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GENERATOR CONDITIONS ON THE FITTING SUBGROUP OF A POLYCYCLIC GROUP

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In an earlier paper (3), polycyclic groups in which every subgroup can be generated by d, or fewer, elements were studied. In this paper we investigate the structure of those polycyclic groups G such that every abelian normal subgroup of $F(G)$, the Fitting subgroup of G, can be generated by at most d elements.

In Section 1, we prove the existence of a function f such that if G is a finitely generated nilpotent group in which every abelian normal subgroup can be generated by at most d elements, then every subgroup of G can be generated by $f(d)$, or fewer, elements. It is shown in Section 2 that for G a polycyclic group in which $F(G)$ can be generated by at most d elements, $G/F(G)$ may be regarded as a subgroup of a direct product of linear groups of degree at most d. Section 3 contains the results on those polycyclic groups G such that every abelian normal subgroup of $F(G)$ can be generated by at most d elements.

1.

We first introduce some notation. Given a positive integer d, as in (3) we denote by $\mathscr{X}_d$ the class of soluble groups G such that every subgroup of G can be generated by at most d elements. Given a group G, we write $d(G)=d$ to mean that G has d elements in a minimal generating set and $d_n(G)\leq d$ to mean that every abelian normal subgroup of G can be generated by d, or fewer, elements. The purpose of this section is to prove the following result.

Theorem 1. *Let G be a finitely generated nilpotent group with $d_n(G)\leq d$. Then $G\in\mathscr{X}_{f(d)}$ where $f(d)=3d^2+[d^2/4]$ and $[x]$ denotes the integer part of x.*

Before proving Theorem 1, we need several lemmas.

Lemma 1. *Let A be an abelian group with $d(A)=d$. Let B be the torsion subgroup of A. Define*

$$G=\{\theta\in \text{Aut}\, A \mid b\theta=b \textit{ for all } b\in B \textit{ and } a\theta\equiv a \bmod B \quad \textit{for all } a\in A\}.$$
Then $G\in\mathscr{X}_{[d^2/4]}$.

Proof. Since G is an abelian group, we only need show that G can be generated by at most $[d^2/4]$ elements. Let $\{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ be a generating set for A chosen such that $r+s=d$ and $\{x_1, \ldots, x_r\}$ generate B. For $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, define a map $\theta_{ij}: A \to A$ by

$$x_k\theta_{ij} = x_k, \qquad y_j\theta_{ij} = y_jx_i, \qquad y_l\theta_{ij} = y_l,$$

where $1 \leqslant k \leqslant r$, $1 \leqslant l \leqslant s$ and $l \neq j$. Then θ_{ij} is an automorphism of A in G and it is clear that any automorphism of A in G can be written as a product of suitable powers of the θ_{ij} $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$. Thus G can be generated by $rs = r(d-r)$ elements. The result follows since $r(d-r) \leqslant [d^2/4]$.

Lemma 2. *Let G be a finitely generated nilpotent group, A be a maximal abelian normal subgroup of G and T be the torsion subgroup of A. Let P be a Sylow p-subgroup of T and C be the centralizer of P in G. Then G/C is a finite p-group.*

Proof. Since G is polycyclic, P is a finite group and so G/C is a finite nilpotent group. Suppose G/C is not a p-group so that there exists a prime q different from p and an element g of G such that $g \notin C$ but $g^q \in C$. Let $H = \langle g, P\rangle$ so that P is a normal subgroup of H with cyclic factor group and $g^q \in Z(H)$. Thus $H/\langle g^q\rangle$ is a finite nilpotent group with Sylow q-subgroup of order q. Hence, for any $x \in P$, there exists an integer n such that

$$x^g = xg^{nq}.$$

If g has infinite order, this equation can only hold if $n=0$ in which case $g \in C$. Thus we may suppose that g has finite order so that H is a finite nilpotent group. It then follows that $H/C_H(P)$ is a p-group and so there exists an integer n such that $g^{p^n} \in C_H(P) \leqslant C$. Since $g^q \in C$, we deduce that $g \in C$ contrary to assumption. Thus G/C is a p-group.

Lemma 3. *Let G be a finitely generated nilpotent group and A be a free abelian normal subgroup of G of rank n. Then*

$$G/C_G(A) \in \mathscr{X}^1_{\frac{1}{2}n(n-1)}.$$

Proof. Write $B = Z(G) \cap A$ so that B is a normal subgroup of G and $B > 1$. We first prove that A/B is torsion-free. For, if not, let C/B be the torsion subgroup of A/B. Thus, given $g \in C$, there exists an integer m such that $g^m \in B$. Since C is a normal subgroup of G, given any $x \in G$ there exists an element z of C such that $g^x = gz$. Then

$$(g^m)^x = g^m z^m = g^m,$$

since $g^m \in B \leqslant Z(G)$. Thus $z = 1$ as required.

Repeating this argument and using the fact that A has finite rank we

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obtain a series

$$1 = B_0 \leqslant B = B_1 \leqslant \cdots \leqslant B_r = A,$$

with B_i a free abelian normal subgroup of G, A/B_i torsion-free and $B_i/B_{i-1} \leqslant Z(G/B_{i-1})$ $(1 \leqslant i \leqslant r)$. Now taking a generating set of A based on this series, we see that $G/C_G(A)$ is isomorphic to a subgroup of STL$(n, \mathbf{Z})$, the group of lower unitriangular $n \times n$ matrices with integer entries. However, there is a series for STL$(n, \mathbf{Z})$, obtained by refining its lower central series, which is a series of length $\frac{1}{2}n(n-1)$ in which each factor is cyclic. Thus $G \in \mathscr{X}_{\frac{1}{2}n(n-1)}$ as required.

Proof of Theorem 1. Let A be a maximal abelian normal subgroup of G so that $A = C_G(A)$ by Lemma 2.19.1 of (**5**). Let T be the torsion subgroup of A and suppose $T = P_1 \times \cdots \times P_r$ where P_i is a Sylow p_i-subgroup of T. Lemma 2 implies that $G/C(P_i)$ is a p_i-group. Since $d(P_i) \leqslant d(T) \leqslant d$, a result of P. Hall (see (**5**, Lemma 7.44)) gives that every subgroup of $G/C(P_i)$ can be generated by at most $\frac{1}{2}d(5d-1)$ elements. Thus since

$$C(T) = \bigcap_{i=1}^{r} C(P_i)$$

we have that $G/C(T)$ is a subgroup of a direct product of p_i-groups and so $G/C(T) \in \mathscr{X}_{\frac{1}{2}(5d-1)}$.

Now writing $\bar{G} = G/T$ and $\bar{A} = A/T$, Lemma 3 gives that $\bar{G}/C(\bar{A}) \in \mathscr{X}_{\frac{1}{2}d(d-1)}$. Defining K by $K/T = C(\bar{A})$, we have therefore that $G/K \cap C \in \mathscr{X}_{d(3d-1)}$ where $C = C(T)$. It follows from the definition of K that $K \cap C$ consists of those elements g of G such that $b^g = b$ for all $b \in T$ and $a^g \equiv a \bmod T$ for all $a \in A$. Thus since $A = C_G(A)$ and $A \leqslant K \cap C$, Lemma 1 implies that $K \cap C/A \in \mathscr{X}_{[d^2/4]}$. It now follows that $G \in \mathscr{X}_{f(d)}$ where

$$f(d) = d(3d-1) + [d^2/4] + d = 3d^2 + [d^2/4].$$

Example. Let n be a positive integer and $p_1, \ldots, p_n$ be n distinct odd primes. Let $C_i = \langle x_i \rangle$ be cyclic of order p_i $(1 \leqslant i \leqslant n)$ and $C = C_1 \times C_2 \times \cdots \times C_n$. For $1 \leqslant i \leqslant n$, define a map $\theta_i \colon C \to C$ by

$$x_i\theta_i = x_i^{-1}; \qquad x_j\theta_i = x_j \quad \text{for } j \neq i.$$

Then θ_i is an automorphism of C of order 2. Let $H = \langle \theta_1, \ldots, \theta_n \rangle$ so that H is elementary abelian of order 2^n and define G to be the semi-direct product of C by H. It is clear from construction that C is the largest normal nilpotent subgroup of G.

Thus G is an example of a finite supersoluble group with every abelian normal subgroup cyclic but with a subgroup H which cannot be generated by fewer than n elements. This example shows, therefore, that the assumption of nilpotency in Theorem 1 is of crucial importance.

2.

Following Robinson (5; Part I, p. 66) a normal series of a polycyclic group G will be called a weak chief series if each finite factor H/K is a minimal normal subgroup of G/K while each infinite factor is a rationally irreducible G-module, that is each such factor has no non-trivial G-admissible subgroups of infinite index. Our first result in this section is an analogue of a well-known result for finite groups.

Lemma 4. *Let G be a polycyclic group and*

$$1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_r = G$$

be a weak chief series for G. Then

$$F(G) = \bigcap_{i=1}^{r} C_i$$

where C_i denotes the centralizer of G_i/G_{i-1} in G $(1 \leqslant i \leqslant r)$.

Proof. Write $D = \bigcap_{i=1}^{r} C_i$ and define $D_j = D \cap G_j$ $(0 \leqslant j \leqslant r)$. Since each C_i is a normal subgroup of G, D is a normal subgroup of G and

$$[D_i, D] \leqslant [G_i, C_i] \cap D \leqslant G_{i-1} \cap D = D_{i-1}.$$

Thus D has a central series and so is nilpotent. Hence $D \leqslant F(G)$.

Conversely, we will show that $F(G) \leqslant C_i$ for $1 \leqslant i \leqslant r$. Since $F(G)G_{i-1}/G_{i-1}$ is a normal nilpotent subgroup of G/G_{i-1}, we will suppose for convenience of notation that $i = 1$. Since G_1 is an abelian normal subgroup of G, $G_1 \leqslant F(G)$ and so defining Z_1 to be $G_1 \cap Z(F(G))$ we have that $Z_1 > 1$. If G_1 is a finite group, minimality of G_1 forces Z_1 to equal G_1 so that $F(G)$ centralizes G_1 as required. Thus we may suppose that G_1 is free abelian of rank s, say and that Z_1 also has rank s. By the fundamental theorem of finitely generated abelian groups, there exist non-zero integers $d_1, \ldots, d_s$ and elements $x_1, \ldots, x_s$ of G such that $\{x_1, \ldots, x_s\}$ generate G_1 and $\{x_1^{d_1}, \ldots, x_s^{d_s}\}$ generate Z_1. For any $g \in F(G)$, define elements $w_i(g)$ of G_1 by

$$x_i^g = x_i w_i(g); \quad (i \leqslant i \leqslant s).$$

Then, since $x_i^{d_i} \in Z(F(G))$, $(x_i^{d_i})^g = x_i^{d_i}$. However

$$(x_i^{d_i})^g = (x_i^g)^{d_i} = x_i^{d_i}(w_i(g))^{d_i}.$$

Thus $(w_i(g))^{d_i} = 1$ which means that $w_i(g) = 1$ and so g centralizes each x_i as required.

The main result of this section is the following.

Theorem 2. *Let G be a polycyclic group and suppose that $F(G)$ can be generated by d elements. Then $G/F(G)$ is isomorphic to a subgroup of a direct product $G_1 \times \cdots \times G_r$, where G_i is either an irreducible subgroup of*

$\mathrm{GL}(d_i, p_i)$ *for some prime* p_i *or a rationally irreducible subgroup of* $\mathrm{GL}(d_i, \mathbf{Z})$ *where* $d_i \leq d$ $(1 \leq i \leq r)$.

Proof. By a result of Hirsch, $\phi(G)$, the Frattini subgroup of G, is nilpotent (**5**; Part 2, p. 196). Refine the series

$$1 \leq \phi(G) \leq F(G) \leq G$$

to a weak chief series and suppose that the terms between $\phi(G)$ and $F(G)$ in this refinement are

$$\phi(G) = G_0 \leq G_1 \leq \cdots \leq G_t = F(G).$$

Since $F(G)/\phi(F(G))$ is abelian and $\phi(F(G)) \leq \phi(G)$, $F(G)/\phi(G)$ is abelian with at most d generators. Defining C_i to be the centralizer of G_i/G_{i-1} in G $(1 \leq i \leq t)$ and $D = \bigcap_{i=1}^{t} C_i$, we only need show that $D = F(G)$ to complete the proof of the theorem.

A theorem of P. Hall (**2**; Theorem 2) gives that $F(G)/\phi(G) = F(G/\phi(G))$ and so we may suppose without loss of generality that $\phi(G) = 1$. Lemma 4 now implies that $D \geq F(G)$. If $D \neq F(G)$, refine the series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_t = F(G) < D \leq \cdots \leq G$$

to a weak chief series of G and let G_{t+1} be the first term above $F(G)$ in this refinement. It is then clear that the centralizer of each factor in this weak chief series will contain G_{t+1} contrary to Lemma 4. Thus $D = F(G)$ as required.

Remark. The theorem of Hall referred to in the proof of Theorem 2 applies to a much wider class of groups than polycyclic. It is perhaps of interest to point out that Hall's theorem has a relatively easy proof for polycyclic groups. This proof uses similar methods to the standard proof for finite groups together with a theorem of Hirsch (**5**; Theorem 10.51) that a non-nilpotent polycyclic group has a finite epimorphic image which is non-nilpotent.

3.

In this section we prove two results.

Theorem 3. *There exists a function g such that if G is a polycyclic group with* $d_n(F(G)) \leq d$ *then the Fitting length of G is at most* $g(d)$.

Proof. Frick and Newman prove in (**1**) that a soluble linear group of degree d has Fitting length at most

$$s(d) = 4 + 2r(d) + [(2d-1)/8 \cdot 3^{r(d)}],$$

where $r(d) = [\log_3 (2d-1)/4]$. The result follows, using Theorems 1 and 2, taking $g(d) = s(f(d)) + 1$.

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We need some extra notation to state our second result. Let G be a polycyclic group and

$$1 = G_0 \leqslant \cdots \leqslant G_n = G$$

be a weak chief series for G which we denote by $\mathscr{C}$. Denote by $r_{\mathscr{C}}(G)$ the maximum of the integers $d(G_i/G_{i-1})$ $(1 \leqslant i \leqslant n)$ and let $r(G)$ be the maximum of $r_{\mathscr{C}}(G)$ as $\mathscr{C}$ ranges over all weak chief series for G. Our final result is

Theorem 4. *Let G be a polycyclic group with $d_n(F(G)) \leqslant d$. Then $r(G) \leqslant f(d) = 3d^2 + [d^2/4]$.*

Proof. Theorem 1 implies that any weak chief factor below $F(G)$ can be generated by at most $f(d)$ elements. By Theorem 2, $G/F(G)$ is isomorphic to a subgroup of a direct product $G_1 \times \cdots \times G_r$ where G_i is either an irreducible subgroup of $\mathrm{GL}(d_i, p_i)$ for some prime p_i or a rationally irreducible subgroup of $\mathrm{GL}(d_i, \boldsymbol{Z})$ where $d_i \leqslant f(d)$ $(1 \leqslant i \leqslant r)$. If G_i is a rationally irreducible subgroup of $\mathrm{GL}(d_i, \boldsymbol{Z})$, then G is irreducible regarded as a subgroup of $\mathrm{GL}(d_i, \boldsymbol{Q})$ (see remarks on pages 80–81 of Part I of (5)). A theorem of Huppert (**4**, Satz 12) states that a finite soluble linear group G of degree n which is completely reducible over an algebraically closed field has $r(G) \leqslant n$. In fact Huppert's result extends to polycyclic linear groups provided one uses results of Suprunenko (**6**) to handle the case where the group is primitive. Thus $r(G) \leqslant f(d)$ as required.

Remark. There is no upper bound in general for the derived length of a polycyclic group G with $d_n(F(G)) \leqslant d$. Theorem 1 of (**3**) shows that for any positive integer n there is a finite p-group $K(n)$ of derived length n all of whose subgroups can be generated by 3, or fewer, elements.

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