow{\varphi} M$ is a $\mathcal{W}$-precover if and only if $\mathrm{Hom}_S(V, W) \to \mathrm{Hom}_S(V, M) \to 0$ is exact, since, if $\mathrm{Hom}_S(V \otimes_R P', V \otimes_R P) \to \mathrm{Hom}_S(V \otimes_R P', M) \to M \to 0$ is exact, taking $P' = R$, then $\mathrm{Hom}_S(V, V \otimes_R P) \to \mathrm{Hom}_S(V, M) \to 0$ is exact. Conversely, if $\mathrm{Hom}_S(V, V \otimes_R P) \to \mathrm{Hom}_S(V, M) \to 0$ is exact and $f \in \mathrm{Hom}_S(V \otimes_R P', M)$, then, given $p' \in P' \cong \mathrm{Hom}_S(V, V \otimes_R P')$, $fp' \in \mathrm{Hom}_S(V, M)$, which implies that there is $g \in \mathrm{Hom}_S(V, V \otimes_R P) \cong P$ such that $fp' = \varphi g$. Therefore, given $p' \in P'$, we can assign $g \in P$ to it. So we have an element of $\mathrm{Hom}_S(P, P') \cong \mathrm{Hom}_S(V \otimes_R P', V \otimes_R P)$, as desired.

Now let $M \in R$-Mod and embed $V \otimes_R M$ into an injective E. Then the composition $M \to \mathrm{Hom}_S(V, V \otimes_R M) \to \mathrm{Hom}_S(V, E)$ is an $\mathcal{U}$-pre-envelope. To show this, if $U \in \mathcal{U}$

and $M \to U$ is a morphism, then we have a morphism $V \otimes_R M \to V \otimes_R U \cong V \otimes_R \mathrm{Hom}_S(V, E') \cong E'$ for some injective $E' \in S$-Mod. Therefore, the map $V \otimes_R M \to E'$ may be extended to $E \to E'$, which gives a morphism $\mathrm{Hom}_S(V, E) \to \mathrm{Hom}_S(V, E')$ such that the composition $M \to \mathrm{Hom}_S(V, E) \to \mathrm{Hom}_S(V, E')$ is the morphism $M \to \mathrm{Hom}_S(V, E') \cong U$. □

Proposition 3.11. *Let $N \in \mathcal{B}(S)$. The following assertions are equivalent.*

(i) $N \in \mathcal{B}(S)$.

(ii) *There exists an exact sequence $\cdots \to W_1 \to W_0 \to E^0 \to E^1 \to \cdots$ in S-Mod, where every E^i is injective, every $W_i \in \mathcal{W}$, $N = \mathrm{Ker}(E^0 \to E^1)$ and $\mathrm{Hom}_S(V, \cdot)$ leaves it exact.*

(iii) *There exists an exact $\mathcal{W}$-resolution $\cdots \to W_1 \to W_0 \to N \to 0$, which remains exact when we apply $\mathrm{Hom}_S(W, \cdot)$ for every $W \in \mathcal{W}$ and $\mathrm{Hom}_S(V, \cdot)$ leaves exact every injective resolution of N.*

Proof. (i) ⇒ (ii). Let $0 \to N \to E^0 \to E^1 \to \cdots$ be an injective resolution of N. Then $0 \to \mathrm{Hom}_S(V, N) \to \mathrm{Hom}_S(V, E^0) \to \cdots$ is exact since $\mathrm{Ext}^i_S(V, N) = 0$ for all $i \geqslant 1$. Now let $\cdots \to P_1 \to P_0 \to \mathrm{Hom}_S(V, N) \to 0$ be a projective resolution of $\mathrm{Hom}_S(V, N)$ in R-Mod and let $W_i = V \otimes_R P_i \in \mathcal{W}$. Then we have the complex $\cdots \to W_1 \to W_0 \to V \otimes_R \mathrm{Hom}_S(V, N) \to 0$, which is exact since $\mathrm{Tor}^R_i(V, \mathrm{Hom}_S(V, N)) = 0$ for all $i \geqslant 1$ and $N \cong V \otimes_R \mathrm{Hom}_S(V, N)$.

(ii) ⇒ (i). $\mathrm{Hom}_S(V, \cdot)$ leaves $0 \to N \to E^0 \to E^1 \to \cdots$ exact, which implies that $\mathrm{Ext}^i_S(V, N) = 0$ for all $i \geqslant 1$. On the other hand, if $W_i = V \otimes_R P_i$ for some projective P_i, then $\mathrm{Hom}_S(V, W_i) \cong \mathrm{Hom}_S(V, V \otimes_R P_i) \cong P_i$. Therefore, the natural morphism $V \otimes_R \mathrm{Hom}_S(V, W_i) \to W_i$ is an isomorphism and

$$\cdots \to \mathrm{Hom}_S(V, W_1) \to \mathrm{Hom}_S(V, W_0) \to \mathrm{Hom}_S(V, N) \to 0$$

is a projective resolution of $\mathrm{Hom}_S(V, N)$. But then the complex

$$\cdots \to V \otimes_R \mathrm{Hom}_S(V, W_1) \to V \otimes_R \mathrm{Hom}_S(V, W_0) \to V \otimes_R \mathrm{Hom}_S(V, N) \to 0$$

is equivalent to the exact sequence $\cdots \to W_1 \to W_0 \to N \to 0$ and so $V \otimes_R \mathrm{Hom}_S(V, N) \cong N$ and $\mathrm{Tor}^R_i(V, \mathrm{Hom}_S(V, N)) = 0$ for all $i \geqslant 1$, and therefore $N \in \mathcal{B}(S)$.

(ii) ⇔ (iii). $\cdots \to W_1 \to W_0 \to N \to 0$ remains exact when we apply $\mathrm{Hom}_S(W, \cdot)$ for every $W \in \mathcal{W}$ if and only if $\mathrm{Hom}_S(V \otimes_R P, \cdot)$ leaves it exact for every projective P, if and only if $\mathrm{Hom}_R(P, \mathrm{Hom}_S(V, \cdot))$ leaves it exact for every projective P, if and only if $\mathrm{Hom}_S(V, \cdot)$ leaves it exact. Finally, $0 \to N \to E^0 \to E^1 \to \cdots$ remains exact when we apply $\mathrm{Hom}_S(V, \cdot)$ if and only if this functor leaves exact any injective exact resolution of N, since $\mathrm{Ext}^i_S(V, N) = 0$ for all $i \geqslant 1$. □

We can also prove in a dual manner the following characterization of modules in $\mathcal{A}(R)$.

Proposition 3.12. *Let $M \in R$-Mod. The following assertions are equivalent.*

(i) $M \in \mathcal{A}(R)$.

(ii) *There exists an exact sequence* $\cdots \to P_1 \to P_0 \to U^0 \to U^1 \to \cdots$ *in* R-Mod *where every* P_i *is projective, every* $U^i \in \mathcal{U}$, $M = \operatorname{Ker}(U^0 \to U^1)$ *and* $V \otimes_R \cdot$ *leaves it exact.*

(iii) *There exists an exact* $\mathcal{U}$*-resolution* $0 \to M \to U^0 \to U^1 \to \cdots$ *which remains exact when we apply* $\operatorname{Hom}_S(\cdot, U)$ *for every* $U \in \mathcal{U}$ *and* $V \otimes_R \cdot$ *leaves exact every projective resolution of* M.

Proposition 3.13. *Let* $0 \to N' \to N \to N'' \to 0$ *and* $0 \to M' \to M \to M'' \to 0$ *be exact in* S-Mod *and* R-Mod*, respectively. If two any of* N', N *and* N'' *(respectively,* M', M *and* M''*) are in* $\mathcal{B}(S)$ *(respectively,* $\mathcal{A}(R)$*), then so is the third.*

Proof. Let $0 \to N' \to N \to N'' \to 0$ be exact in S-Mod. If $N' \in \mathcal{B}(S)$, then

$$0 \to \operatorname{Hom}_S(V, N') \to \operatorname{Hom}_S(V, N) \to \operatorname{Hom}_S(V, N'') \to 0$$

is exact since $\operatorname{Ext}^1_S(V, N') = 0$. If $N \in \mathcal{B}(S)$, then $\operatorname{Ext}^1_S(V, N) = 0$ and we have an exact sequence

$$0 \to \operatorname{Hom}_S(V, N') \to \operatorname{Hom}_S(V, N) \to \operatorname{Hom}_S(V, N'') \to \operatorname{Ext}^1_S(V, N') \to 0$$

and so

$$V \otimes_R \operatorname{Hom}_S(V, N) \to V \otimes_R \operatorname{Hom}_S(V, N'') \to V \otimes_R \operatorname{Ext}^1_S(V, N') \to 0$$

is exact. If $N, N'' \in \mathcal{B}(S)$, then

$$N \to N'' \to V \otimes_R \operatorname{Ext}^1_S(V, N') \to 0$$

is exact and so $\operatorname{Ext}^1_S(V, N') = 0$. Therefore, if two of N, N' and N'' are in $\mathcal{B}(S)$, then

$$0 \to \operatorname{Hom}_S(V, N') \to \operatorname{Hom}_S(V, N) \to \operatorname{Hom}_S(V, N'') \to 0$$

is exact. But this is equivalent to

$$0 \to \operatorname{Hom}_S(V \otimes_R P, N') \to \operatorname{Hom}_S(V \otimes_R P, N) \to \operatorname{Hom}_S(V \otimes_R P, N'') \to 0$$

being exact for every projective right R-module P. But this means that the functor $\operatorname{Hom}_S(W, \cdot)$ leaves $0 \to N' \to N \to N'' \to 0$ exact for any $W \in \mathcal{W}$. Now, by [**6**, Lemma 8.2.1], given two $\mathcal{W}$-resolutions for N' and N'', we can get a $\mathcal{W}$-resolution for N and the same holds for injective resolutions. If we paste them together, we get an exact sequence of complexes such that remains exact when $\operatorname{Hom}_S(V, \cdot)$ is applied to it. If two of these complexes are exact, then so is the third, so, by Proposition 3.11, we get the desired result.

The proof of the other assertion is dual. □

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Corollary 3.14. *Let $N \in S$-Mod and $M \in R$-Mod. If $\mathrm{id}(N) < \infty$, then $N \in \mathcal{B}(S)$, and if $\mathrm{pd}(M) < \infty$, then $M \in \mathcal{A}(R)$.*

In the rest of the section, R and S will denote right and left Noetherian rings, R will be a left n-perfect ring and ${}_SV_R$ will be a dualizing module such that $\mathrm{id}({}_SV), \mathrm{id}(V_R) \leqslant r$.

Lemma 3.15. *If $N \in \mathcal{B}(S)$ and $0 \to N \to E^0 \to \cdots \to E^{r+n} \to G \to 0$ is exact with every E^i injective, then G is Gorenstein injective.*

Proof. Since $N \in \mathcal{B}(S)$, then any injective resolution of N remains exact whenever $\mathrm{Hom}_S(V, \cdot)$ is applied. Now, if E, E' are injective left S-modules, then

$$\mathrm{Hom}_S(E, E') \cong \mathrm{Hom}_R(\mathrm{Hom}_S(V, E), \mathrm{Hom}_S(V, E')).$$

Since $\mathrm{Ext}^i_S(V, V) = 0$ for all $i \geqslant 1$, then $\mathrm{Ext}^i_R(\mathrm{Hom}_S(V, E), \mathrm{Hom}_S(V, E')) = 0$ for all $i \geqslant 1$. Moreover, $\mathrm{id}({}_SV) \leqslant r$ implies that $\mathrm{fd}(\mathrm{Hom}_S(V, E)) \leqslant r$ and so, since R is left n-perfect, $\mathrm{pd}(\mathrm{Hom}_S(V, E)) \leqslant n + r$. Therefore, $\mathrm{Ext}^{n+r+i}_R(\mathrm{Hom}_S(V, E), \mathrm{Hom}_S(V, N)) = 0$ for all $i \geqslant 1$ and for every injective $E \in S$-Mod. Then

$$0 \to \mathrm{Hom}_S(V, N) \to \mathrm{Hom}_S(V, E^0) \to \cdots \to \mathrm{Hom}_S(V, E^{n+r}) \to \cdots$$

becomes exact from the term

$$\mathrm{Hom}_R(\mathrm{Hom}_S(V, E), \mathrm{Hom}_S(V, E^{n+r}))$$

whenever $\mathrm{Hom}_R(\mathrm{Hom}_S(V, E) \cdot)$ is applied for every injective $E \in S$-Mod. But then we get that

$$0 \to \mathrm{Hom}_S(E, G) \to \mathrm{Hom}_S(E, E^{n+r}) \to \mathrm{Hom}_S(E, E^{n+r+1}) \to \cdots$$

is exact for every injective $E \in S$-Mod.

Now let us construct an exact injective resolution for G on the left by using injective precovers. Since $E^{n+r} \to G$ is surjective, then an injective precover $E \xrightarrow{\varphi} G$ is also surjective. Now let $K = \mathrm{Ker}(\varphi)$. Then $\mathrm{Ext}^i_S(E', K) = 0$ for all $i \geqslant 1$ and every injective E', which gives that $\mathrm{Ext}^1_S(L, K) = 0$ for every $L \in S$-Mod such that $\mathrm{id}(L) < \infty$. Let $P \to \mathrm{Hom}_S(V, K) \to 0$ be exact, with P projective. Then $V \otimes_R P \to V \otimes_R \mathrm{Hom}_S(V, K) \to 0$ is exact. But $V \otimes_R \mathrm{Hom}_S(V, K) \cong K$, since $K \in \mathcal{B}(S)$ and $\mathrm{id}(V \otimes_R P) < \infty$. Now if $E' \to K$ is an injective precover of K, then there exists a commutative diagram

$$\begin{array}{ccc} & & V \otimes_R P \\ & \swarrow & \downarrow \\ E' & \longrightarrow & K \end{array}$$

where $E' \to K$ is surjective and so N is Gorenstein injective. □

Corollary 3.16. *$N \in \mathcal{B}(S)$ if and only if there is $n \geqslant 0$ such that there exists an exact sequence*

$$0 \to N \to G^0 \to G^1 \to \cdots \to G^k \to 0,$$

where every G^i is Gorenstein injective. Moreover, in that case, $k \leqslant r + n$.

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Proof. Apply Propositions 3.8 and 3.13 and Lemma 3.15. □

Theorem 3.17. *Let N be a left S-module. The following assertions are equivalent.*

(i) *N is Gorenstein injective.*

(ii) *$N \in \mathcal{B}(S)$ and $\operatorname{Ext}^i_S(L, N) = 0$ for all $i \geqslant 1$ and every $L \in S$-Mod such that $\operatorname{id}(L) < \infty$.*

(iii) *There is an exact sequence in S-Mod $\cdots \to E_1 \to E_0 \to N \to 0$ where every E_i is injective such that $\operatorname{Hom}_S(E, \cdot)$ leaves it exact for every injective $E \in S$-Mod and $\operatorname{Ext}^i_S(L, N) = 0$ for all $i \geqslant 1$ and every $L \in S$-Mod such that $\operatorname{id}(L) < \infty$.*

(iv) *There is an exact sequence in S-Mod $0 \to K \to E_{r+n} \cdots \to E_1 \to E_0 \to N \to 0$ where every E_i is injective and $K \in \mathcal{B}(S)$.*

(v) *There is an exact sequence in S-Mod $0 \to N \to G^0 \to G^1 \to \cdots \to G^m \to 0$ for some $m \geqslant 0$ where every G^i is Gorenstein injective and $\operatorname{Ext}^i_S(L, N) = 0$ for all $i \geqslant 1$ and every L such that $\operatorname{id}(L) < \infty$.*

Proof. (i) ⇒ (ii). It follows from [**6**, Proposition 10.1.3] and Proposition 3.8.

(ii) ⇒ (iii). This can be proved by using the same argument in Lemma 3.15.

(iii) ⇒ (i). Immediate by the definition of Gorenstein injective.

(i) ⇒ (iv). If $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ is the complete resolution for N, then take $K = \operatorname{Ker}(E_n \to E_{n-1})$. K is Gorenstein injective and, by Proposition 3.8, $K \in \mathcal{B}(S)$.

(iv) ⇒ (i). Immediate from Lemma 3.15.

(ii) ⇔ (v). It follows from the preceding corollary. □

Dual arguments give the following results.

Lemma 3.18. *If $M \in \mathcal{A}(R)$ and $0 \to G \to P_{r-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is exact with every P_i projective, then G is Gorenstein projective.*

Corollary 3.19. *$M \in \mathcal{A}(R)$ if and only if there is $k \geqslant 0$ such that there exists an exact sequence*

$$0 \to G_k \to G_{k-1} \to \cdots \to G_0 \to M \to 0$$

where every G_i is Gorenstein projective. Moreover, in that case, $k \leqslant r$.

Theorem 3.20. *Let M be a left R-Mod. The following assertions are equivalent.*

(i) *M is Gorenstein projective.*

(ii) *$M \in \mathcal{A}(R)$ and $\operatorname{Ext}^i_R(M, L) = 0$ for all $i \geqslant 1$ and every $L \in R$-Mod such that $\operatorname{pd}(L) < \infty$.*

(iii) *There is an exact sequence in* R-Mod $0 \to M \to P^0 \to P^1 \to \cdots$ *where every* P_i *is projective such that* $\mathrm{Hom}_S(\cdot, P)$ *leaves it exact for every projective* $P \in R$-Mod *and* $\mathrm{Ext}^i_R(M, L) = 0$ *for all* $i \geqslant 1$ *and every* $L \in R$-Mod *such that* $\mathrm{pd}(L) < \infty$.

(iv) *There is an exact sequence in* R-Mod $0 \to M \to P^0 \to P^1 \to \cdots \to P^{r-1} \to C \to 0$ *where every* P^i *is projective and* $C \in \mathcal{A}(R)$.

(v) *There is an exact sequence in* R-Mod $0 \to G_k \to G_{k-1} \to \ldots \to G_1 \to G_0 \to M \to 0$ *for some* $k \geqslant 0$ *where every* G_i *is Gorenstein projective and* $\mathrm{Ext}^i_R(M, L) = 0$ *for all* $i \geqslant 1$ *and every* L *such that* $\mathrm{pd}(L) < \infty$.

Corollary 3.21. $G = G_1 \oplus G_2$ *is Gorenstein injective (respectively, Gorenstein projective) if and only if* G_1 *and* G_2 *are.*

Theorem 3.22. *If* $N \in \mathcal{B}(S)$, *then* N *has a Gorenstein injective pre-envelope* $N \to G$ *such that* $\mathrm{id}(G/N) \leqslant r + n$.

Proof. Let $0 \to N \to E^0 \to \cdots \to E^{r+n} \to C \to 0$ be exact. Then, by Lemma 3.15, is Gorenstein injective. So let

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

be the complete injective resolution for C with $C = \mathrm{Ker}(I^0 \to I^1)$. Let

$$0 \to D \to I_{r+n} \to \cdots \to E_0 \to C \to 0$$

be exact. Then we have the commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow \cdots \longrightarrow & E^{r+n} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & I_{r+n} & \longrightarrow \cdots \longrightarrow & I_0 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Now the associated complex to this diagram

$$0 \to D \oplus E^0 \to \cdots \to I_0 \oplus C \to C \to 0$$

is exact and has the exact sequence

$$0 \to C \to C \to 0$$

as a subcomplex, and so the quotient complex

$$0 \to N \to D \oplus E^0 \to \cdots \to I_0 \to 0$$

is exact. Since all of its terms are injective, except perhaps $D \oplus E^0$, if $0 \to D \oplus E^0 \to L \to 0$ is exact with $\mathrm{id}(L) < \infty$ and $D \oplus E^0$ is Gorenstein injective and $\mathrm{Ext}^1_S(L, X) = 0$ for every Gorenstein injective module X, then $N \to D \oplus E^0$ is the desired pre-envelope. $\square$

A dual reasoning gives the following result.

Theorem 3.23. *If* $M \in \mathcal{A}(R)$, *then* M *has a Gorenstein projective precover* $G \to M$ *whose kernel* K *has* $\mathrm{pd}(K) \leqslant r$.

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4. V-Gorenstein modules

The aim of this section is to extend the class of modules studied in [**7**] and [**8**] that generalizes Gorenstein modules. We show the existence of pre-envelopes and precovers by these classes of module.

Let C, D and E be abelian categories and $T : C \times D \to E$ be an additive functor covariant in the first variable and contravariant in the second. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of C and D, respectively. Then T is said to be left balanced by $\mathcal{F} \times \mathcal{G}$ if, for each object M of C, there is a complex $\cdots \to F_1 \to F_0 \to M \to 0$ that becomes exact when $T(\cdot, G)$ is applied for every $G \in \mathcal{G}$ and if, for every object N in D, there is a complex $0 \to N \to G^0 \to G^1 \to \cdots$ that becomes exact when $T(F, \cdot)$ is applied for every $F \in \mathcal{F}$. It is possible then to define left-derived functors of T by using either left $\mathcal{F}$-resolutions or right $\mathcal{G}$-resolutions (cf. [**6**, Chapter 8]). Now, let $\mathcal{U}$ denote the same class as in the preceding section.

Proposition 4.1. *Let R and S be right and left Noetherian rings; R is a left n-perfect and ${}_SV_R$ is a dualizing module. Then $\mathrm{Hom}_R(\cdot,\cdot)$ is left balanced by $\mathcal{U} \times \mathcal{U}$ on R-Mod $\times$ R-Mod.*

Proof. By Proposition 3.10, if $M \in R$-Mod, then there exists an exact sequence $0 \to M \to U^0 \to U^1 \to \cdots$ such that $\mathrm{Hom}_R(\cdot, U)$ leaves it exact for every $U \in \mathcal{U}$. Now let us show that every left R-module has an $\mathcal{U}$-precover, which will finish the proof.

Since S is left Noetherian, then there exists a set of representatives of the indecomposable injective left S-modules $\{E_k\}$ that gives a set of representatives of the modules in $\mathcal{U}$, $\{U_k = \mathrm{Hom}_S(V, E_k)\}$. Now, if $M \in R$-Mod, let $s(M) = \oplus U_k^{\mathrm{Hom}_R(U_k, M)}$. Since ${}_SV$ is finitely generated and S is left Noetherian,

$$\mathrm{Hom}_S(V, \varinjlim(\cdot)) \cong \varinjlim \mathrm{Hom}_S(V, \cdot),$$

and so $s(M) \in \mathcal{U}$. The evaluation map $s(M) \to M$ is a $\mathcal{U}$-precover. □

This allows us to give the following definition. In the rest of the section, R and S will be right and left Noetherian rings, such that R is left n-perfect, and ${}_SV_R$ will be a dualizing module such that $\mathrm{id}({}_SV), \mathrm{id}(V_R) \leqslant r$.

Definition 4.2. A left R-module M is said to be V-Gorenstein injective if there exists an exact resolution

$$\cdots \to U_1 \to U_0 \to U^0 \to U^1 \to \cdots,$$

with every U_i and U^i in $\mathcal{U}$, $M = \mathrm{Ker}(U^0 \to U^1)$ and such that it remains exact whenever $\mathrm{Hom}_R(U, \cdot)$ is applied for every $U \in \mathcal{U}$.

Proposition 4.3. *If $M \in R$-Mod is V-Gorenstein injective, then $\mathrm{Ext}^i_R(U, M) = 0$ for all $i \geqslant 1$ for every $U \in \mathcal{U}$.*

Proof. If $U' \in \mathcal{U}$, then

$$\mathrm{Ext}^i_R(U, U') = \mathrm{Ext}^i_R(\mathrm{Hom}_S(V, E), \mathrm{Hom}_S(V, E')) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_i(V, \mathrm{Hom}_S(V, E)), E'),$$

and since $E \in \mathcal{B}(S)$, it follows that $\mathrm{Ext}^i_R(U, U') = 0$ for all $i \geqslant 1$, and so $\mathrm{Ext}^i_R(U, M) = 0$ for all $i \geqslant 1$ for every $U \in \mathcal{U}$. □

Proposition 4.4. *Let $M \in R$-Mod. Then the following hold.*

(i) *If $0 \to M \to U^0 \to U^1 \to \cdots$ is an exact right $\mathcal{U}$-resolution and*

$$C^i = \mathrm{Ker}(U^i \to U^{i+1}),$$

then C^i is V-Gorenstein injective for $i \geqslant r$.

(ii) *If $\cdots \to U_1 \to U_0 \to M \to 0$ is an exact left $\mathcal{U}$-resolution and*

$$C_i = \mathrm{Coker}(U_{i+1} \to U_i),$$

then C_i is V-Gorenstein injective for $i \geqslant r - 1$.

Proof. (i) If R is left n-perfect, then $\mathrm{pd}(\mathrm{Hom}_S(V, E)) \leqslant n$ for every injective $E \in S$-Mod, since $\mathrm{id}({}_SV) < \infty$, i.e. $\mathrm{pd}(U) \leqslant n$ for every $U \in \mathcal{U}$. Therefore, $\mathrm{Ext}^i_R(U, M) = 0$ for all $i \geqslant n + 1$, or, equivalently, if $0 \to U^0 \to U^1 \to \cdots$ is a right $\mathcal{U}$-resolution of M, then

$$0 \to C^i \to U^i \to U^{i+1} \to \cdots \qquad (*)$$

is exact and remains exact whenever $\mathrm{Hom}_R(U, \cdot)$ is applied for every $U \in \mathcal{U}$. Now let

$$0 \to V \to E^0 \to \cdots \to E^r \to 0$$

be exact in S-Mod. Then

$$0 \to \mathrm{Hom}_S(V, V) \to \mathrm{Hom}_S(V, E^0) \to \cdots \to \mathrm{Hom}_S(V, E^r) \to 0$$

is exact since $\mathrm{Ext}^i_S(V, V) = 0$ for all $i \geqslant 1$. But $\mathrm{Hom}_S(V, V) \cong R$. In this way, if $\cdots \to U_1 \to U_0 \to M \to 0$ is a left $\mathcal{U}$-resolution of M, since, by the preceding, the left derived functors of $\mathrm{Hom}_R(\cdot, \cdot)$ using $\mathcal{U}$-resolutions on both variables, $\mathrm{Ext}^R_k(R, M) = 0$ for every $k \geqslant r - 1$, we get that the considered left $\mathcal{U}$-resolution for M is exact at U_k, $k \geqslant r - 1$.

In the case $r = 1$ and $0 \to R \to U^0 \to U^1 \to 0$ is exact, we have that

$$0 \to \mathrm{Hom}_R(U^1, M) \to \mathrm{Hom}_R(U^0, M) \to \mathrm{Hom}_R(M, M)$$

is exact. Then $\mathrm{Ext}^R_k(R, M) = 0$ for all $k \geqslant 1$ and the natural morphism

$$\mathrm{Ext}^R_0(R, M) \to M$$

is monic. Now if we calculate $\mathrm{Ext}^R_0(R, M)$ using a left $\mathcal{U}$-resolution of M, then $U_1 \to U_0 \to M$ is exact at U_0 and so $\cdots \to U_1 \to U_0 \to M \to 0$ is exact at U_k, $k \geqslant 0$.

Finally, if V is injective, then $0 \to R \to U^0 \to 0$ is exact and so every $\mathcal{U}$-precover is surjective and so $\cdots \to U_1 \to U_0 \to M \to 0$ is exact.

From the preceding, it follows that every left $\mathcal{U}$-resolution of C^i is exact, so if we paste $(*)$ to it we get that C^i is V-Gorenstein injective.

(ii) The proof is analogous to (i) using the fact that if $\cdots \to U_1 \to U_0 \to M \to 0$ is a left $\mathcal{U}$-resolution, then $\cdots \to U_{r+1} \to U_r \to U_{r-1} \to U_{r-2}$ is exact, i.e. $C_i \to E_{i-1}$ is a monomorphism for every $i \geqslant r-1$. □

Theorem 4.5. *Let $M \in \mathcal{A}(R)$. Then M is V-Gorenstein injective if and only if $V \otimes_R M$ is Gorenstein injective.*

Proof. Let

$$\cdots \to U_1 \to U_0 \to U^0 \to U^1 \to \cdots$$

be exact with every U_i and U^i in $\mathcal{U}$, $M = \operatorname{Ker}(U^0 \to U^1)$ and such that this sequence remains exact whenever $\operatorname{Hom}_R(U, \cdot)$ is applied for every $U \in \mathcal{U}$. If we apply $V \otimes_R -$, we get a complex

$$\cdots \to V \otimes_R \operatorname{Hom}_S(V, E_1) \to V \otimes_R \operatorname{Hom}_S(V, E_0) \to V \otimes_R \operatorname{Hom}_S(V, E^0) \to \cdots,$$

which is isomorphic to

$$\cdots \to \operatorname{Hom}_S(\operatorname{Hom}_R(V,V), E_1) \to \operatorname{Hom}_S(\operatorname{Hom}_R(V,V), E_0) \\ \to \operatorname{Hom}_S(\operatorname{Hom}_R(V,V), E^0) \to \operatorname{Hom}_S(\operatorname{Hom}_R(V,V), E^1) \to \cdots,$$

and since $\operatorname{Hom}_R(V,V) \cong S$, we get a complex in S-Mod,

$$\cdots \to E_1 \to E_0 \to E^0 \to \cdots,$$

such that $V \otimes_R M \cong \operatorname{Ker}(E^0 \to E^1)$. Now, if E is injective in S-Mod, then $E \in \mathcal{B}(S)$ and $U = \operatorname{Hom}_S(V, E) \in \mathcal{A}(R)$. Therefore, $\operatorname{Tor}_i^R(V, U) = 0$ for all $i \geqslant 1$ for every $U \in \mathcal{U}$. Moreover, $M \in \mathcal{A}(R)$ by hypothesis, and so the complex

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

is exact. If $U, U' \in \mathcal{U}$, then

$$\operatorname{Hom}_R(U, U') \cong \operatorname{Hom}_R(\operatorname{Hom}_S(V, E), \operatorname{Hom}_S(V, E')) \cong \operatorname{Hom}_S(E, E'),$$

and so the functor $\operatorname{Hom}_S(E, \cdot)$ leaves it exact and therefore $V \otimes_R M$ is Gorenstein injective.

Conversely, if $V \otimes_R M$ is Gorenstein injective and if $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ is the complete injective resolution for $V \otimes_R M$, then $\operatorname{Hom}_S(V, \cdot)$ leaves it exact since $\operatorname{id}({}_S V) < \infty$ and therefore we get an exact sequence $\cdots \to U_1 \to U_0 \to U^0 \to U^1 \to \cdots$, where every $U_i, U^i \in \mathcal{U}$ and $\operatorname{Ker}(U^0 \to U^1) = \operatorname{Ker}(\operatorname{Hom}_S(V, E^0) \to \operatorname{Hom}_S(V, E^1)) = \operatorname{Hom}_S(V, V \otimes_R M) \cong M$ since $M \in \mathcal{A}(R)$. Finally, let $U \in \mathcal{U}$, $U = \operatorname{Hom}_S(V, E)$, where E is injective. Then

$$\ldots \to \operatorname{Hom}_R(U, U_1) \to \operatorname{Hom}_R(U, U_0) \to \operatorname{Hom}_R(U, U^0) \to \operatorname{Hom}_R(U, U^1) \to \cdots$$

is isomorphic to

$$\ldots \to \operatorname{Hom}_S(E, E_1) \to \operatorname{Hom}_S(E, E_0) \to \operatorname{Hom}_S(E, E^0) \to \operatorname{Hom}_S(E, E^1) \to \cdots.$$

But this complex is exact, since $V \otimes_R M$ is Gorenstein injective and therefore M is V-Gorenstein injective. □

An analogous reasoning to the one used in Theorem 3.22 gives the following result.

Theorem 4.6. *Every $M \in \mathcal{A}(R)$ has a Gorenstein injective pre-envelope $0 \to M \to G \to L \to 0$ such that there exists an exact sequence $0 \to L \to U^0 \to U^1 \to \cdots \to U^k \to 0$ with $k \leqslant r-1$ if $r \geqslant 1$ and every $U^i \in \mathcal{U}$ for $i = 0, \dots, k$.*

Now let $\mathcal{W}$ be the class of left S-modules M that are isomorphic to $V \otimes_R P$ for some projective $P \in R$-Mod and let us denote by $\mathcal{B}(S)_{\mathrm{fg}}$ and $\mathcal{W}_{\mathrm{fg}}$ the classes of finitely generated left S-modules that are in $\mathcal{B}(S)$ and $\mathcal{W}$, respectively. Then using the fact that every finitely generated left R-module has a finitely generated projective pre-envelope, since R is right coherent, we get the following result.

Proposition 4.7. $\mathrm{Hom}_S(\cdot, \cdot)$ *is left balanced on $\mathcal{B}(S)_{\mathrm{fg}} \times \mathcal{B}(S)_{\mathrm{fg}}$ by $\mathcal{W}_{\mathrm{fg}} \times \mathcal{W}_{\mathrm{fg}}$.*

This suggests the following definition.

Definition 4.8. A left S-module N is said to be V-Gorenstein projective if there exists an exact resolution

$$\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots ,$$

with every W_i and W^i in $\mathcal{W}$, $N = \mathrm{Ker}(W^0 \to W^1)$ and such that this sequence remains exact whenever $\mathrm{Hom}_R(\cdot, W)$ is applied for every $W \in \mathcal{W}$.

There exist similar results for V-Gorenstein projective finitely generated modules to those obtained for V-Gorenstein injective.

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