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ON AN EXTENSION OF A THEOREM OF SATO

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Let $f(z)$ be a n-valued algebroid function of order $\lambda(0 < \lambda < 1)$, Sato obtained an elliptic theorem for $f(z)$ with a condition. In this paper, we prove that Sato's theorem is true without conditions and give a generalisation.

1. INTRODUCTION

Let $f(z)$ be a n-valued algebroid function defined by an irreducible equation

$$A_n(z)f^n + A_{n-1}(z)f^{n-1} + \cdots + A_1(z)f + A_0(z) = 0 \tag{1}$$

where $A_j(z)(j = 0, 1, \ldots, n)$ are entire functions without common zeros.

Let $T(r, f)$ be the characteristic function of $f(z)$ and $a(z)$ a rational function. Define

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, a, f)}{T(r, f)},$$
$$N(r, \infty, f) = N(r, f).$$

With this notation, Sato [2] obtained the following result in 1981.

THEOREM A. *Let $f(z)$ be a n-valued algebroid function of order $\lambda(0 < \lambda < 1)$, defined by the irreducible equation (1), and suppose that 0 is not a Valiron deficient value for $A_n(z)$. Let a_j, $j = 1, 2, \ldots, n$, be mutually distinct values, and put*

$$u_j = 1 - \delta(a_n, f) \text{ and } \nu = 1 - \delta(\infty, f),\ 0 \leqslant u_j,\ \nu \leqslant 1.$$

Then there is at least one a_i, $1 \leqslant i \leqslant n$, such that

$$u_i^2 + \nu^2 - 2u_i\nu\cos\pi\lambda \geqslant n^{-2}\sin^2(\pi\lambda).$$

If $u_i < n^{-1}\cos\pi\lambda$, then $\nu \geqslant 1/n$; if $\nu < n^{-1}\cos\pi\lambda$, then $u_i \geqslant 1/n$.

In this paper we shall prove that Sato's theorem is also true when $a_j(j = 1, 2, \ldots, n)$ are rational functions and without conditions.

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2. LEMMAS

LEMMA 1. *Let $f(z)$ be an n-valued transcendental algebroid function defined by equation (1), and set*

$$A(z) = \max_{0 \leqslant i \leqslant n} |A_i(z)|,$$
$$\mu(r, A) = \frac{1}{2n\pi} \int_0^{2\pi} \log A(re^{i\theta}) d\theta.$$

Then

$$|T(r, f) - \mu(r, A)| = O(1).$$

PROOF: See [**3**].

LEMMA 2. *Let $f(z)$ be a n-valued transcendental algebroid function defined by equation (1) and let $a_i(z)\,(i = 1, 2, \ldots, n)$ be rational functions, which are mutually distinct. Set*

$$g_i(z) = A_n(z)a_i^n + A_{n-1}(z)a_i^{n-1} + \cdots + A_0(z),\ i = 1, 2, \ldots, n,$$
$$g_0(z) = A_n(z),\ g(z) = \max_{0 \leqslant i \leqslant n} |g_i(z)|, \quad (|z| = r > r_0),$$
$$\mu(r, g) = \frac{1}{2n\pi} \int_0^{2\pi} \log g(re^{i\theta}) d\theta.$$

Then

$$|\mu(r, g) - \mu(r, A)| = o(T(r, f)).$$

PROOF: By the definition of a μ-function

$$\begin{aligned}\mu(r,g) - \mu(r,A) &= \frac{1}{2n\pi} \int_0^{2\pi} \log \frac{\max_{0 \leqslant i \leqslant n} |g_i(z)|}{\max_{0 \leqslant i \leqslant n} |A_i(z)|} d\theta \ (z = re^{i\theta}) \\ &\leqslant \frac{1}{2n\pi} \int_0^{2\pi} \log^+ \sum_{i=0}^{n} \left(|a_i^n| + |a_i|^{n-1} + \cdots + 1\right) d\theta \\ &= o(T(r,f)) \quad r \to \infty.\end{aligned}$$

On the other hand, since

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ a_1^n & a_1^{n-1} & a_1^{n-2} & \ldots & a_1 & 1 \\ a_2^n & a_2^{n-1} & a_2^{n-2} & \ldots & a_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n^n & a_n^{n-1} & a_n^{n-2} & \ldots & a_n & 1 \end{pmatrix} \begin{pmatrix} A_n \\ A_{n-1} \\ A_{n-2} \\ \vdots \\ A_0 \end{pmatrix} \tag{2}$$

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and the coefficient determinant is not equal to zero $(|z| = r > r_0)$, we have

$$A_i(z) = b_{i0}g_0(z) + b_{i1}g_1(z) + \cdots + b_{in}g_n(z) \quad (0 \leqslant i \leqslant n),$$

where the b_{ij} are rational functions of the $a_i (0 \leqslant i \leqslant n)$.

Therefore, by the same reasoning, we have

$$\mu(r, A) - \mu(r, g) \leqslant o(T(r, f))$$

Lemma 2 is thus proved. □

LEMMA 3. *Let $g_i(z)\,(0 \leqslant i \leqslant n)$ be as defined in Lemma 2, then*

$$T(r, g_i/g_j) - o(T(r, f)) \leqslant nT(r, f) \leqslant \sum_{i \neq j} T(r, g_i/g_j) + o(T(r, f)).$$

PROOF: Let

$$U(z) = \max_{0 \leqslant i \leqslant n} (\log |g_i(z)|),$$
$$U_{ij}(z) = \max(\log |g_i(z)|, \log |g_j(z)|).$$

Since

$$U_{ij}(z) = \log^+ |g_i/g_j| + \log |g_j|,$$

we have

$$\begin{aligned}
\mu(r, g) &= \frac{1}{2n\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta \\
&\geqslant \frac{1}{2n\pi} \int_0^{2\pi} U_{ij}(re^{i\theta}) d\theta \\
&= \frac{1}{2n\pi} \int_0^{2\pi} \log^+ |g_i/g_j| \, d\theta + \frac{1}{2n\pi} \int_0^{2\pi} \log |g_j| \, d\theta \\
&= \frac{1}{n} m(r, g_i/g_j) + \frac{1}{n} N(r, 0, g_j) + o(T(r, f)) \\
&\geqslant \frac{1}{n} T(r, g_i/g_j) + o(T(r, f)).
\end{aligned}$$

By Lemmas 1 and 2, we have proved the left hand inequality.

To prove the second inequality, we assume, without loss of generality, that $j = 0$.

$$\begin{aligned} n\mu(r, g) &= \frac{1}{2\pi}\int_0^{2\pi} \log \max_{0\leqslant i\leqslant n} |g_i(z)|\, d\theta \quad (z = re^{i\theta}) \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=1}^{n} \log^+ |g_i/g_j| + \log|g_0|\right) d\theta \\ &= \sum_{i=1}^{n} m(r, g_i/g_0) + N(r, 0, g_0) + O(1) \\ &= \sum_{i=1}^{n} T(t, g_i/g_0) + N(r, 0, g_0) - \sum_{i=1}^{n} N(r, g_i/g_0) + O(1). \end{aligned}$$

Since (2) and $A_j (j = 0, 1, \cdots, n)$ have no common zeros, it follows that the common zeros of $g_j (j = 0, 1, \cdots, n)$ must be the zeros of the determinant

$$D(z) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a_1^n & a_1^{n-1} & a_1^{n-2} & \dots & a_1 & 1 \\ a_2^n & a_2^{n-1} & a_2^{n-2} & \dots & a_2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n^n & a_n^{n-1} & a_n^{n-2} & \dots & a_n & 1 \end{vmatrix} (D(z) \not\equiv 0).$$

By

$$T(r, 1/D(z)) = o(T(r, f)), \quad r \to \infty,$$

we have

$$N(r, 0, g_0) \leqslant \sum_{i=1}^{n} N(r, g_i/g_0) + o(T(r, f)).$$

Therefore

$$nT(r, f) \sim n\mu(r, g) \leqslant \sum_{i=1}^{n} T(r, g_i/g_0) + o(T(r, f)),$$

this completes the proof. ∎

3. Results

THEOREM 1. *Let $f(z)$ be a n-valued algebroid function of order $\lambda (0 < \lambda < 1)$, defined by the equation (1), $a_j(z) (j = 0, 1, \cdots, n)$ be n mutually distinct rational functions. Put*

$$u_j = 1 - \delta(a_j, f), \quad \nu = 1 - \delta(\infty, f), \quad j = 1, 2, \cdots, n.$$

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Then, there is at least one $a_i (1 \leqslant i \leqslant n)$ such that

$$u_i^2 + \nu^2 - 2u_i\nu\cos\pi\lambda \geqslant n^{-2}\sin^2\pi\lambda.$$

If $u_i < n^{-1}\cos\pi\lambda$, then $\nu \geqslant 1/n$; if $\nu < n^{-1}\cos\pi\lambda$, then $u_i \geqslant 1/n$.

PROOF: Let $g_i(z)(i = 0, 1, \cdots, n)$ be defined in Lemma 2. By Lemma 2 the functions $g_i/g_0\ (i = 1, \cdots, n)$ are of order at most $\lambda (0 < \lambda < 1)$. We use Edrei and Fuchs's idea [1] and their well-known representation. Then

$$T(r, g_i/g_0) \leqslant \int_0^\infty N(t, 0, g_i/g_0)P(t, r, \beta_i)dt + \int_0^\infty N(t, g_i/g_0)P(t, r, \pi - \beta_i)dt, \tag{3}$$

where

$$P(t, r, \alpha) = \frac{1}{\pi}\frac{r\sin\alpha}{t^2 + 2rt\cos\alpha + r^2}, \quad (0 < \alpha < \pi).$$

Since

$$N(r, 0, g_i/g_0) \leqslant N(r, 0, g_i) + o(T(r, f)) = nN(r, a_i, f) + o(T(r, f))$$
$$N(r, g_i/g_0) \leqslant N(r, 0, A_n) + o(T(r, f)) = nN(r, f) + o(T(r, f)),$$

therefore, by the definition of u_i and ν, given $\varepsilon > 0$, there exists $t_0 > 0$, such that for $t \geqslant t_0$

$$N(t, 0, g_i/g_0) \leqslant n(u_i + \varepsilon)T(t, f) \quad (1 \leqslant i \leqslant n),$$
$$N(t, g_i/g_0) \leqslant n(\nu + \varepsilon)T(t, f).$$

By Lemma 3 and (3), we have

$$T(r, f) \leqslant \sum_{i=1}^{n}\int_{t_0}^\infty (u_i + \varepsilon)T(t, f)P(t, r, \beta_i)dt + \sum_{i=1}^{n}\int_{t_0}^\infty (\nu + \varepsilon)T(t, f)P(t, r, \pi - \beta_i)dt + o(T(r, f)).$$

Using this inequality and by adopting the arguments used by Sato [2] or Edrei and Fuchs [1], Theorem 1 follows. □

COROLLARY. *If 0 is a Valiron deficient value for $A_n(z)$, Theorem A is also true.*

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