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SPHERE-LIKE ISOPARAMETRIC HYPERSURFACES IN DAMEK–RICCI SPACES

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Abstract Locally harmonic manifolds are Riemannian manifolds in which small geodesic spheres are isoparametric hypersurfaces, i.e., hypersurfaces whose nearby parallel hypersurfaces are of constant mean curvature. Flat and rank one symmetric spaces are examples of harmonic manifolds. Damek–Ricci spaces are non-compact harmonic manifolds, most of which are non-symmetric. Taking the limit of an 'inflating' sphere through a point p in a Damek–Ricci space as the center of the sphere runs out to infinity along a geodesic half-line γ starting from p, we get a horosphere. Similarly to spheres, horospheres are also isoparametric hypersurfaces. In this paper, we define the sphere-like hypersurfaces obtained by 'overinflating the horospheres' by pushing the center of the sphere beyond the point at infinity of γ along a virtual prolongation of γ. They give a new family of isoparametric hypersurfaces in Damek–Ricci spaces connecting geodesic spheres to some of the isoparametric hypersurfaces constructed by J. C. Díaz-Ramos and M. Domínguez-Vázquez [17] in Damek–Ricci spaces. We study the geometric properties of these isoparametric hypersurfaces, in particular their homogeneity and the totally geodesic condition for their focal varieties.

1. Introduction

A hypersurface in a Riemannian manifold is called isoparametric if its nearby parallel hypersurfaces have constant mean curvature. B. Segre [40] proved that isoparametric hypersurfaces of the Euclidean space $\mathbb{R}^n$ are the tubes about a k-dimensional subspace

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for some $0 \leqslant k \leqslant n-1$. A systematic study of isoparametric hypersurfaces was initiated by É. Cartan [4]. Cartan proved that in spaces of constant curvature, a hypersurface is isoparametric if and only if the multiset of its principal curvatures is constant and he also classified these hypersurfaces in hyperbolic spaces, where, in addition to tubes about totally geodesic subspaces, the family of isoparametric hypersurfaces contains also horospheres. The classification of isoparametric hypersurfaces in the sphere turned out to be a much more subtle problem. This is related to the fact that although all isoparametric hypersurfaces in the Euclidean and hyperbolic spaces are homogeneous, H. Ozeki and M. Takeuchi [35], [36], and D. Ferus, H. Karcher, and H.-F. Münzner [23] found infinitely many non-homogeneous isoparametric hypersurfaces in spheres. The classification of isoparametric hypersurfaces in spheres has been achieved in a sequence of papers by J. Dorfmeister and E. Neher [20], T. E. Cecil, Q.-S. Chi, and G. R. Jensen [5], Q.-S. Chi [6], [7], [8], [9], and R. Miyaoka [32], [33]. A detailed survey of the history of the classification of isoparametric hypersurfaces in spaces of constant curvature can be found in Q.-S. Chi [10].

There are also many results on the classification of isoparametric hypersurfaces in rank one symmetric spaces. Regular orbits of isometric cohomogeneity one actions are always isoparametric, so it is a natural step to classify such actions. As for rank one symmetric spaces of non-compact type, J. Berndt and H. Tamaru [2] classified these actions on the complex hyperbolic spaces and on the Cayley hyperbolic plane, and later J. C. Díaz-Ramos, M. Domínguez-Vázquez, and A. Rodríguez-Vázquez [18] complemented their result by a classification of cohomogeneity one isometric actions on the quaternionic hyperbolic space up to orbit equivalence. Later J. C. Díaz-Ramos and M. Domínguez-Vázquez [16] constructed a family of non-homogeneous isoparametric hypersurfaces in complex hyperbolic spaces, based on which J. C. Díaz-Ramos, M. Domínguez-Vázquez, and V. Sanmartín-López [19] could complete the classification of isoparametric hypersurfaces in complex hyperbolic spaces.

Harmonic manifolds have many properties in common with flat and rank one symmetric spaces, which are harmonic as well, so it is natural to investigate isoparametric hypersurfaces also in harmonic manifolds. Locally harmonic manifolds were introduced by E. T. Copson and H. S. Ruse [11] as Riemannian manifolds admitting a non-constant harmonic function in a punctured neighbourhood of any point p which depends only on the distance of the variable point from p. A. J. Ledger [30] showed that a locally symmetric space is locally harmonic if and only if it is flat or has rank one. In 1944 A. Lichnerowicz [31] conjectured that locally harmonic manifolds of dimension 4 are necessarily locally symmetric spaces and posed the question whether this holds in higher dimensions as well. The Lichnerowicz conjecture was proved by A. G. Walker [42] in dimension 4, and by Y. Nikolayevsky [34] in dimension 5. Z. I. Szabó [41] proved the Lichnerowicz conjecture for manifolds having compact universal covering space. G. Knieper [28] confirmed the Lichnerowicz conjecture for all compact harmonic manifolds without focal points or with Gromov hyperbolic fundamental groups. As for the non-compact case, the answer to the question of Lichnerowicz is negative in infinitely many dimensions starting at 7. E. Damek and F. Ricci [15] noticed that certain solvable extensions of some Heisenberg-type Lie groups become globally harmonic manifolds if we choose a

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suitable left invariant Riemannian metric on them, but they happen to be symmetric only if the used Heisenberg-type group has a center of dimension 1, 3, or 7. We refer to the book [3] by J. Berndt, F. Tricerri, and L. Vanhecke for more details on Damek–Ricci spaces. At present harmonic symmetric spaces and Damek–Ricci spaces are the only known examples of harmonic manifolds. In 2006 J. Heber [25] showed that a simply connected homogeneous harmonic manifold is either flat, or a rank one symmetric space, or a Damek–Ricci space. The existence of non-homogeneous harmonic manifolds is still an open problem.

There are many characterisations of harmonic manifolds. We refer to [3, Section 2.6] for a list of the most important ones. E. T. Copson and H. S. Ruse [11] proved that local harmonicity holds if and only if small geodesic spheres are isoparametric hypersurfaces. For a non-compact, complete, connected, simply connected, and locally harmonic manifold, the exponential map at any point p is a diffeomorphism between the tangent space at p and the manifold; in particular, there are no conjugate points along geodesic curves, and all geodesic spheres are isoparametric hypersurfaces.

If M is a complete, connected, and simply connected Riemannian manifold with no conjugate points, $\xi \in T_pM$ is a unit tangent vector, $\gamma\colon \mathbb{R} \to M$ is the geodesic curve with initial velocity $\xi = \gamma'(0)$, then the Busemann functions b_ξ^+ and b_ξ^- of ξ are defined as

$$b_\xi^\pm(x) = \lim_{t\to\pm\infty} b_{\xi,t}(x), \text{ where } b_{\xi,t}(x) = d(x,\gamma(t)) - |t|.$$

Horospheres are the level sets of Busemann functions. For $r \in \mathbb{R}$, the equation $b_{\xi,r}(x) = 0$ defines the geodesic sphere Σ_r^γ of radius $|r|$ centered at $\gamma(r)$, respectively. As r tends to $\pm\infty$, these ‘inflating’ spheres tend to the opposite horospheres $\Sigma_{\pm\infty}^\gamma$ with equation $b_\xi^\pm(x) = 0$. This family of geodesic spheres and horospheres belong to a one-parameter family of hypersurfaces parameterised by $r \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. (Strictly speaking, spheres of radius 0 are not hypersurfaces, but we think of them as degenerate hypersurfaces.) In general, one can prove only $\mathcal{C}^1$-differentiability of the Busemann functions and the horospheres [21]; however, A. Ranjan and H. Shah [38] proved that in a non-compact, complete, connected, simply connected, and harmonic manifold, both Busemann functions and horospheres are analytic.

In the case of a harmonic manifold, the union of the family Σ_r^γ, $r \in \bar{\mathbb{R}}$ is the union of two opposite horoballs tangent to one another at p. Applying the maximum principle for hypersurfaces [22, Theorem 1], these two horoballs cover the whole space if and only if $\Sigma_{-\infty}^\gamma = \Sigma_{+\infty}^\gamma$ and the horospheres are minimal hypersurfaces, which happens if and only if the harmonic manifold is Euclidean [28, Proposition 2.4]. Thus, in the non-flat case, there is a gap between the two opposite horoballs.

In the case of the real hyperbolic space, using Poincaré’s conformal model on a Euclidean ball, the spheres and horospheres Σ_r^γ, $r \in \bar{\mathbb{R}}$ are those members of a parabolic pencil of Euclidean spheres that are contained in the closure of the model. The gap between the two opposite horoballs is covered by those members of the pencil that are sticking out of the model. Intrinsically, the intersection of a protruding sphere with the model is a parallel hypersurface of a hyperplane orthogonal to the geodesic curve γ.

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Having in mind the example of the hyperbolic space, the following question seems to be interesting for an arbitrary non-compact, non-flat, simply connected complete harmonic manifold.

Question 1.1. Is there a kind of natural (analytic) prolongation of the family Σ_r^γ that fills the gap between the horospheres $\Sigma_{-\infty}^\gamma$ and $\Sigma_{+\infty}^\gamma$?

There are several ways to define what we mean by an analytical prolongation of the family Σ_r^γ. One approach, which will be used below is that we try to write the equation of Σ_r^γ in a form $F(x,\phi(r)) = 0$, where $\phi\colon \bar{\mathbb{R}} \to [-1,1]$ is a homeomorphism, which is analytic on $\mathbb{R}$, and $F\colon M \times [-1,1] \to \mathbb{R}$ is an analytic function. Then we extend the domain of F by analytic continuation as far as possible, and prolong the family of spheres Σ_r^γ with the hypersurfaces $\tilde{\Sigma}_\theta^\gamma$ defined by the equations $F(x,\theta) = 0$ for $|\theta| > 1$.

Question 1.2. If there is an analytic extension, then what can we say about the geometry of the hypersurfaces $\tilde{\Sigma}_\theta^\gamma$ for $|\theta| > 1$?

For example, it was proved by Z. I. Szabó [41] that in a harmonic manifold, the volume of the intersection of two geodesic balls of small radii depends only on the radii and the distance between the centers. The authors proved in [13, 14] that this property characterises harmonic manifolds even if this property is assumed only for balls of the same radius. It seems to be an interesting question whether analogous theorems can be proved for overinflated spheres. Some results in this direction were obtained by S. Kim and J. H. Park [27].

The main goal of this paper is to construct the family of 'overinflated spheres' $\tilde{\Sigma}_\theta^\gamma$ for $|\theta| > 1$ in the Damek–Ricci spaces, and study their geometric properties. As it can be expected, all the hypersurfaces $\tilde{\Sigma}_\theta^\gamma$ are isoparametric. The overinflated spheres are tubes about their focal varieties, which are known to be minimal submanifolds.

The paper is structured as follows. In Sections 2, 3, and 4, we collect preliminaries that will be needed later on isoparametric functions, on Damek–Ricci spaces and on the J^2-condition for $\mathfrak{v}$-vectors in the Damek–Ricci Lie algebra. Most of the facts listed here are known, but we add some proofs for the sake of the reader.

As the underlying Lie group of Damek–Ricci spaces is an exponential Lie group, Damek–Ricci spaces can be modelled on the Lie algebra of this Lie group. In Section 5, we introduce the so-called half-space model of Damek–Ricci spaces, which will be more useful for our constructions. For example, in Theorem 5.1, we show that geodesic lines are represented in the half-space model by the intersection of the model with a conic section or a straight line sticking out of the model. Thus, the corresponding conic section or straight line provides a virtual continuation of the geodesic line beyond its points at infinity.

Concentric geodesic spheres are the level sets of the distance function d_{x_0} from the common center x_0 of the spheres. In Section 6, we find a modification D_{x_0} of the distance function d_{x_0} such that the modified function has the same level sets as d_{x_0}, but the modified function D_{x_0} makes sense also if the point x_0 is moving out of the half-space model into the complementary half-space. In Theorem 6.1, we verify that the functions D_{x_0} are isoparametric for every point x_0 of the affine space containing the half-space model. In Section 7, we compute the equation of the focal varieties $\mathcal{F}_{x_0}$ of the isoparametric hypersurfaces obtained as the regular level sets of the functions D_{x_0}.

Based on the results of preceding sections, Section 8 answers Question 1.1 in Damek–Ricci spaces by constructing an analytic prolongation of the family of spheres Σ_r^γ explicitly. Members of the prolongation are level sets of the functions D_{x_0} passing through $p = \gamma(0)$, where x_0 is running over those points of the conic section or straight line containing the geodesic line γ that do not belong to the half-space model. When γ is contained in an ellipse in the half-space model, the prolongation yields a continuous transition between the opposite horospheres $\Sigma_{\pm\infty}^\gamma$. However, if γ is contained in a parabola or a straight line such a transition is obstructed by the lack of the definition of the function D_{x_0} for the case when x_0 is a point at infinity of the projective closure of the affine space containing the half-space model. To fix this problem, we compute the limit of a suitable rescaling of the function D_{x_0} as x_0 tends to infinity along the parabola or straight line containing γ. It turns out that the limit depends on γ, so it will be denoted by $D_{\circledast}^\gamma$. The limit functions $D_{\circledast}^\gamma$ are isoparametric, consequently their regular level sets are tubes about the singular level set $\mathcal{F}_{\circledast}^\gamma$ of $D_{\circledast}^\gamma$. The regular level sets of the functions $D_{\circledast}^\gamma$ belong to the family of isoparametric hypersurfaces constructed by J. C. Díaz-Ramos and M. Domínguez-Vázquez [17].

In Section 9, we study the family of geodesic curves meeting a given focal variety $\mathcal{F}_{x_0}$ or $\mathcal{F}_{\circledast}^\gamma$ orthogonally. These geodesic curves intersect each tube about the focal variety orthogonally. In the case of $\mathcal{F}_{x_0}$, we prove that the prolongations of these geodesics meet at the point x_0, and conversely, any geodesic curve, the prolongation of which goes through x_0 intersects the focal variety $\mathcal{F}_{x_0}$ orthogonally at some point p. We also prove that the points at infinity of the geodesic separate the points x_0 and p harmonically along the prolongation of γ. This implies that for a given geodesic curve γ for all points p of γ, except for at most one point p^*, there is a unique focal variety of the type $\mathcal{F}_{x_0}$ that meets γ at p orthogonally, and the focal varieties of the form $\mathcal{F}_{\circledast}^\eta$ can meet γ orthogonally only at the exceptional point p^*. If the exceptional point p^* exists, then $\mathcal{F}_{\circledast}^\gamma$ is defined and meets γ orthogonally at p^*. We prove that in a symmetric Damek–Ricci space, no other focal varieties of the type $\mathcal{F}_{\circledast}^\eta$ can intersect γ orthogonally at p^*, but if the space is not symmetric, such focal varieties can exist.

If a Damek–Ricci space is symmetric, then the focal varieties constructed in this paper are totally geodesic submanifolds. The general case is considered in Section 10. We prove that if the space is not symmetric, then none of the focal varieties of the form $\mathcal{F}_{x_0}$ are totally geodesic. However, each focal variety $\mathcal{F}_{x_0}$ has at least one point p such that $\mathcal{F}_{x_0}$ is the image of $T_p\mathcal{F}_{x_0}$ under the exponential map $\exp_p$. The set of such points is homeomorphic to the set of vectors satisfying the J^2-condition. Focal varieties of the form $\mathcal{F}_{\circledast}^\eta$ behave differently. They are homogeneous, so they are either totally geodesic or do not have such a point. It will be proved that the focal variety $\mathcal{F}_{\circledast}^\eta$ is totally geodesic if and only if a certain vector which defines it uniquely up to left translation satisfies the J^2-condition. We remark that totally geodesic submanifolds of Damek–Ricci spaces have been classified by S. Kim, Y. Nikolayevsky, and J. H. Park [26]. They showed that they are either subgroups ('smaller' Damek–Ricci spaces) or isometric to rank-one symmetric spaces of negative curvature. When the ambient Damek–Ricci space is not symmetric, totally geodesic focal varieties $\mathcal{F}_{\circledast}^\eta$ belong to the first group.

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Section 11 is devoted to the study of homogeneity of the overinflated spheres. They are all homogeneous in the symmetric case. In a non-symmetric Damek–Ricci space, it will be proved that tubes about a focal variety are homogeneous exactly in those cases, when the focal variety is totally geodesic.

Finally, in Section 12, we give an explicit formula for the mean curvature of the overinflated spheres as a function of their tube radius about their focal variety.

2. Isoparametric functions

Definition 2.1. A smooth function $F\colon M\to\mathbb{R}$ defined on a Riemannian manifold M is said to be isoparametric if there exist a continuous function $a\colon F(M)\to\mathbb{R}$ and a $\mathcal{C}^2$ function $b\colon F(M)\to\mathbb{R}$ such that

$$\Delta F = a\circ F \qquad \text{and} \qquad \|\nabla F\|^2 = b\circ F. \tag{1}$$

The geometrical meaning of the second, so-called transnormality condition is that nearby regular level sets of F are parallel hypersurfaces. The mean curvature H of a hypersurface Σ can be expressed as $H=\frac{1}{\dim\Sigma}h$, where h is the trace of the shape operator of Σ (with respect to a fixed unit normal). The following proposition shows that the regular level sets of an isoparametric function are isoparametric hypersurfaces and gives a formula for their mean curvature.

Proposition 2.2. *If F is an isoparametric function satisfying equations (1), then the trace h of the shape operator of a regular level set $F^{-1}(c)$ of F with respect to the unit normal vector field $\mathbf{N}=\frac{\nabla F}{\sqrt{b\circ F}}$ is expressed by*

$$h=\frac{-2a(c)+b'(c)}{2\sqrt{b(c)}}.$$

Proof. In an open neighbourhood of any point $p\in F^{-1}(c)$, the vector field $\mathbf{N}$ can be extended to an orthonormal frame $E_1,\dots,E_{n-1},E_n=\mathbf{N}$, where n is the dimension of the manifold. Then using the equations

$$\langle\nabla_{E_n}\mathbf{N},\mathbf{N}\rangle=0;\quad \nabla_{E_i}F=0 \text{ for } 1\leqslant i<n;\quad \text{and}\quad \nabla_{E_n(p)}F=\langle\nabla F(p),E_n(p)\rangle=\sqrt{b(c)},$$

we obtain

$$\begin{aligned} h(p)&=\sum_{i=1}^{n-1}\langle-\nabla_{E_i(p)}\mathbf{N},E_i(p)\rangle=\sum_{i=1}^{n}\langle-\nabla_{E_i(p)}\mathbf{N},E_i(p)\rangle=\sum_{i=1}^{n}\left\langle-\nabla_{E_i(p)}\left(\frac{\nabla F}{\sqrt{b\circ F}}\right),E_i(p)\right\rangle\\ &=-\frac{\Delta F(p)}{\sqrt{b(c)}}-\nabla_{E_n(p)}\left(\frac{1}{\sqrt{b\circ F}}\right)\langle\nabla F(p),E_n(p)\rangle=-\frac{a(c)}{\sqrt{b(c)}}+\frac{b'(c)}{2\sqrt{b(c)}}\end{aligned}$$

□

Definition 2.3. The singular level sets of an isoparametric function F are called the focal varieties of F.

There are some fundamental results of Q. M. Wang [43] and J. Ge and Z. Tang [24] on isoparametric functions.

Theorem 2.4 [43],[24]. *For an isoparametric function F on a connected and complete Riemannian manifold,*

(i) *only the minimal and maximal values of F can be singular;*
(ii) *the focal varieties of F are smooth minimal submanifolds;*
(iii) *the regular level sets of F are tubes about either of the focal varieties, having constant mean curvature.*

The following proposition is useful if we want to compute the radius of the tubes appearing in case (iii) of Theorem 2.4.

Proposition 2.5 [43]. *Let F be an isoparametric function which attains its minimal value c_0, and let $c > c_0$ be an arbitrary regular value of F. Then the level set $F^{-1}(c)$ is a tube of radius $r(c)$ about the focal variety $F^{-1}(c_0)$, where the radius $r(c)$ is given by the converging improper integral*

$$r(c) = \int_{c_0}^{c} \frac{\mathrm{d}x}{\sqrt{b(x)}}.$$

3. Damek–Ricci spaces

Damek–Ricci spaces are solvable Lie groups equipped with a left-invariant Riemannian metric. To construct a Damek–Ricci space, we have to fix

- a Euclidean linear space $(\mathfrak{s}, \langle\,,\rangle)$ with an orthogonal decomposition $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a 1-dimensional subspace spanned by a given unit vector $A \in \mathfrak{a}$;
- a representation $J\colon \mathrm{Cl}(\mathfrak{z}, q) \to \mathrm{End}(\mathfrak{v})$ of the Clifford algebra $\mathrm{Cl}(\mathfrak{z}, q)$ of the quadratic form $q\colon \mathfrak{z} \to \mathbb{R}$, $q(Z) = -\langle Z, Z\rangle$ such that

$$\|J_Z V\| = \|Z\| \|V\| \qquad \forall\, Z \in \mathfrak{z}, V \in \mathfrak{v}. \tag{2}$$

Equation (2) implies also the identities

$$\langle J_Z V_1, J_Z V_2\rangle = \|Z\|^2 \langle V_1, V_2\rangle \quad \text{and} \quad \langle J_Z V_1, V_2\rangle = -\langle V_1, J_Z V_2\rangle \qquad \forall\, Z \in \mathfrak{z}, V_1, V_2 \in \mathfrak{v}.$$

We can equip the linear space $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with a Lie algebra structure such that $[\mathfrak{n}, \mathfrak{z}] = \{0\}$ and $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$, defining the Lie bracket of $U, V \in \mathfrak{v}$ by

$$\langle [U, V], Z\rangle = \langle J_Z U, V\rangle \qquad \forall\, Z \in \mathfrak{z}. \tag{3}$$

If $\mathfrak{v} = \{0\}$ or $\mathfrak{z} = \{0\}$, then $\mathfrak{n}$ is commutative; otherwise $\mathfrak{n}$ is a 2-step nilpotent Lie algebra with center $\mathfrak{z}$.

Equation (3) implies immediately that $\ker_{\mathfrak{v}}(\mathrm{ad}\, U) = (J_{\mathfrak{z}} U)^{\perp} \cap \mathfrak{v}$, where $\ker_{\mathfrak{v}}(\mathrm{ad}\, U)$ abbreviates the intersection $\ker(\mathrm{ad}\, U) \cap \mathfrak{v}$. Hence, $\mathfrak{v}$ has an orthogonal direct sum decomposition

$$\mathfrak{v} = \ker_{\mathfrak{v}}(\mathrm{ad}\, U) \oplus J_{\mathfrak{z}} U = \mathbb{R}U \oplus \left(\ker_{\mathfrak{v}}(\mathrm{ad}\, U) \cap U^{\perp}\right) \oplus J_{\mathfrak{z}} U \tag{4}$$

for any $U \in \mathfrak{v}$.

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We can introduce a solvable Lie algebra structure on $\mathfrak{s}$ with the Lie bracket

$$[V+Z+sA,U+X+tA] = \left(\frac{s}{2}U - \frac{t}{2}V\right) + ([U,V]+sX-tZ).$$

The simply connected, connected Lie group S with Lie algebra $\mathfrak{s}$, equipped with the left invariant Riemannian metric induced by $\langle,\rangle$ is a Damek–Ricci space. We shall denote the normal Lie subgroup of S corresponding to the Lie algebra $\mathfrak{n}$ by $N \lhd S$.

There is a classification of Damek–Ricci spaces (i.e., a classification of the possible input data for the construction of a Damek–Ricci space). Every Damek–Ricci space is harmonic. The Damek–Ricci spaces corresponding to the degenerate cases $\dim\mathfrak{v} = 0$ or $\dim\mathfrak{z} = 0$ are isometric with a real hyperbolic space $\mathbb{R}\mathbf{H}^n$. The further rank one symmetric spaces $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$, and $\mathbb{O}\mathbf{H}^2$ are also among the Damek–Ricci spaces, with $\dim\mathfrak{z} = 1,3,7$, respectively, but none of the other Damek–Ricci spaces are symmetric. See [3, Sections 3.1.2, 4.1.2, 4.4] for details.

We collect some useful formulae in $\mathfrak{s}$. Denote by n and m the dimension of $\mathfrak{v}$ and $\mathfrak{z}$, respectively. Let $E_1,\dots,E_n$ be an orthonormal basis of $\mathfrak{v}$, $F_1,\dots,F_m$ be an orthonormal basis of $\mathfrak{z}$, and $A \in \mathfrak{a}$ be the unit vector introduced above. The Lie algebra structure on $\mathfrak{n}$ is given by the structure constants $C_{i,j,\alpha}(1 \leqslant i,j \leqslant n,\ 1 \leqslant \alpha \leqslant m)$ appearing in the decomposition $[E_i,E_j] = \sum_{\alpha=1}^m C_{i,j,\alpha}F_\alpha$. When there is no danger of confusion, we write $C_{ij\alpha}$ instead of $C_{i,j,\alpha}$.

Lemma 3.1. *Setting $J_\alpha = J_{F_\alpha}$, we have $J_\alpha(E_i) = \sum_{j=1}^n C_{ij\alpha}E_j$.*

Proof. The formula follows from $\langle J_\alpha(E_i),E_k\rangle = \langle [E_i,E_k],F_\alpha\rangle$. □

Lemma 3.2. *The structure constants satisfy the identities*

$$C_{ij\alpha} = -C_{ji\alpha}, \qquad \sum_{k=1}^n C_{ik\alpha}C_{kj\alpha} = -\delta_{ij} \quad \forall\, i,j,\alpha.$$

Proof. The first equation follows from the skew-symmetry of the Lie bracket, the second identity can be obtained by evaluating the identity $J_Z^2 = -\|Z\|^2\mathrm{Id}_{\mathfrak{v}}$ on the basis vectors $Z = F_\alpha$. □

Lemma 3.3 [3, p. 25]. *For any $U,V \in \mathfrak{v}$ and for any $X \in \mathfrak{z}$, we have*

$$[J_XU,V] - [U,J_XV] = -2\langle U,V\rangle X.$$

Proof. Polarising the identity $J_Z^2 = -\|Z\|^2\mathrm{Id}_{\mathfrak{v}}$, we obtain

$$J_XJ_Y + J_YJ_X = -2\langle X,Y\rangle \mathrm{Id}_{\mathfrak{v}} \tag{5}$$

If $Y \in \mathfrak{z}$ is an arbitrary element, then this gives

$$\begin{aligned}\langle [J_XU,V] - [U,J_XV],Y\rangle &= \langle J_YJ_XU,V\rangle - \langle J_YU,J_XV\rangle \\ &= \langle J_YJ_XU,V\rangle + \langle J_XJ_YU,V\rangle = -2\langle U,V\rangle\langle X,Y\rangle\end{aligned}$$

and this implies the statement. □

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Equation (5) provides also the identity

$$\left\langle J_X V, J_Y V \right\rangle = -\left\langle \tfrac{1}{2}(J_Y J_X + J_X J_Y)V, V \right\rangle = \langle X, Y\rangle \|V\|^2. \tag{6}$$

Lemma 3.4. *For $V \in \mathfrak{v}$, let $P_V : \mathfrak{v} \to J_{\mathfrak{z}}V$ be the orthogonal projection onto $J_{\mathfrak{z}}V$. Then for any $V, V_1, V_2 \in \mathfrak{v}$, we have*

$$\langle [V, V_1], [V, V_2]\rangle = \|V\|^2 \langle P_V(V_1), P_V(V_2)\rangle.$$

Proof. The statement is true for $V = 0$. If $V \neq 0$, then $\left\{ \frac{1}{\|V\|} J_\alpha V : 1 \leqslant \alpha \leqslant m \right\}$ is an orthonormal basis of $J_{\mathfrak{z}}V$ by (6), and

$$\begin{aligned} \langle [V, V_1], [V, V_2]\rangle &= \sum_{\alpha=1}^{m} \langle F_\alpha, [V, V_1]\rangle \cdot \langle F_\alpha, [V, V_2]\rangle \\ &= \sum_{\alpha=1}^{m} \langle J_\alpha V, V_1\rangle \cdot \langle J_\alpha V, V_2\rangle = \|V\|^2 \langle P_V(V_1), P_V(V_2)\rangle. \end{aligned}$$

□

Rewriting the obtained identity in the form

$$\langle J_{[V,V_1]} V, V_2\rangle = \langle [V, V_1], [V, V_2]\rangle = \|V\|^2 \langle P_V(V_1), P_V(V_2)\rangle = \langle \|V\|^2 P_V(V_1), V_2\rangle,$$

we get the following corollary.

Corollary 3.5 [12, Eq. (1.8)]. *For any $V, V_1 \in \mathfrak{v}$, we have*

$$J_{[V,V_1]} V = \|V\|^2 P_V(V_1).$$

Lemma 3.6. *We have also the identity*

$$\sum_{i=1}^{n} \langle [E_i, V], [E_i, W]\rangle = m\langle V, W\rangle \quad \forall V, W \in \mathfrak{v}.$$

Proof. Write V and W as a linear combination $V = \sum_{j=1}^{n} V_j E_j$ and $W = \sum_{k=1}^{n} W_k E_k$. Then

$$\begin{aligned} \sum_{i=1}^{n} \langle [E_i, V], [E_i, W]\rangle &= \sum_{i,j,k=1}^{n} \langle [E_i, E_j], [E_i, E_k]\rangle V_j W_k = \sum_{i,j,k=1}^{n} \sum_{\alpha=1}^{m} C_{ij\alpha} C_{ik\alpha} V_j W_k \\ &= \sum_{\alpha=1}^{m} \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} -C_{ji\alpha} C_{ik\alpha} \right) V_j W_k = \sum_{\alpha=1}^{m} \sum_{j,k=1}^{n} \delta_{jk} V_j W_k = m\langle V, W\rangle. \end{aligned}$$

□

4. The J^2-condition

The J^2-condition for generalised Heisenberg type groups was introduced by M. Cowling, A. H. Dooley, Á. Korányi, and F. Ricci [12]. Their definition can be adapted to Damek–Ricci spaces.

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Definition 4.1. In a Damek–Ricci space, we say that a vector $v\in\mathfrak{v}$ satisfies the J^2-condition if for any $z_1,z_2\in\mathfrak{z}$, $z_1\perp z_2$, there exists an element $z_3\in\mathfrak{z}$ such that $J_{z_1}J_{z_2}v=J_{z_3}v$.

Definition 4.2. A Damek–Ricci space satisfies the J^2-condition if every vector in $\mathfrak{v}$ satisfies the J^2-condition.

Lemma 4.3. *A vector $v\in\mathfrak{v}$ satisfies the J^2-condition if and only if the $\mathrm{Cl}(\mathfrak{z},q)$-submodule $\mathrm{Cl}(\mathfrak{z},q)v$ of $\mathfrak{v}$ generated by the element v coincides with $\mathbb{R}v\oplus J_{\mathfrak{z}}v$.*

Proof. It is clear that $v\in\mathbb{R}v\oplus J_{\mathfrak{z}}v\leqslant\mathrm{Cl}(\mathfrak{z},q)v$ for any v, so we need to show that $\mathbb{R}v\oplus J_{\mathfrak{z}}v$ is a $\mathrm{Cl}(\mathfrak{z},q)$-module if and only if v satisfies the J^2-condition.

Assume first that v satisfies the J^2-condition and show that $\mathbb{R}v\oplus J_{\mathfrak{z}}v$ is a $\mathrm{Cl}(\mathfrak{z},q)$-module. Since $\mathrm{Cl}(\mathfrak{z},q)$ is generated by the elements of $\mathfrak{z}\subset\mathrm{Cl}(\mathfrak{z},q)$, it suffices to prove that $J_{z_1}(\mathbb{R}v\oplus J_{\mathfrak{z}}v)\subseteq\mathbb{R}v\oplus J_{\mathfrak{z}}v$ for all $z_1\in\mathfrak{z}$. Choose an arbitrary element $\lambda v+J_zv$ of $\mathbb{R}v\oplus J_{\mathfrak{z}}v$ and decompose z as $z=\mu z_1+z_2$, where $z_1\perp z_2$. Then there is an element $z_3\in\mathfrak{z}$ such that $J_{z_1}J_{z_2}v=J_{z_3}v$, thus $J_{z_1}(\lambda v+J_zv)=-\mu\|z_1\|^2v+J_{\lambda z_1+z_3}v\in\mathbb{R}v\oplus J_{\mathfrak{z}}v$, as we wanted to show.

Conversely, if $\mathbb{R}v\oplus J_{\mathfrak{z}}v$ is a $\mathrm{Cl}(\mathfrak{z},q)$-module, and $z_1\perp z_2$ are two elements of $\mathfrak{z}$, then $J_{z_2}v\in\mathbb{R}v\oplus J_{\mathfrak{z}}v$ implies $J_{z_1}J_{z_2}v\in\mathbb{R}v\oplus J_{\mathfrak{z}}v$, so there exist $\lambda\in\mathbb{R}$ and z_3 such that $J_{z_1}J_{z_2}v=\lambda v+J_{z_3}v$. Since

$$\langle v,J_{z_3}v\rangle=\langle z_3,[v,v]\rangle=0\quad\text{and}\quad\langle v,J_{z_1}J_{z_2}v\rangle=-\langle J_{z_1}v,J_{z_2}v\rangle=-\langle z_1,z_2\rangle\|v\|^2=0,$$

λ must vanish, therefore $J_{z_1}J_{z_2}v=J_{z_3}v$. □

Corollary 4.4. *A vector $v\in\mathfrak{v}$ satisfies the J^2-condition if and only if $\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$ is a $\mathrm{Cl}(\mathfrak{z},q)$-submodule of $\mathfrak{v}$. This is also equivalent to the condition $J_{\mathfrak{z}}(\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp})\subseteq\ker_{\mathfrak{v}}(\operatorname{ad}v)$.*

Proof. The first part follows from the fact that the orthogonal complement of a $\mathrm{Cl}(\mathfrak{z},q)$-submodule of $\mathfrak{v}$ is also a $\mathrm{Cl}(\mathfrak{z},q)$-submodule of $\mathfrak{v}$ as the operators J_z are skew adjoint. To show the second part, it is enough to check that if $w\in\ker_{\mathfrak{v}}(\operatorname{ad}v)$, then $J_zw\perp v$ for any $z\in\mathfrak{z}$. However, this follows from $\langle J_zw,v\rangle=\langle[w,v],z\rangle=0$. □

Proposition 4.5. *If there is a non-zero vector $v\in\mathfrak{v}$ which satisfies the J^2-condition, then $m=\dim\mathfrak{z}\in\{0,1,3,7\}$ and $\mathbb{R}v\oplus J_{\mathfrak{z}}v$ is a non-trivial irreducible $\mathrm{Cl}(\mathfrak{z},q)$-module.*

Proof. Recall the classification of Clifford modules over $\mathrm{Cl}(\mathfrak{z},q)$ (see [3, Sec. 3.1.2] or [29, Ch. I, §. 5]).

(a) If $m\not\equiv 3\pmod 4$, then there exists a unique (up to isomorphism) irreducible $\mathrm{Cl}(\mathfrak{z},q)$-module $\mathfrak{d}$. Every $\mathrm{Cl}(\mathfrak{z},q)$-module $\mathfrak{v}$ is isomorphic to a k-fold direct sum of $\mathfrak{d}$, that is, $\mathfrak{v}\cong\bigoplus^k\mathfrak{d}$.

(b) If $m\equiv 3\pmod 4$, then there exists exactly two non-isomorphic irreducible $\mathrm{Cl}(\mathfrak{z},q)$-modules $\mathfrak{d}_1$ and $\mathfrak{d}_2$. Every $\mathrm{Cl}(\mathfrak{z},q)$-module $\mathfrak{v}$ is isomorphic to the direct sum $\mathfrak{v}\cong\left(\bigoplus^{k_1}\mathfrak{d}_1\right)\oplus\left(\bigoplus^{k_2}\mathfrak{d}_2\right)$ for some k_1 and k_2. The modules $\mathfrak{d}_1$ and $\mathfrak{d}_2$ have the same dimension.

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The formula for the dimension n_0 of the modules $\mathfrak{d}$, $\mathfrak{d}_1$, and $\mathfrak{d}_2$ depends on the modulo 8 residue class of m and is given in the following table.

m	$8p$	$8p+1$	$8p+2$	$8p+3$	$8p+4$	$8p+5$	$8p+6$	$8p+7$
n_0	2^{4p}	2^{4p+1}	2^{4p+2}	2^{4p+2}	2^{4p+3}	2^{4p+3}	2^{4p+3}	2^{4p+3}

If there exists a non-zero vector $v \in \mathfrak{v}$ which satisfies the J^2-condition, then $\mathrm{Cl}(\mathfrak{z},q)v = \mathbb{R}v \oplus J_{\mathfrak{z}}v$. The dimension of $\mathrm{Cl}(\mathfrak{z},q)v$ is a multiple of n_0, the dimension of $\mathbb{R}v \oplus J_{\mathfrak{z}}v$ is $m+1$, so we must have $n_0 \leqslant m+1$. As $8p+7 < 2^{4p}$ if $p > 0$, inequality $n_0 \leqslant m+1$ can hold only if $p=0$. Among the eight values of m corresponding to $p=0$, exactly the values $0,1,3,7$ satisfy the inequality $n_0 \leqslant m+1$. Since in these four cases we have $n_0 = m+1$ in fact, $\mathbb{R}v \oplus J_{\mathfrak{z}}v$ is a non-trivial irreducible $\mathrm{Cl}(\mathfrak{z},q)$-module. □

The following proposition describes the set of vectors satisfying the J^2-condition in those cases when this set contains a non-zero vector.

Theorem 4.6. *For a given $\mathfrak{z}$, $\dim \mathfrak{z} = m$, let $\mathfrak{d}$ or $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be the irreducible $\mathrm{Cl}(\mathfrak{z},q)$-modules appearing in the previous proof.*

(i) *If $m \in \{0,1\}$ and $\mathfrak{v} = \bigoplus^k \mathfrak{d}$, then all elements of $\mathfrak{v}$ satisfy the J^2-condition.*

(ii) *If $m=3$ and $\mathfrak{v} = \left(\bigoplus^{k_1} \mathfrak{d}_1\right) \oplus \left(\bigoplus^{k_2} \mathfrak{d}_2\right)$ then a vector $v \in \mathfrak{v}$ satisfies the J^2-condition if and only if v is isotypic, i.e., v is either in $\bigoplus^{k_1} \mathfrak{d}_1$ or in $\bigoplus^{k_2} \mathfrak{d}_2$.*

(iii) *If $m=7$ and $\mathfrak{v} = \left(\bigoplus^{k_1} \mathfrak{d}_1\right) \oplus \left(\bigoplus^{k_2} \mathfrak{d}_2\right)$ then a vector $v \in \mathfrak{v}$ satisfies the J^2-condition if and only if v is isotypic, and if $i \in \{1,2\}$ is the index for which $v \in \bigoplus^{k_i} \mathfrak{d}_i$, then v has the form $v = (\lambda_1 w, \dots, \lambda_{k_i} w)$, where w is an element of $\mathfrak{d}_i$ and the coefficients $\lambda_1, \dots, \lambda_{k_i}$ are real numbers.*

Proof. We consider all cases simultaneously, writing $\mathfrak{v} = \mathfrak{D}_1 \oplus \cdots \oplus \mathfrak{D}_K$, where $K = k$ and $\mathfrak{D}_i = \mathfrak{d}$ for all i in case (i), while in cases (ii) and (iii), we set $K = k_1 + k_2$, $\mathfrak{D}_i = \mathfrak{d}_1$ for $1 \leqslant i \leqslant k_1$ and $\mathfrak{D}_i = \mathfrak{d}_2$ for $k_1 < i \leqslant k_1 + k_2$.

Assume that v satisfies the J^2-condition. Let $\pi_i \colon \mathfrak{v} \to \mathfrak{D}_i$ be the projection onto the ith component and $v_i = \pi_i(v)$. The restriction π_i^v of π_i onto the submodule $\mathrm{Cl}(\mathfrak{z},q)v$ is a module homomorphism between two irreducible modules, hence it is either the 0-homomorphism or an isomorphism of modules. This implies that if $v_i \neq 0$, then $\mathfrak{D}_i \cong \mathrm{Cl}(\mathfrak{z},q)v$. Hence v must be isotypic.

Furthermore, if $i < j$ are two indices for which $v_i \neq 0$ and $v_j \neq 0$, then $\pi_j \circ (\pi_i^v)^{-1} \colon \mathfrak{D}_i \to \mathfrak{D}_j$ is a module isomorphism mapping v_i to v_j. Actually, the condition that for any pair of indices $i < j$ for which $v_i \neq 0$ and $v_j \neq 0$, there exists a module isomorphism $\mathfrak{D}_i \to \mathfrak{D}_j$ mapping v_i to v_j is also sufficient for an isotypic vector to satisfy the J^2-condition.

To understand what this characterisation of the J^2-condition means in different cases, we need a description of the module automorphisms of the irreducible Clifford modules. Let $\mathbb{K}$ denote $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ corresponding to the cases $m = 0,1,3,7$, respectively. In each case, we can construct a realisation of the modules $\mathfrak{d}$ or $\mathfrak{d}_1$ and $\mathfrak{d}_2$ on the linear space

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$\mathbb{K}$ thinking of $\mathfrak{z}$ as the linear space of purely imaginary elements $\mathbb{K}$. Then the operator $J_z\colon \mathbb{K}\to\mathbb{K}$ for $z\in\mathfrak{z}$ is the multiplication by z in the commutative cases $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. When $\mathbb{K}$ is not commutative, J_z can act on $\mathbb{K}$ both by left and by right multiplications by z, providing the two non-isomorphic Clifford module structures $\mathfrak{d}_1$ and $\mathfrak{d}_2$ of $\mathbb{K}$. In each case when J_z equals the left multiplication by z, a module automorphism $\phi\colon \mathbb{K}\to\mathbb{K}$ is a right multiplication by a non-zero element of the nucleus of $\mathbb{K}$. Recall that the nucleus of an alternative algebra consists of elements x satisfying the associativity identity $(ab)x=a(bx)$ for every a, b. Similarly, when J_z is the right multiplication by z, a module automorphism $\phi\colon \mathbb{K}\to\mathbb{K}$ is a left multiplication by a non-zero element of the nucleus. It is clear that the nucleus of an associative algebra equals the whole algebra, and it is known that the nucleus of the algebra of octonions is $\mathbb{R}$. This means that in the associative cases $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, the automorphism groups of the irreducible Clifford modules act transitively on non-zero vectors, while in the case of $\mathbb{O}$, two non-zero vectors belong to the same orbit of the automorphism group if and only if they are real multiples of one another. This completes the proof. □

Remark 4.7. The above characterisation of vectors having the J^2-condition implies a theorem of M. Cowling et al. [12] saying that a Damek–Ricci space satisfies the J^2-condition if and only if it is a symmetric space.

Proposition 4.8. *The following statements are equivalent for a Damek–Ricci space:*

(i) *The space is symmetric.*

(ii) *The space satisfies the J^2-condition.*

(iii) *If for the non-zero vectors $v_1,v_2\in\mathfrak{v}$ the intersection $(J_\mathfrak{z}v_1\oplus\mathbb{R}v_1)\cap(J_\mathfrak{z}v_2\oplus\mathbb{R}v_2)$ has a non-zero element, then $J_\mathfrak{z}v_1\oplus\mathbb{R}v_1=J_\mathfrak{z}v_2\oplus\mathbb{R}v_2$.*

Proof. By the above remark, the equivalence (i) $\Longleftrightarrow$ (ii) is proved in [12] for Damek–Ricci spaces and it also follows from Theorem 4.6.

Implication (ii)$\Longrightarrow$(iii) follows from the fact that if $v_1,v_2\in\mathfrak{v}\setminus\{0\}$ satisfy the J^2-condition, then $J_\mathfrak{z}v_1\oplus\mathbb{R}v_1$ and $J_\mathfrak{z}v_2\oplus\mathbb{R}v_2$ are irreducible Clifford modules, so their intersection, which is a submodule of both, is either 0 or equal to both.

Now we prove (iii)$\Longrightarrow$(ii). By Lemma 4.3, we need to show that for any non-zero vector $v\in\mathfrak{v}$, $J_\mathfrak{z}v\oplus\mathbb{R}v$ is a $\mathrm{Cl}(\mathfrak{z},q)$-submodule. Choose an arbitrary non-zero element $w\in J_\mathfrak{z}v\oplus\mathbb{R}v$. Then w is in the intersection $(J_\mathfrak{z}v\oplus\mathbb{R}v)\cap(J_\mathfrak{z}w\oplus\mathbb{R}w)$, hence $J_\mathfrak{z}v\oplus\mathbb{R}v=J_\mathfrak{z}w\oplus\mathbb{R}w$, therefore $J_\mathfrak{z}w\subseteq J_\mathfrak{z}v\oplus\mathbb{R}v$. □

5. The half-space model of Damek–Ricci spaces

A convenient model of Damek–Ricci spaces can be built on the linear space $\mathfrak{n}\oplus\mathbb{R}$ by pulling back the Riemannian metric of S by the diffeomorphism $\Phi\colon \mathfrak{n}\oplus\mathbb{R}\to S$ defined by

$$\Phi(Q,\tau)=\exp(Q)\exp(\tau A),$$

where exp is the exponential map of the Lie group S. Fixing orthonormal bases $E_1,\dots,E_n\in\mathfrak{v}$ and $F_1,\dots,F_m\in\mathfrak{z}$, the map Φ provides a global coordinate system $(v_1,\dots,v_n;z_1,\dots,z_m;\tau)\colon S\to\mathbb{R}^{n+m+1}$ by

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$$\Phi^{-1}(p) = \left(\sum_{i=1}^{n} v_i(p)E_i + \sum_{\alpha=1}^{m} z_\alpha(p)F_\alpha, \tau(p)\right) \qquad \text{for } p \in S.$$

The basis vector fields induced by this chart on S will be denoted by $\partial_{v_1},\dots,\partial_{v_n};\partial_{z_1},\dots,\partial_{z_m};\partial_\tau$.

Every Damek–Ricci space is an Hadamard manifold; therefore its ideal boundary can be defined in the usual manner. To deal with the ideal boundary $\partial_\infty S$ of S, we shall prefer to model the Damek–Ricci space on the open upper half-space $\mathfrak{n}\times\mathbb{R}_+ \subset \mathfrak{n}\oplus\mathbb{R}$ obtained by the modification

$$\Psi\colon \mathfrak{n}\times\mathbb{R}_+ \to S, \qquad \Psi(Q,t) = \Phi(Q,\ln t)$$

of the diffeomorphism Φ. (We use the sign $\times$ in the expression $\mathfrak{n}\times\mathbb{R}_+$ instead of the sign $\oplus$ since $\mathbb{R}_+$ is not a linear space.)

Using this half-space model, the ideal boundary $\partial_\infty S$ of S can be identified with the one-point compactification of the hyperplane $\mathfrak{n}\times\{0\}$.

Furthermore, it is easy to rewrite known formulae computed in the model $S \cong \mathfrak{n}\oplus\mathbb{R}$ to the half-space model by the simple coordinate transformation $t = e^\tau$.

For example, rewriting the multiplication rule of the group S computed in [3, Sec. 4.1.3] to the half-space model $S \overset{\Psi}{\cong} \mathfrak{v}\oplus\mathfrak{z}\times\mathbb{R}_+$, we obtain

$$(V_1,Z_1,t_1)\cdot(V_2,Z_2,t_2) = (V_1+\sqrt{t_1}V_2, Z_1+t_1Z_2+\tfrac{1}{2}\sqrt{t_1}[V_1,V_2], t_1t_2).$$

It is clear from this equation that the left translation

$$L_{(\bar V,\bar Z,\bar t)}\big((V,Z,t)\big) = \left(\sqrt{\bar t}V, \bar t Z + \tfrac{1}{2}\sqrt{\bar t}\,\mathrm{ad}\,\bar V(V), \bar t t\right) + \left(\bar V,\bar Z,0\right). \tag{7}$$

by an arbitrary element $(\bar V,\bar Z,\bar t)\in S$ extends to the whole space $\mathfrak{n}\oplus\mathbb{R}$ as an affine transformation.

Any geodesic of the Damek–Ricci space can be obtained as a left translation of a geodesic starting from the identity element $e=(0,0,1)$ of S. If $\xi=(v,z,s)\in\mathfrak{s}\cong T_eS$ is a unit tangent vector, then by [3, Sec. 4.1.11, Thm. 1], the geodesic $\hat\gamma$ with initial velocity ξ is given by $\hat\gamma(t)=\gamma(\tanh(t/2))$, where

$$\gamma(\theta) = \left(\frac{2\theta(1-s\theta)}{\chi(\theta)}v + \frac{2\theta^2}{\chi(\theta)}J_zv, \frac{2\theta}{\chi(\theta)}z, \frac{1-\theta^2}{\chi(\theta)}\right), \qquad \chi(\theta) = (1-s\theta)^2+\|z\|^2\theta^2. \tag{8}$$

Formula (8) can be evaluated for any real number θ, for which $\chi(\theta)\neq 0$, but for $|\theta|\geqslant 1$, $\gamma(\theta)$ is lying in the closed lower half-space $\mathfrak{n}\times(-\infty,0]$. Let $\mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$ be the projective space obtained by adding points at infinity to the affine space $\mathfrak{s}$, which is naturally isomorphic to the projective space associated to the linear space $\mathfrak{s}\oplus\mathbb{R}$. The point at infinity of the straight line $\mathfrak{a}$ will play a special role in this paper and will be denoted by $\circledast$. The curve γ extends continuously to a map $\mathbb{R}\mathbf{P}^1\to\mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$, which we denote by the same symbol γ. It is clear that

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$$\gamma(\infty) = \begin{cases} \left(-\dfrac{2s}{s^2+\|z\|^2}v + \dfrac{2}{s^2+\|z\|^2}J_z v, 0, -\dfrac{1}{s^2+\|z\|^2}\right) & \text{if } \|v\| \neq 1, \\ \circledast & \text{if } \|v\| = 1. \end{cases} \tag{9}$$

We shall call the map $\gamma\colon \mathbb{R}\mathbf{P}^1 \to \mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$ the prolongation of the geodesic curve $\hat{\gamma}$.

Theorem 5.1. *The map $\gamma\colon \mathbb{R}\mathbf{P}^1 \to \mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$ is a birational equivalence onto*

(i) *an ellipse in $\mathfrak{s}$ if $z \neq 0$;*

(ii) *the closure in $\mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$ of a parabola in $\mathfrak{s}$ with axis parallel to $\mathfrak{a}$ if $z = 0$, but $v \neq 0$;*

(iii) *the closure in $\mathbf{P}(\mathfrak{s}\oplus\mathbb{R})$ of a straight line parallel to $\mathfrak{a}$ if $\xi \in \mathfrak{a}$.*

Proof. Observe that the geodesic is contained in the linear subspace generated by the pairwise orthogonal vectors $v, J_z v, z, A$. Choose an orthonormal system of vectors $E_v, E_J, E_z \in \mathfrak{n}$ such that

$$v = \|v\|E_v, \quad J_z v = \|J_z v\|E_J, \quad z = \|z\|E_z,$$

and denote by $X(\theta), Y(\theta), Z(\theta), W(\theta)$ the coefficients of $\gamma(\theta)$ in the decomposition

$$\gamma(\theta) = X(\theta)E_v + Y(\theta)E_J + Z(\theta)E_z + W(\theta)A.$$

As χ is a quadratic polynomial of θ, the functions X, Y, Z, W are rational functions of θ:

$$X(\theta) = \frac{2\theta(1-s\theta)}{\chi(\theta)}\|v\|, \quad Y(\theta) = \frac{2\theta^2}{\chi(\theta)}\|z\|\|v\|, \quad Z(\theta) = \frac{2\theta}{\chi(\theta)}\|z\|, \quad W(\theta) = \frac{1-\theta^2}{\chi(\theta)}.$$

Using the relation $\|\xi\|^2 = \|v\|^2 + \|z\|^2 + s^2 = 1$, a simple algebraic computation shows that the functions X, Y, Z, W satisfy the equations

$$\|z\|X + sY - \|v\|Z = 0, \tag{10}$$

$$\left(1 - \frac{\|v\|^2}{2}\right)Y - s\|v\|Z + \|v\|\|z\|W = \|v\|\|z\|, \tag{11}$$

$$\|z\|(X^2+Y^2) - 2\|v\|Y = 0. \tag{12}$$

Case (i): If $z \neq 0$, we distinguish two cases depending on v.

If $v \neq 0$, then we can express Z and W as affine functions of the vector $XE_v + YE_J$ from the equations (10) and (11); therefore, the image of γ is contained in an affine image of the linear subspace spanned by E_v and E_J. Since (12) defines a circle in this linear subspace, the image of γ is an ellipse. In this case, we can express θ as the rational function $\frac{Y}{Z\|v\|}$ of the coordinates of $\gamma(\theta)$, thus γ is a birational equivalence.

If $v = 0$, then $X = Y = 0$ and γ is in the linear subspace spanned by E_z and A. It can be verified that in this case, the coordinate functions Z and W satisfy the quadratic equation

$$Z^2 + W^2 - \frac{2s}{\|z\|}Z = 1,$$

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which defines a circle. The parameter θ can be expressed as the rational function $\frac{Z}{sZ+\|z\|W+\|z\|}$ of the coordinates of $\gamma(\theta)$, hence γ is a birational equivalence.

Case (ii): If $z=0$, but $v\neq 0$, then $Y=Z=0$, thus, the image of γ is in the linear subspace spanned by E_v and A, and the coordinates of $\gamma(\theta)$ satisfy equation

$$4\|v\|^2W+X^2-(2\|v\|+sX)^2=0,$$

which defines a parabola with axis parallel to $\mathfrak{a}$. As $\theta=\frac{X(\theta)}{2\|v\|+sX(\theta)}$, γ is a birational equivalence.

Case (iii): If $z=0$ and $v=0$, then $s=1$ and $\gamma(\theta)=\left(0,0,\frac{1+\theta}{1-\theta}\right)$, so γ parameterises the projective line containing $\mathfrak{a}$. □

As the group S acts on itself simply transitively and isometrically by left translations, any unit speed geodesic in S can be written uniquely as a map $\hat{\eta}\colon t\mapsto\eta(\tanh(t/2))$, where $\eta=L_{(\bar V,\bar Z,\bar t)}\circ\gamma$ is the composition of the left translation $L_{(\bar V,\bar Z,\bar t)}$ by $(\bar V,\bar Z,\bar t)\in S$ and the curve γ defined by (8) from a fixed unit tangent vector $(v,z,s)\in T_eS$. Left translations (7) act on the half-space model as an affine transformation fixing the point at infinity $\circledast$ in the direction of A. Therefore, any regular geodesic curve is represented in the half-space model either by an arc of an ellipse, or by an arc of a parabola with axis parallel to A, or by a half-line parallel to A, and the affine type of the representing curve is invariant under left translations.

6. Distance-like isoparametric functions in Damek–Ricci spaces

Denote by d the distance function on S induced by the Riemannian metric. As it is shown in [39, Sec. 4.4, Eq. (21)], the distance of the points $x_i=(V_i,Z_i,t_i)\in\mathfrak{v}\oplus\mathfrak{z}\times\mathbb{R}_+\cong S$, $(i=0,1)$ satisfies the equation

$$4\sinh^2\left(\frac{d(x_1,x_0)}{2}\right)=\left(\frac{t_1}{t_0}+\frac{t_0}{t_1}-2\right)+\frac{t_1+t_0}{2t_1t_0}\|V_1-V_0\|^2 + \frac{1}{t_1t_0}\left(\left\|Z_1-Z_0+\frac{1}{2}[V_1,V_0]\right\|^2+\frac{\|V_1-V_0\|^4}{16}\right).$$

This equation can be compressed to the form

$$4\cosh^2\left(\frac{d(x_1,x_0)}{2}\right)=\frac{1}{t_1t_0}\left(\left(t_1+t_0+\left\|\frac{V_1-V_0}{2}\right\|^2\right)^2+\left\|Z_1-Z_0+\frac{1}{2}[V_1,V_0]\right\|^2\right).$$

For any given center $x_0=(V_0,Z_0,t_0)\in\mathfrak{v}\oplus\mathfrak{z}\times\mathbb{R}_+$, the distance function $d_{x_0}(.)=d(.,x_0)$ has the same level sets as the smooth 'distorted distance' function

$$D_{x_0}((V,Z,t))=\frac{1}{t}\left(\left(t+t_0+\left\|\frac{V-V_0}{2}\right\|^2\right)^2+\left\|Z-Z_0+\frac{1}{2}[V,V_0]\right\|^2\right), \tag{13}$$

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which is related to the function d_{x_0} by the formula

$$4\cosh^2\left(\frac{d_{x_0}}{2}\right) = \frac{1}{t_0}D_{x_0}.$$

Observe that $tD_{x_0}((V,Z,t))$ is a quartic polynomial function on the linear space $\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$.

As the Damek–Ricci spaces are harmonic and the regular level sets of the function D_{x_0} are the geodesic spheres about x_0, they are parallel hypersurfaces of one another and each of them has constant mean curvature. This implies that the function D_{x_0} is isoparametric.

The key observation is that in contrast to the function d_{x_0}, which diverges when x_0 tends to a point at infinity, the distorted distance function makes sense also for the points not belonging to the upper half-space, i.e., for points with non-positive t_0.

Theorem 6.1. *For any $x_0 = (V_0,Z_0,t_0) \in \mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$, the function D_{x_0} is an isoparametric function on S.*

Proof. It suffices to show that there exist smooth functions a, b such that $\Delta D_{x_0} = a\circ D_{x_0}$ and $\|\nabla D_{x_0}\|^2 = b\circ D_{x_0}$. We refer to [3, Sec. 4.4, Lemma] for the computation of the Laplace operator Δ of S. We note that there is an unnecessary coefficient $\frac{1}{2}$ in the formula in [3]. The correct formula and its transcription to the half-space model are

$$\begin{aligned}\Delta &= e^\tau\sum_{i=1}^n \partial_{v_i}^2 + e^\tau\left(e^\tau + \frac{1}{4}\sum_{i=1}^n v_i^2\right)\sum_{\alpha=1}^m \partial_{z_\alpha}^2 + \partial_\tau^2 - \left(m+\frac{n}{2}\right)\partial_\tau + e^\tau\sum_{i,j=1}^n\sum_{\alpha=1}^m C_{ij\alpha}v_i\partial_{v_j}\partial_{z_\alpha}\\ &= t\sum_{i=1}^n \partial_{v_i}^2 + t\left(t+\frac{1}{4}\sum_{i=1}^n v_i^2\right)\sum_{\alpha=1}^m \partial_{z_\alpha}^2 + t^2\partial_t^2 - \left(m+\frac{n}{2}-1\right)t\partial_t + t\sum_{i,j=1}^n\sum_{\alpha=1}^m C_{ij\alpha}v_i\partial_{v_j}\partial_{z_\alpha},\end{aligned}$$

where the second line is obtained from the preceding line using $\partial_\tau = t\partial_t$ and $\partial_\tau^2 = t^2\partial_t^2 + t\partial_t$.

Evaluating ΔD_{x_0} at $x = (V,Z,t) \in \mathfrak{v}\oplus\mathfrak{z}\times\mathbb{R}_+$, where $V = \sum_{i=1}^n V^iE_i$, we get

$$\begin{aligned}\Delta D_{x_0}(x) = &\sum_{i=1}^n\left(t+t_0+\frac{\|V-V_0\|^2}{4}+\frac{1}{2}\langle E_i,V-V_0\rangle^2+\frac{1}{2}\|[E_i,V_0]\|^2\right)\\ &+\left(t+\frac{1}{4}\|V\|^2\right)\left(\sum_{\alpha=1}^m 2\right)\\ &+\frac{2}{t}\left(\left(t_0+\frac{\|V-V_0\|^2}{4}\right)^2+\left\|Z-Z_0+\frac{1}{2}[V,V_0]\right\|^2\right)\\ &-\left(m+\frac{n}{2}-1\right)\left(t-\frac{1}{t}\left(\left(t_0+\frac{\|V-V_0\|^2}{4}\right)^2+\left\|Z-Z_0+\frac{1}{2}[V,V_0]\right\|^2\right)\right)\\ &+\sum_{i,j=1}^n\sum_{\alpha=1}^m C_{ij\alpha}V^i\langle F_\alpha,[E_j,V_0]\rangle.\end{aligned} \tag{14}$$

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It is clear that

$$\sum_{i=1}^{n} \langle E_i, V - V_0\rangle^2 = \|V - V_0\|^2.$$

Furthermore, Lemma 3.6 yields

$$\sum_{i=1}^{n} \|[E_i, V_0]\|^2 = m\|V_0\|^2,$$

and from Lemma 3.2, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n}\sum_{\alpha=1}^{m} C_{ij\alpha}V^i\langle F_\alpha, [E_j, V_0]\rangle &= \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{m} C_{ij\alpha}V^i C_{jk\alpha}\langle V_0, E_k\rangle \\ &= \sum_{i,k=1}^{n}\sum_{\alpha=1}^{m} -\delta_{ik}V^i\langle V_0, E_k\rangle = -m\langle V, V_0\rangle.\end{aligned}$$

Plugging these equations into (14), a simple algebraic rearrangement gives

$$\Delta D_{x_0}(x) = \left(m + \frac{n}{2} + 1\right) D_{x_0}(x) - 2(m+1)t_0.$$

Consider now the squared norm of the gradient of D_{x_0}. Denote by $\mathbf{E}_1, \ldots, \mathbf{E}_n; \mathbf{F}_1, \ldots, \mathbf{F}_m; \mathbf{A}$ the left-invariant vector fields corresponding to the orthonormal basis $E_1, \ldots, E_n; F_1, \ldots, F_m; A \in \mathfrak{s}$. These vector fields are computed in [3, Sec. 4.1.5]. The derivative of D_{x_0} with respect to these vector fields are

$$\begin{aligned}\mathbf{E}_i D_{x_0}(x) &= \sqrt{t}\partial_{v_i} D_{x_0}(x) - \frac{1}{2}\sqrt{t}\sum_{\alpha=1}^{m}\sum_{j=1}^{n} C_{ij\alpha}v_j\partial_{z_\alpha}D_{x_0}(x) \\ &= \frac{1}{\sqrt{t}}\left(t + t_0 + \left\|\frac{V - V_0}{2}\right\|^2\right)\langle V - V_0, E_i\rangle + \frac{1}{\sqrt{t}}\left\langle Z - Z_0 + \frac{1}{2}[V, V_0], [E_i, V_0]\right\rangle \\ &\quad - \sum_{\alpha=1}^{m}\sum_{j=1}^{n} C_{ij\alpha}v_j\frac{1}{\sqrt{t}}\left\langle Z - Z_0 + \frac{1}{2}[V, V_0], F_\alpha\right\rangle \\ &= \frac{1}{\sqrt{t}}\left(t + t_0 + \left\|\frac{V - V_0}{2}\right\|^2\right)\langle V - V_0, E_i\rangle + \frac{1}{\sqrt{t}}\left\langle Z - Z_0 + \frac{1}{2}[V, V_0], [E_i, V_0]\right\rangle \\ &\quad - \frac{1}{\sqrt{t}}\left\langle Z - Z_0 + \frac{1}{2}[V, V_0], [E_i, V]\right\rangle \\ &= \frac{1}{\sqrt{t}}\left(t + t_0 + \left\|\frac{V - V_0}{2}\right\|^2\right)\langle V - V_0, E_i\rangle + \frac{1}{\sqrt{t}}\left\langle J_{Z - Z_0 + \frac{1}{2}[V, V_0]}(V - V_0), E_i\right\rangle, \\ \mathbf{F}_\alpha D_{x_0}(x) &= t\partial_{z_\alpha}D_{x_0}(x) = 2\left\langle Z - Z_0 + \frac{1}{2}[V, V_0], F_\alpha\right\rangle, \\ \mathbf{A}D_{x_0}(x) &= \partial_\tau D_{x_0}(x) = t\partial_t D_{x_0}(x) = -D_{x_0}(x) + 2\left(t + t_0 + \left\|\frac{V - V_0}{2}\right\|^2\right).\end{aligned}$$

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The squared norm of the gradient is

$$\begin{aligned}\left\|\nabla D_{x_0}(x)\right\|^2 &= \sum_{i=1}^{n}\left(\mathbf{E}_i D_{x_0}(x)\right)^2+\sum_{\alpha=1}^{m}\left(\mathbf{F}_\alpha D_{x_0}(x)\right)^2+\left(\mathbf{A}D_{x_0}(x)\right)^2\\ &= \frac{1}{t}\left(t+t_0+\left\|\frac{V-V_0}{2}\right\|^2\right)^2\|V-V_0\|^2+\frac{1}{t}\left\|Z-Z_0+\frac{1}{2}[V,V_0]\right\|^2\|V-V_0\|^2\\ &\quad +4\left\|Z-Z_0+\frac{1}{2}[V,V_0]\right\|^2\\ &\quad +D_{x_0}^2(x)-4D_{x_0}(x)\left(t+t_0+\left\|\frac{V-V_0}{2}\right\|^2\right)+4\left(t+t_0+\left\|\frac{V-V_0}{2}\right\|^2\right)^2\\ &= D_{x_0}^2(x)-4t_0D_{x_0}(x).\end{aligned}$$

□

7. The focal varieties of the functions D_{x_0}

Let $x_0=(V_0,Z_0,t_0)\in\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$ be an arbitrary point. Since $D_{x_0}((V,Z,t))\geqslant t+2t_0$, the function D_{x_0} has no maximal value. To describe the focal variety $\mathcal{F}_{x_0}$ of the function D_{x_0} corresponding to its minimal value (if it exists), we distinguish three cases depending on the sign of t_0.

When t_0 is positive, the minimal value of D_{x_0} is $4t_0$, and the minimum is attained at the point x_0, so $\mathcal{F}_{x_0}$ consists of a single point. This result is consistent with the fact that the regular level sets of D_{x_0} are the geodesic spheres centered at x_0.

If $t_0=0$, the infimum of the range of D_{x_0} is 0, but the 0 value is not attained, so D_{x_0} has no focal varieties. The level sets are parallel horospheres of the space.

The case $t_0<0$ is more interesting. Then the minimum of the function D_{x_0} is 0, and the focal variety $\mathcal{F}_{x_0}$ is defined by the system of equations

$$\|V-V_0\|^2=-4(t+t_0),\qquad Z=Z_0-\tfrac{1}{2}[V,V_0]. \tag{15}$$

The first equation defines a downward opening paraboloid of revolution in the space $\mathfrak{v}\oplus\mathbb{R}$. The focal surface can be obtained as the intersection of the upper half-space $\mathfrak{n}\times\mathbb{R}_+$ with the image of this paraboloid under the affine map $\mathfrak{v}\oplus\mathbb{R}\to\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$, $(V,t)\mapsto(V,Z_0-\frac{1}{2}[V,V_0],t)$. By Theorem 2.4, we conclude that $\mathcal{F}_{x_0}$ is an n-dimensional minimal submanifold of the Damek–Ricci space, which is diffeomorphic to $\mathbb{R}^n$, and the isoparametric hypersurfaces obtained as the regular level sets of the function D_{x_0} are the tubes about $\mathcal{F}_{x_0}$, in particular, the regular level sets are diffeomorphic to $\mathbb{R}^n\times S^m$, see Figure 1.

A straightforward computation using (7) shows the following lemma.

Proposition 7.1. *Let L_p be the affine transformation (7) extending the left translation by the element $p=(\bar V,\bar Z,\bar t)\in S$ to $\mathfrak{n}\oplus\mathbb{R}$. Then $D_{x_0}\circ L_p=\bar t D_{L_{p^{-1}}(x_0)}$ holds for any $x_0\in\mathfrak{n}\oplus\mathbb{R}$.*

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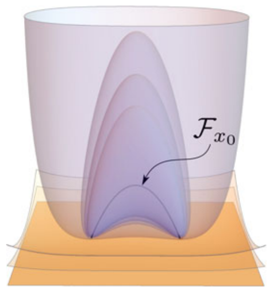

Figure 1. The level sets of the function D_{x_0} for $t_0 < 0$ are tubes about the focal variety $\mathcal{F}_{x_0}$. In fact, the figure depicts only a 3-dimensional slice of the level sets.

The lemma allows us to describe the left action of the group S on the focal varieties $\mathcal{F}_{x_0} = D_{x_0}^{-1}(0)$.

Corollary 7.2. *We have $L_p(\mathcal{F}_{x_0}) = \mathcal{F}_{L_p(x_0)}$, in particular, the family of the focal varieties $\mathcal{F}_{x_0}$ is invariant under left translations.*

8. Prolongation of the family of spheres tangent to one another at one point

Consider the unit speed geodesic curve $\hat{\eta}\colon t \mapsto \eta(\tanh(t/2))$ starting from the point $p = (\bar{V},\bar{Z},\bar{t}) = \eta(0)$, where $\eta = L_{(\bar{V},\bar{Z},\bar{t})} \circ \gamma$, and γ is the prolongation (8) of the unit speed geodesic $\hat{\gamma}$ starting with initial velocity $\hat{\gamma}'(0) = (v,z,s) \in T_eS$. For $r \in \mathbb{R}$, the geodesic sphere $\Sigma_r^{\hat{\eta}}$ of radius $|r|$ centered at $\eta(\theta)$, where $\theta = \tanh(r/2)$, is defined by the equation $D_{\eta(\theta)}(x) = D_{\eta(\theta)}(p)$. As r is running over $\mathbb{R}$, θ is varying in the interval $(-1,1)$, however, the equation $D_{\eta(\theta)}(x) = D_{\eta(\theta)}(p)$ makes sense and defines an isoparametric hypersurface $\tilde{\Sigma}_\theta^{\hat{\eta}}$ also in the case, when θ is an arbitrary element of $\mathbb{R}\mathbf{P}^1$ for which $\eta(\theta)$ is not a point at infinity. Thus, following the strategy described in the introduction, we may call the family $\tilde{\Sigma}_\theta^{\hat{\eta}}$ the natural analytic prolongation of the family of spheres passing through p and centered at a point of the geodesic $\hat{\eta}$, see Figure 2.

By Theorem 5.1, the map $\eta\colon \mathbb{R}\mathbf{P}^1 \to \mathbf{P}(\mathfrak{s} \oplus \mathbb{R})$ parameterises either an ellipse or the projective closure of a parabola or a straight line. In the case of an ellipse, the hypersurfaces $\tilde{\Sigma}_\theta^{\hat{\eta}}$ are defined for all $\theta \in \mathbb{R}\mathbf{P}^1$. However, in the other two cases, there is a value of θ, for which $\eta(\theta) = \circledast$, and for this value, we do not have a definition of $\tilde{\Sigma}_\theta^{\hat{\eta}}$ at the moment. The curve η or the curve γ goes through the point at infinity $\circledast$ if and only if the initial velocity (v,z,s) of $\hat{\gamma}$ has vanishing z component, and in that case, the point at infinity corresponds to the parameter $\theta = 1/s \in \mathbb{R}\mathbf{P}^1$. To eliminate the exceptional role of the parameter $\theta = 1/s$, we compute the limit of the hypersurfaces $\tilde{\Sigma}_\theta^{\hat{\eta}}$ as θ tends to $1/s$. The hypersurfaces $\tilde{\Sigma}_\theta^{\hat{\eta}}$ are algebraic hypersurfaces of degree 4. Non-trivial polynomial equations of degree at most 4 up to a non-zero constant multiplier form a projective space with a natural topology. We shall say that the hypersurfaces $\tilde{\Sigma}_\theta^{\hat{\eta}}$ tend to the hypersurface $\tilde{\Sigma}$ as θ tends to $1/s$ if the equations of them tend to the equation of $\tilde{\Sigma}$ in the projective space of at most quartic equations.

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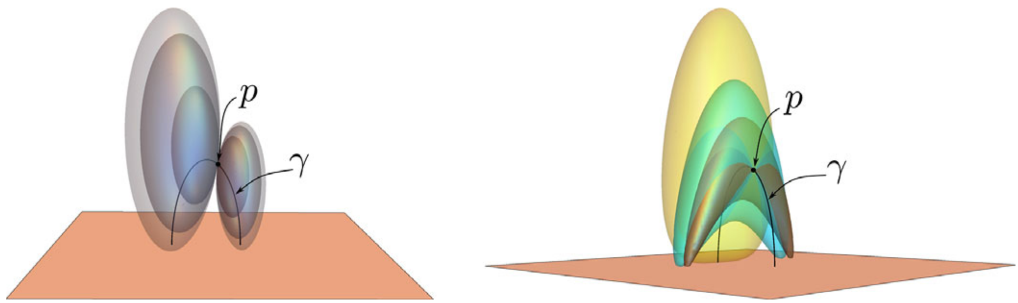

Figure 2. Left: Limit horospheres of inflating spheres intersecting a geodesic γ orthogonally at $p=\gamma(0)$. Right: A horosphere and two isoparametric hypersurfaces belonging to the analytic prolongation of the family of inflating spheres.

Proposition 8.1. *Using the above notations, if $z=0$, then the quartic hypersufaces $\tilde{\Sigma}^{\hat{\eta}}_{\theta}$ tend to the hypersurface defined by the quadratic equation*

$$\frac{1}{t}\left(\left(2s\sqrt{\bar{t}}-\langle V-\bar{V},v\rangle\right)^2+\left\|[V-\bar{V},v]\right\|^2\right)=4s^2 \tag{16}$$

as θ tends to $1/s$.

We shall denote the limit hypersurface by $\tilde{\Sigma}^{\hat{\eta}}_{1/s}$ and by

$$D^{\eta}_{\circledast}(V,Z,t)=\frac{1}{t}\left(\left(2s\sqrt{\bar{t}}-\langle V-\bar{V},v\rangle\right)^2+\left\|[V-\bar{V},v]\right\|^2\right)$$

the function on the left hand side of equation (16).

Proof. The coefficients of the polynomial function $tD_{\eta(\theta)}((V,Z,t))$ diverge as θ tends to $1/s$, (we set $1/s=\infty$ if $s=0$), but if we multiply $D_{\eta(\theta)}$ with the constant $(1/\theta-s)^2$ to slow down the increase of the coefficients, the normalised polynomials will converge to a non-zero polynomial. Using the relation $s^2+\|v\|^2=1$, one can bring $(1/\theta-s)^2tD_{\eta(\theta)}((V,Z,t))$ into the form

$$\left(\left(\frac{1}{\theta}-s\right)\left(t+\bar{t}+\left\|\frac{\bar{V}-V}{2}\right\|^2\right)+2s\bar{t}-\sqrt{\bar{t}}\langle V-\bar{V},v\rangle\right)^2+\left\|\left(\frac{1}{\theta}-s\right)\left(Z-\bar{Z}+\frac{1}{2}[V,\bar{V}]\right)+\sqrt{\bar{t}}[V-\bar{V},v]\right\|^2.$$

Thus, we have

$$\lim_{\theta\to 1/s}(1/\theta-s)^2D_{\eta(\theta)}((V,Z,t))=\frac{\bar{t}}{t}\left(\left(2s\sqrt{\bar{t}}-\langle V-\bar{V},v\rangle\right)^2+\left\|[V-\bar{V},v]\right\|^2\right),$$

which coincides with $D^{\eta}_{\circledast}(V,Z,t)$ up to the constant multiplier $\bar{t}$. Then (16) is the equation of the level set of $D^{\eta}_{\circledast}$ passing through the point p. □

The proof of the following statement is straightforward from the above formulas.

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Proposition 8.2. *The function $D^\eta_\circledast$ is isoparametric for any prolonged geodesic $\eta = L_{(\bar V,\bar Z,\bar t)}\circ\gamma$, where γ is the prolongation (8) of the unit speed geodesic $\hat\gamma$ starting with initial velocity $\hat\gamma'(0) = (v,z,s)\in T_eS$.*

(i) *The function $D^\eta_\circledast$ has no maximal value.*

(ii) *If $v\neq 0$, that is, the image of η is a parabola, then the minimal value of $D^\eta_\circledast$ is equal to 0. In particular, its focal variety is*

$$\mathcal{F}^\eta_\circledast = \{(V,Z,t)\in\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R} \mid [V-\bar V,v]=0 \text{ and } \langle V-\bar V,v\rangle = 2s\sqrt{\bar t}\}.$$

In the half-space model, the minimal submanifold $\mathcal{F}^\eta_\circledast$ is represented by the intersection of the half-space model and the translation of the linear subspace $(\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^\perp)\oplus\mathfrak{z}\oplus\mathbb{R} = (J_{\mathfrak{z}}v\oplus\mathbb{R}v)^\perp$ with the vector $\bar V + (2s\sqrt{\bar t}/\|v\|^2)v$, which is an affine subspace of dimension $n = \dim(\ker\operatorname{ad}v)-1$.

(iii) *If $v=0$ and $s=\pm1$, meaning that $\operatorname{im}\eta$ is a straight line perpendicular to the boundary $\mathfrak{n}$ of the model, then the function $D^\eta_\circledast(V,Z,t) = 4\bar t/t$ does not possess any minimal values. As a result, the function has an empty focal variety. The level sets of the function $D^\eta_\circledast$ are parallel horospheres, represented by parallel hyperplanes perpendicular to $\mathfrak{a}$ in the half-space model.*

The functions $D^\eta_\circledast$ and their focal varieties inherit the following invariance properties from the functions D_{x_0} (cf. Proposition 8.3 and Corollary 7.2).

Proposition 8.3. *Let $L_{\hat p}$ be the left translation by the element $\hat p = (\hat V,\hat Z,\hat t)\in S$, and η be an arbitrary parabola-shaped pregeodesic as in Proposition 8.1. Then $D^\eta_\circledast\circ L_{\hat p} = \hat t D^{L_{\hat p^{-1}}\circ\eta}_\circledast$ holds.*

In particular, we have $L_{\hat p}(\mathcal{F}^\eta_\circledast) = \mathcal{F}^{L_{\hat p}\circ\eta}_\circledast$, thus, the family of the focal varieties $\mathcal{F}^\eta_\circledast$ is invariant under left translations.

The following proposition expresses the function $D^\eta_\circledast$ in terms of the Euclidean distance function.

Proposition 8.4. *Let η be the parabola-shaped pregeodesic curve considered in Proposition 8.2 (ii). Then we have*

$$tD^\eta_\circledast(V,Z,t) = (2s\sqrt{\bar t}-\langle V-\bar V,v\rangle)^2 + \left\|[V-\bar V,v]\right\|^2 = \|v\|^2\delta\left((V,Z,t),\mathcal{F}^\eta_\circledast\right)^2,$$

where $\delta(p,\mathcal{F}^\eta_\circledast)$ denotes the Euclidean distance of a point p from the focal variety $F^\eta_\circledast$.

Proof. Choose an orthonormal basis $F_1,\dots,F_m$ of $\mathfrak{z}$ and set $J_\alpha = J_{F_\alpha}$. Then $\bar v, J_1\bar v,\dots,J_m\bar v$ is an orthonormal basis of $J_{\mathfrak{z}}v\oplus\mathbb{R}v$, where $\bar v = v/\|v\|$. Thus,

$$\begin{aligned}\|v\|^2\delta(V,\mathcal{F}^\eta_\circledast)^2 &= \left\langle V-\bar V-\frac{2s\sqrt{\bar t}}{\|v\|^2}v,v\right\rangle^2 + \sum_{\alpha=1}^m\left\langle V-\bar V-\frac{2s\sqrt{\bar t}}{\|v\|^2}v,J_\alpha v\right\rangle^2\\ &= \left(\langle V-\bar V,v\rangle-2s\sqrt{\bar t}\right)^2 + \sum_{\alpha=1}^m\left\langle V-\bar V,J_\alpha v\right\rangle^2.\end{aligned}$$

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The proof is completed by the identity

$$\sum_{\alpha=1}^{m}\langle V-\bar{V},J_\alpha v\rangle^2=\sum_{\alpha=1}^{m}\langle [V-\bar{V},v],F_\alpha\rangle^2=\|[V-\bar{V},v]\|^2. \qquad \square$$

Remark 8.5. Extending the orthonormal system $E_1=J_1\bar{v},\dots,E_m=J_m\bar{v},E_{m+1}=\bar{v}$ to an orthonormal basis $E_1,\dots,E_n$ of $\mathfrak{v}$, we obtain that $D^{\gamma}_{\circledast}(V,Z,t)=\frac{1}{t}\sum_{i=1}^{m+1}\langle V,E_i\rangle^2$ is an isoparametric function. More generally, if $E_1,\dots,E_n$ is an arbitrary orthonormal basis of $\mathfrak{v}$ and $I\subseteq\{1,\dots,n\}$ is an arbitrary subset, then the function $F(V,Z,t)=\frac{1}{t}\sum_{i\in I}\langle V,E_i\rangle^2$ is also isoparametric, since one can prove the identities

$$\Delta F=\left(m+\frac{n}{2}+1\right)F+2|I| \quad\text{and}\quad \|\nabla F\|^2=F^2+4F \tag{17}$$

by a computation analogous to the proof of Theorem 6.1. The focal variety of F has the form $\exp(\mathfrak{w}\oplus\mathfrak{z}\oplus\mathfrak{a})$, where $\mathfrak{w}$ is the linear subspace spanned by $\{E_i\mid i\notin I\}$. Isoparametric functions of this type and the corresponding isoparametric hypersurfaces were studied by J. C. Díaz-Ramos and M. Domínguez-Vázquez [17].

Remark 8.6. The anonymous reviewer called our attention to an interesting topological property of the foliation of the linear space $\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$ defined by the regular level sets of the function D_{x_0}, where $x_0=(v_0,z_0,t_0)$ with $t_0>0$. The range of this function on the open upper half-space is $[4t_0,+\infty)$, its range on the open lower half-space is $(-\infty,0]$ and $|D_{x_0}|$ converges to ∞ as t tends to 0. To eliminate the singularity of D_{x_0} along the hyperplane $t=0$, we may compose D_{x_0} with the fractional linear transformation $\varphi(x)=\frac{2t_0}{2t_0-x}$. The composition $\tilde{D}_{x_0}=\varphi\circ D_{x_0}$ is real analytic, has the same level sets as D_{x_0}, and its range is the closed interval $[-1,1]$.

The function $\tilde{D}_{x_0}$ has two critical values, ±1. The minimum -1 is attained at the isolated point x_0 and the Hessian of $\tilde{D}_{x_0}$ at x_0 is positive definite. The maximal value 1 is attained along the n-dimensional paraboloid parameterised by the elements v of $\mathfrak{v}$ by

$$\mathcal{P}_{x_0}=\left\{(v,z,t)\,\middle|\, v\in\mathfrak{v},\, z=z_0-\tfrac{1}{2}[v,v_0],\, t=-t_0-\left\|\frac{v-v_0}{2}\right\|^2\right\}.$$

The parallel affine subspaces $\mathfrak{F}_v=\{v\}\times\mathfrak{z}\times\mathbb{R}$, $(v\in\mathfrak{v})$ foliate the space and intersect $\mathcal{P}_{x_0}$ transversally. Each subspace $\mathfrak{F}_v$ intersects $\mathcal{P}_{x_0}$ at exactly one point. The special form of the restriction of the function D_{x_0} onto $\mathfrak{F}_v$ allows us to check easily that Hessian of $\tilde{D}_{x_0}$ is negative definite on the tangent space of $\mathfrak{F}_v$ at the intersection point $\mathfrak{F}_v\cap\mathcal{P}_{x_0}$ and that $\mathfrak{F}_v$ intersects the level surface $\tilde{D}^{-1}_{x_0}(c)$ in an m-dimensional Euclidean sphere for any $0<c<1$. This means that $\tilde{D}^{-1}_{x_0}(c)$ is diffeomorphic to $\mathbb{R}^n\times S^m$ for such values of c, while it is diffeomorphic to the sphere S^{m+n} for $-1<c<0$. Hence the topology of the regular level sets $\tilde{D}^{-1}_{x_0}(c)$ changes as c goes through 0. At the moment of the transition, $\tilde{D}^{-1}_{x_0}(0)$ is diffeomorphic to $\mathbb{R}^{m+n}$, see Figure 3.

By a fundamental construction of C. Qian and Z. Tang [37, Theorem 1.1], if M is a closed connected smooth manifold, and f is a Morse–Bott function on M with critical set $M_+\sqcup M_-$, where M_+ and M_- are both closed connected submanifolds of codimensions more than 1, then there exists a Riemannian metric g on M so that f is an isoparametric

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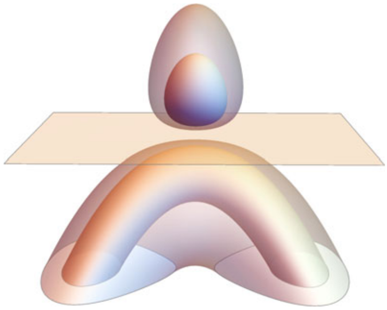

Figure 3. Topology of the level sets of the function $\tilde{D}_{x_0}$ on $\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$ for $t_0 > 0$.

function. In our example, the function $f = \tilde{D}_{x_0}$ satisfies all the conditions of this theorem except for the compactness of the space $M = \mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$ and the critical submanifold $M_+ = \mathcal{P}_{x_0}$. However, there is no Riemannian metric on $\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}$ which makes $\tilde{D}_{x_0}$ an isoparametric function, since the regular sets of $f = \tilde{D}_{x_0}$ are not diffeomorphic. This shows that compactness assumptions are crucial in the theorem of Qian and Tang. On the other hand, there is a flat Riemannian metric on the complement of the hyperplane $t = 0$ which makes the restriction of $\tilde{D}_{x_0}$ onto the complement isoparametric.

Similar bifurcation of the topology can be observed in the family Σ_θ^γ at the parameters $\theta = \pm 1$, corresponding to the horospheres of the family. However, members of this family can intersect one another not only at the point $\gamma(0)$ and it is not clear if they cover the whole Damek–Ricci space, so they do not give rise to a foliation of the space.

9. Geodesic curves orthogonal to a focal variety

Theorem 9.1. *Let $\mathcal{F}_{x_0}$ be the focal variety of the function D_{x_0} for a point $x_0 = (V_0, Z_0, t_0)$ with $t_0 < 0$. Then the prolongation of any geodesic curve that intersects $\mathcal{F}_{x_0}$ orthogonally goes through the point x_0. Similarly, the prolongation of any geodesic curve intersecting $\mathcal{F}_{\circledast}^{\eta}$ orthogonally goes through the point $\circledast$.*

Proof. Consider a focal variety $\mathcal{F}_{x_0}$ and a geodesic curve intersecting it orthogonally at the point p. Applying the left translation $L_{p^{-1}}$ to the configuration of the focal variety and the geodesic, we see by Corollary 7.2, that it suffices to prove the theorem for the case $p = e \in \mathcal{F}_{x_0}$. Condition $e \in \mathcal{F}_{x_0}$ is equivalent to the restrictions

$$t_0 = -\tfrac{1}{4}\|V_0\|^2 - 1, \qquad Z_0 = 0 \tag{18}$$

on the point $x_0 = (V_0, Z_0, t_0)$. The system of equations of such a focal submanifold $\mathcal{F}_{x_0}$ is

$$t = 1 - \tfrac{1}{4}\|V\|^2 + \tfrac{1}{2}\langle V, V_0\rangle, \qquad Z = -\tfrac{1}{2}[V, V_0], \tag{19}$$

from which its tangent space at the identity is

$$T_e\mathcal{F}_{x_0} = \left\{(v', z', s') \in \mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R} \mid s' = \tfrac{1}{2}\langle v', V_0\rangle,\, z' = -\tfrac{1}{2}[v', V_0]\right\}. \tag{20}$$

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The prolongation of a geodesic starting from e is parameterised by a map γ of the form (8). The initial velocity of γ is $\gamma'(0) = 2(v,z,s)$. The pregeodesic γ intersects $\mathcal{F}_{x_0}$ orthogonally if and only if

$$0 = \left\langle 2(v,z,s), \left(v', -\tfrac{1}{2}[v',V_0], \tfrac{1}{2}\langle v',V_0\rangle\right)\right\rangle = \langle v', 2v + sV_0\rangle - \langle z,[v',V_0]\rangle = \langle v', 2v + sV_0 + J_zV_0\rangle$$

for all $v' \in \mathfrak{v}$. This is equivalent to the equation

$$v = -\tfrac{1}{2}(sV_0 + J_zV_0), \tag{21}$$

consequently $s^2 + \|z\|^2 \neq 0$, otherwise we would have $(v,z,s) = 0$, contradicting $\|(v,z,s)\| = 1$. Using the assumption $\|v\|^2 + \|z\|^2 + s^2 = 1$, equation (21) gives

$$\frac{1}{s^2 + \|z\|^2} = 1 + \tfrac{1}{4}\|V_0\|^2 = -t_0. \tag{22}$$

Combination of equation (9) for the case $\|v\| \neq 1$, and equations (21), (22),(18) provides

$$\gamma(\infty) = \left(\frac{(s^2V_0 + sJ_zV_0) - (sJ_zV_0 - \|z\|^2V_0)}{s^2 + \|z\|^2}, 0, -\frac{1}{s^2 + \|z\|^2}\right) = (V_0, Z_0, t_0) = x_0.$$

Consider now a focal variety $\mathcal{F}^{\eta}_{\circledast}$ and a geodesic curve meeting it orthogonally. Referring to Proposition 8.3, we may assume that they meet at the identity element e. Let us write η as the left translate $L_{(\bar{V},\bar{Z},\bar{t})} \circ \gamma$ of a pregeodesic γ of the type (8) with initial velocity $\gamma'(0) = 2(v,0,s)$. Then $e \in \mathcal{F}^{\eta}_{\circledast}$ holds if and only if $[\bar{V}, v] = 0$ and $\langle \bar{V}, v\rangle = -2s\sqrt{\bar{t}}$. In this case, the tangent space $T_e\mathcal{F}^{\eta}_{\circledast}$ is the space $(J_{\mathfrak{z}}v \oplus \mathbb{R}v)^{\perp}$. Then any geodesic curve intersecting $\mathcal{F}^{\eta}_{\circledast}$ orthogonally at e starts with initial velocity in $J_{\mathfrak{z}}v \oplus \mathbb{R}v \subseteq \mathfrak{v}$. The prolongation of any such geodesic goes through $\circledast$ by Theorem 5.1. □

Theorem 9.2. *If the prolongation η of a unit speed geodesic curve $\hat{\eta}$ goes through the point x_0 with negative last coordinate, then the geodesic $\hat{\eta}$ intersects the focal variety $\mathcal{F}_{x_0}$ at a point p orthogonally. This point p is the unique intersection points of $\hat{\eta}$ with $\mathcal{F}_{x_0}$. Furthermore, the points x_0, p, and the two intersection points of* $\operatorname{im}\eta$ *with the boundary of the half-space model form a harmonic range on the quadric* $\operatorname{im}\eta$ *(which may degenerate to a straight line), i.e., the cross-ratio of these four points with respect to the quadric* $\operatorname{im}\eta$ *is* -1.

Proof. The points of $\operatorname{im}\eta$ in the boundary hyperplane of our model are $\eta(\pm 1)$. As η is a birational equivalence, η preserves cross-ratio. The four-tuple $\big(\theta, 1/\theta, 1, -1\big)$ is a harmonic range in $\mathbb{R}\mathbf{P}^1$, therefore, if we write x_0 in the form $\eta(\theta)$, then the unique point $p \in \operatorname{im}\eta$ for which the cross ratio of the points $\big(x_0, p, \eta(1), \eta(-1)\big)$ with respect to $\operatorname{im}\eta$ is equal to -1 is $p = \eta(1/\theta)$. Since $\big(x_0, p, \eta(1), \eta(-1)\big)$ is a harmonic range, the points $\eta(1)$ and $\eta(-1)$ separate the points x_0 and p in $\operatorname{im}\eta$, which implies that $p \in \mathfrak{n} \times \mathbb{R}_+$.

We show that the focal variety $\mathcal{F}_{x_0}$ passes through p, and that η is orthogonal to $\mathcal{F}_{x_0}$ at p. The statement does not depend on the choice of the parameterisation of the geodesic and its prolongation. Thus, we may assume without loss of generality, that $\eta(0) = p$ and $\eta(\infty) = x_0$. The statement is also invariant under left translations by Corollary 7.2;

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therefore we may also assume that $p = e$, and $\eta = \gamma$, where γ is the pregeodesic given by equation (8). Since $x_0 \neq \circledast$, equation (9) yields $\|v\| \neq 1$ and

$$x_0 = (V_0, Z_0, t_0) = \gamma(\infty) = \left(-\frac{2s}{s^2 + \|z\|^2} v + \frac{2}{s^2 + \|z\|^2} J_z v, 0, -\frac{1}{s^2 + \|z\|^2} \right).$$

A simple computation shows that the components of x_0 satisfy the equations in (18); therefore $\mathcal{F}_{x_0}$ is passing through e. The orthogonality condition (21) is also fulfilled, hence γ intersects $\mathcal{F}_{x_0}$ orthogonally.

If t_0 is chosen so that $\tanh(t_0/2) = 1/\theta$, then $\hat{\eta}(t_0) = p$ and the distance of $\hat{\eta}(t)$ from $\mathcal{F}_{x_0}$ is equal to $|t - t_0|$, which implies that p is the only intersection point of $\hat{\eta}$ and the focal variety $\mathcal{F}_{x_0}$. □

Corollary 9.3. *If $\hat{\eta}$ is a geodesic curve, then for each point p of $\hat{\eta}$, there is a unique focal variety of the form $\mathcal{F}_{x_0}$ or $\mathcal{F}^{\eta}_{\circledast}$ that meets $\hat{\eta}$ orthogonally at p, where η is the prolongation of $\hat{\eta}$.*

However, in the general case, there can be focal varieties of the form $F^{\zeta}_{\circledast} \neq F^{\eta}_{\circledast}$ corresponding to another geodesic curve $\hat{\zeta}$ such that $F^{\zeta}_{\circledast}$ and $F^{\eta}_{\circledast}$ meet $\hat{\eta}$ at the same point orthogonally. Namely, we will prove that this will happen if and only if the Damek–Ricci space is not symmetric.

Theorem 9.4. *A Damek–Ricci space is symmetric if and only if it has the property that whenever two focal varieties $F^{\eta_1}_{\circledast}$ and $F^{\eta_2}_{\circledast}$ intersect a geodesic orthogonally at the same point, they coincide.*

Proof. By Proposition 8.3, we may assume that the two focal varieties intersect the geodesic at e. Then Proposition 8.2 (ii) gives that $F^{\eta_1}_{\circledast} = F^{\eta_2}_{\circledast}$ if and only if their tangent spaces at e are equal, that is $(J_{\mathfrak{z}} v_1 \oplus \mathbb{R} v_1)^{\perp} = (J_{\mathfrak{z}} v_2 \oplus \mathbb{R} v_2)^{\perp}$, where v_1 and v_2 are non-zero elements of $\mathfrak{v}$, related to η_1 and η_2 as in Proposition 8.2 (ii). The existence of a geodesic meeting both $F^{\eta_1}_{\circledast}$ and $F^{\eta_2}_{\circledast}$ orthogonally at e is equivalent to the existence of a non-zero vector $w \in (J_{\mathfrak{z}} v_1 \oplus \mathbb{R} v_1) \cap (J_{\mathfrak{z}} v_2 \oplus \mathbb{R} v_2)$ serving for the initial velocity of such a geodesic curve. Thus, the condition that the focal varieties $F^{\eta_1}_{\circledast}$ and $F^{\eta_2}_{\circledast}$ coincide if there is a geodesic which meets both of them orthogonally at a common point is equivalent to condition (iii) of Proposition 4.8, therefore, it is equivalent to the space being symmetric. □

10. Totally geodesic focal varieties and the J^2-condition

Focal varieties of type $\mathcal{F}_{x_0}$ are not totally geodesic unless the Damek–Ricci space is symmetric, but they have at least one point $p \in \mathcal{F}_{x_0}$ such that $\mathcal{F}_{x_0}$ is the exponential image of a $T_p \mathcal{F}_{x_0}$. It turns out that the set of such points of $\mathcal{F}_{x_0}$ is homeomorphic to the set of vectors in $\mathfrak{v}$ satisfying the J^2-condition; therefore, the existence of more than one such point is equivalent to $\dim \mathfrak{z} \in \{0,1,3,7\}$. Moreover, existence of a totally geodesic focal variety $\mathcal{F}_{x_0}$ implies that the space is symmetric and that all the focal varieties are totally geodesic. The key to proving these statements is the following proposition.

Proposition 10.1. *Let $x_0 = (V_0, Z_0, t_0) = (V_0, 0, -1 - \frac{1}{4}\|V_0\|^2)$ be a point satisfying equations (18), guaranteeing that $\mathcal{F}_{x_0}$ goes through the identity e. Then $\mathcal{F}_{x_0}$ contains*

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all geodesic curves starting from e with initial velocity belonging to $T_e\mathcal{F}_{x_0}$ if and only if V_0 satisfies the J^2-condition.

Proof. For the special choice of x_0, the focal variety $\mathcal{F}_{x_0}$ and its tangent space at the identity are defined by equations (19) and (20), respectively. In particular, $(v,z,s)\in T_e\mathcal{F}_{x_0}$ is non-zero if and only if $v\neq 0$. Consider the reparameterisation $\gamma\colon(-1,1)\to S$ of the unit speed geodesic curve $\hat\gamma$ with initial velocity $\hat\gamma'(0)=(v,z,s)\in T_e\mathcal{F}(v_0)$ given by equation (8). The point $\gamma(\theta)$ belongs to $\mathcal{F}_{x_0}$ for a given $\theta\in(-1,1)$ if and only if

$$\frac{1-\theta^2}{\chi(\theta)}=1-\frac{1}{4}\left\|\frac{2\theta(1-s\theta)}{\chi(\theta)}v+\frac{2\theta^2}{\chi(\theta)}J_zv\right\|^2+\frac{1}{2}\left\langle\frac{2\theta(1-s\theta)}{\chi(\theta)}v+\frac{2\theta^2}{\chi(\theta)}J_zv,V_0\right\rangle \tag{23}$$

and

$$\frac{2\theta}{\chi(\theta)}z=-\frac{1}{2}\left[\frac{2\theta(1-s\theta)}{\chi(\theta)}v+\frac{2\theta^2}{\chi(\theta)}J_zv,V_0\right]. \tag{24}$$

Equation (23) is equivalent to the equation

$$1-\theta^2=\chi(\theta)-\frac{\theta^2(1-s\theta)^2}{\chi(\theta)}\|v\|^2-\frac{\theta^4}{\chi(\theta)}\|z\|^2\|v\|^2+\theta(1-s\theta)\langle v,V_0\rangle+\theta^2\langle[v,V_0],z\rangle.$$

As the right hand side of this equation can be simplified as

$$\begin{aligned}&\chi(\theta)-\theta^2\frac{(1-s\theta)^2+\|z\|^2\theta^2}{\chi(\theta)}\|v\|^2+2s\theta(1-s\theta)+\theta^2\langle-2z,z\rangle\\&\quad=(1-s\theta)^2+\|z\|^2\theta^2-\theta^2\|v\|^2+2s\theta(1-s\theta)-2\theta^2\|z\|^2=1-\theta^2(\|v\|^2+s^2+\|z\|^2)=1-\theta^2,\end{aligned}$$

equation (23) is always fulfilled. Equation (24) holds if $\theta=0$. If $\theta\neq0$, then substituting $s=\frac{1}{2}\langle v,V_0\rangle$ and $z=-\frac{1}{2}[v,V_0]$, it can be brought to the equivalent form

$$-\langle v,V_0\rangle[v,V_0]=[J_{[v,V_0]}v,V_0].$$

With the help of Lemma 3.3, this condition can be transformed into the equivalent condition

$$\langle v,V_0\rangle[v,V_0]=[v,J_{[v,V_0]}V_0],$$

which gives by Corollary 3.5 the condition

$$-\langle v,V_0\rangle[v,V_0]=\|V_0\|^2[v,P_{V_0}(v)], \tag{25}$$

where P_{V_0} is the orthogonal projection onto $J_{\mathfrak{z}}V_0$. This computation shows that the focal variety $\mathcal{F}_{x_0}$ contains all geodesic curves starting from e with initial velocity belonging to $T_e\mathcal{F}_{x_0}$ if and only if equation (25) holds for any $v\in\mathfrak{v}$. If $V_0=0$, then V_0 satisfies both equation (25) and the J^2-condition, thus, it is enough to consider the case $V_0\neq 0$.

Decompose v as $v=v_1+v_2+v_3$, where $v_1=P_{V_0}(v)=J_zV_0\in J_{\mathfrak{z}}V_0$, $v_2=\lambda V_0\in\mathbb{R}V_0$, and $v_3\in(J_{\mathfrak{z}}V_0\oplus\mathbb{R}V_0)^{\perp}\cap\mathfrak{v}=\ker_{\mathfrak{v}}(\operatorname{ad}V_0)\cap V_0^{\perp}$. Plugging this decomposition into equation (25), we obtain

$$-\lambda\|V_0\|^2[J_zV_0,V_0]=\|V_0\|^2[\lambda V_0+v_3,J_zV_0].$$

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Since $v_3 \perp V_0$, we have $[v_3, J_z V_0] = [J_z v_3, V_0]$ by Lemma 3.3, thus, the above equation reduces to

$$0 = \|V_0\|^2 [J_z v_3, V_0].$$

Observe that if v is running over $\mathfrak{v}$, then z can be an arbitrary element of $\mathfrak{z}$, and v_3 can be an arbitrary element of $\ker_{\mathfrak{v}}(\operatorname{ad} V_0) \cap V_0^{\perp}$ independently, so condition (25) holds for every v if and only if $J_{\mathfrak{z}}(\ker_{\mathfrak{v}}(\operatorname{ad} V_0) \cap V_0^{\perp}) \subseteq \ker_{\mathfrak{v}}(\operatorname{ad} V_0)$, however, by Corollary 4.4, this latter condition is fulfilled if and only if V_0 satisfies the J^2-condition. □

Consider now the general case.

Theorem 10.2. *Let $x_0 = (V_0, Z_0, t_0) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ be an arbitrary point with $t_0 < 0$. Denote by $B \subset \mathfrak{v}$ the open ball of radius $2\sqrt{-t_0}$ centered at V_0. Using the system of equations (15) defining $\mathcal{F}_{x_0}$, the focal variety $\mathcal{F}_{x_0}$ can be parameterised by the map $\Upsilon \colon B \to \mathcal{F}_{x_0}$,*

$$\Upsilon(\bar{V}) = \left(\bar{V}, Z_0 - \tfrac{1}{2}[\bar{V}, V_0], -t_0 - \tfrac{1}{4}\|V - V_0\|^2\right).$$

Then for $\bar{V} \in B$, the geodesic curves starting from $\Upsilon(\bar{V})$ in a direction tangent to $\mathcal{F}_{x_0}$ at $\Upsilon(\bar{V})$ stay on $\mathcal{F}_{x_0}$ if and only if $\bar{V} - V_0$ satisfies the J^2-condition.

Proof. Apply to $\mathcal{F}_{x_0}$ the left translation by $\Upsilon(\bar{V})^{-1}$. This moves the point $\Upsilon(\bar{V})$ to e and the focal variety $\mathcal{F}_{x_0}$ to $\mathcal{F}_{\Upsilon(\bar{V})^{-1}x_0}$ by Corollary 7.2. Setting $(\bar{V}, \bar{Z}, \bar{t}) = \Upsilon(\bar{V})$, we have

$$\Upsilon(\bar{V})^{-1} x_0 = \left(-\frac{\bar{V}}{\sqrt{\bar{t}}}, -\frac{\bar{Z}}{\bar{t}}, \frac{1}{\bar{t}}\right)(V_0, Z_0, t_0) = \left(\frac{V_0 - \bar{V}}{\sqrt{\bar{t}}}, \frac{Z_0 - \bar{Z} - \frac{1}{2}[\bar{V}, V_0]}{\bar{t}}, \frac{t_0}{\bar{t}}\right) = \left(\frac{V_0 - \bar{V}}{\sqrt{\bar{t}}}, 0, \frac{t_0}{\bar{t}}\right).$$

Since left translations are isometries of the Damek–Ricci space, $\mathcal{F}_{x_0}$ is the exponential image of $T_{\Upsilon(\bar{V})}\mathcal{F}_{x_0}$ if and only if $\mathcal{F}_{\Upsilon(\bar{V})^{-1}x_0}$ is the exponential image of $T_e \mathcal{F}_{\Upsilon(\bar{V})^{-1}x_0}$, and by Proposition 10.1, this is equivalent to the condition that the vector $\frac{V_0 - \bar{V}}{\sqrt{\bar{t}}}$ or simply the vector $\bar{V} - V_0$ satisfies the J^2-condition. □

In contrast to the focal varieties of the type $\mathcal{F}_{x_0}$, a focal variety of the form $\mathcal{F}^{\eta}_{\circledast}$ is either totally geodesic or it has no points $p \in \mathcal{F}^{\eta}_{\circledast}$ for which $\mathcal{F}^{\eta}_{\circledast}$ is the exponential image of $T_p \mathcal{F}^{\eta}_{\circledast}$. Existence of a totally geodesic focal variety of type $\mathcal{F}^{\eta}_{\circledast}$ is equivalent to $\dim \mathfrak{z} \in \{0, 1, 3, 7\}$.

Theorem 10.3. *The focal variety*

$$\mathcal{F}^{\eta}_{\circledast} = \{(V, Z, t) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R} \mid [V - \bar{V}, v] = 0 \text{ and } \langle V - \bar{V}, v \rangle = 2s\sqrt{\bar{t}}\} \tag{26}$$

obtained in Proposition 8.2 (ii) for a non-zero vector v and the further parameters $\bar{V} \in \mathfrak{v}$, $\bar{t} \in \mathbb{R}_+$, and $s = \pm\sqrt{1 - \|v\|^2}$ is totally geodesic if and only if v satisfies the J^2-condition.

Proof. Left translations of the Damek–Ricci space are given by affine transformations (7) in the half-space model, the linear part of which leaves invariant every linear subspace containing the subspace $\mathfrak{z} \oplus \mathfrak{a}$, in particular all subspaces of the form $(J_{\mathfrak{z}} v \oplus \mathbb{R}v)^{\perp}$. As the direction space of the affine half-space representing the focal variety $\mathcal{F}^{\eta}_{\circledast}$ in the half-space model is $(J_{\mathfrak{z}} v \oplus \mathbb{R}v)^{\perp}$, any left translation which maps an arbitrary point of $\mathcal{F}^{\eta}_{\circledast}$ to e maps the focal variety $\mathcal{F}^{\eta}_{\circledast}$ onto the focal variety

$$\mathcal{F}(v) = \{(V, Z, t) \mid [V, v] = 0 \text{ and } \langle V, v \rangle = 0\}. \tag{27}$$

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This implies that the focal variety $\mathcal{F}^{\eta}_{\circledast}$ is totally geodesic if and only if any unit speed geodesic curve $\hat{\gamma}\colon \mathbb{R}\to S$ starting from $\hat{\gamma}(0)=e$ with initial velocity $\hat{\gamma}'(0)\in T_e\mathcal{F}(v)$ stays in $\mathcal{F}(v)$. The tangent space of $\mathcal{F}(v)$ at e is

$$T_e\mathcal{F}(v)=\{(v',z',s')\in\mathfrak{v}\oplus\mathfrak{z}\oplus\mathbb{R}\mid [v',v]=0 \text{ and } \langle v',v\rangle=0\}=(J_{\mathfrak{z}}v\oplus\mathbb{R}v)^{\perp}.$$

The unit speed geodesic curve $\hat{\gamma}$ starting with initial velocity $\hat{\gamma}'(0)=(v',z',s')\in T_e\mathcal{F}(v)$ can be reparameterised by the pregeodesic $\gamma\colon(-1,1)\to S$ given by equation (8) substituting (v',z',s') for (v,z,s). The point $\gamma(\theta)$ belongs to $\mathcal{F}(v)$ if and only if

$$\left[\frac{2\theta(1-s'\theta)}{\chi(\theta)}v'+\frac{2\theta^2}{\chi(\theta)}J_{z'}v',v\right]=0 \quad\text{and}\quad \left\langle\frac{2\theta(1-s'\theta)}{\chi(\theta)}v'+\frac{2\theta^2}{\chi(\theta)}J_{z'}v',v\right\rangle=0,$$

thus, $\hat{\gamma}$ stays in $\mathcal{F}(v)$ if and only if

$$\left[J_{z'}v',v\right]=0 \quad\text{and}\quad \left\langle J_{z'}v',v\right\rangle=0,$$

which is equivalent to $J_{z'}v'\in\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$. We conclude that $\mathcal{F}^{\eta}_{\circledast}$ is totally geodesic if and only if $J_{z'}v'\in\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$ for all $z'\in\mathfrak{z}$ and for all $v'\in\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$, meaning that $\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$ is a $\mathrm{Cl}(\mathfrak{z},q)$-submodule of $\mathfrak{v}$. This completes the proof by Corollary 4.4. □

11. Homogeneity of the sphere-like isoparametric hypersurfaces

Theorem 11.1. *In a symmetric Damek–Ricci space, all the focal varieties $\mathcal{F}_{x_0}$ and all the tubes about them are homogeneous. If a Damek–Ricci space is not symmetric, then none of the focal varieties $\mathcal{F}_{x_0}(x_0\in\mathfrak{n}\times\mathbb{R}_-)$ is homogeneous; consequently, none of the tubes about them is.*

Proof. If the space is symmetric, then it is a hyperbolic space $\mathbb{K}\mathbf{H}^k$ over an algebra $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$, ($k=2$ if $\mathbb{K}=\mathbb{O}$), and its tangent spaces are $\mathbb{K}$-modules. In that case, each focal variety $\mathcal{F}_{x_0}$ is totally geodesic, and its tangent spaces are $\mathbb{K}$-submodules of corank 1. Thus, the focal varieties are congruent to a totally geodesic $\mathbb{K}\mathbf{H}^{k-1}$. Totally geodesic submanifolds of $\mathbb{K}\mathbf{H}^k$ that are the singular orbits of a cohomogeneity one isometric action were classified by J. Berndt and M. Brück [1], and the submanifolds $\mathbb{K}\mathbf{H}^{k-1}$ are contained in their list; therefore the tubes about these focal surfaces are homogeneous.

Theorem 10.2 shows that if the ambient space is not symmetric, then each focal variety $\mathcal{F}_{x_0}$ has at least one point p with the property that $\mathcal{F}_{x_0}$ is the exponential image of the tangent space $T_p\mathcal{F}_{x_0}$, and it also has points which do not have this property. This means that the focal variety cannot be homogeneous. □

Theorem 11.2. *For $v\in\mathfrak{v}\setminus\{0\}$, the tubes about the focal variety $\mathcal{F}^{\eta}_{\circledast}$ defined in Proposition 8.2 (ii) have constant principal curvatures if and only if v satisfies the J^2-condition.*

Proof. The focal varieties are special cases of the construction of [17] with the choice $\mathfrak{w}=\ker_{\mathfrak{v}}(\operatorname{ad}v)\cap v^{\perp}$ of the subspace $\mathfrak{w}\leqslant\mathfrak{v}$. As it is proved in [17], the tubes about $\mathcal{F}^{\eta}_{\circledast}$ have constant principal curvatures if and only if the Kähler angles of the non-zero vectors $u\in\mathfrak{w}^{\perp}=J_{\mathfrak{z}}v\oplus\mathbb{R}v$ are constant. Recall that the Kähler angles of $u\in\mathfrak{w}^{\perp}$ are defined to be the principal angles between the subspaces $J_{\mathfrak{z}}u$ and $\mathfrak{w}^{\perp}$. In our case, $v\in\mathfrak{w}^{\perp}=J_{\mathfrak{z}}v\oplus\mathbb{R}v$

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and $J_{\mathfrak{z}}v \subseteq \mathfrak{w}^{\perp}$, thus, all the Kähler angles of the vector v are equal to 0. For this reason, the Kähler angles of the non-zero vectors of $\mathfrak{w}^{\perp}$ are constant if and only if they are all equal to 0 for any $u \in \mathfrak{w}^{\perp}$. But this happens if and only if $J_{\mathfrak{z}}u \subseteq J_{\mathfrak{z}}v \oplus \mathbb{R}v$ for any $u \in J_{\mathfrak{z}}v \oplus \mathbb{R}v$ and the latter condition is equivalent to the J^2-condition for v. □

Theorem 11.3. *The tubes about the focal variety $\mathcal{F}^{\eta}_{\circledast}$ defined in Proposition 8.2 (ii) are homogeneous if and only if $v \in \mathfrak{v}\backslash\{0\}$ satisfies the J^2-condition.*

Proof. If the tubes about $\mathcal{F}^{\eta}_{\circledast}$ are homogeneous, then they have constant principal curvatures and v satisfies the J^2-condition by Theorem 11.2.

Conversely, assume that $v \neq 0$ satisfies the J^2-condition. The existence of such a vector implies $m = \dim\mathfrak{z} \in \{0,1,3,7\}$ by Proposition 4.5.

If $m \in \{0,1\}$, then the ambient space is isometric with $\mathbb{K}\mathbf{H}^k$ for some k and $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, hence it is symmetric. By Theorem 10.3, the focal varieties $\mathcal{F}^{\eta}_{\circledast}$ are totally geodesic in a symmetric space. The tangent spaces of the ambient space are linear spaces over $\mathbb{K}$, and the tangent space of $\mathcal{F}^{\eta}_{\circledast}$ at any point is a $\mathbb{K}$-linear subspace of codimension 1. Thus, the tubes about $\mathcal{F}^{\eta}_{\circledast}$ are homogeneous by the same reason as in the proof of the first part of Theorem 11.1.

Assume that $m \in \{3,7\}$. Consider two arbitrary points p_1, p_2 on the tube $\mathcal{T}$ of radius r about $\mathcal{F}^{\eta}_{\circledast}$, and denote by $\tilde{p}_1$ and $\tilde{p}_2$ the points of the focal variety $\mathcal{F}^{\eta}_{\circledast}$ lying nearest to them. The left translations by $\tilde{p}_1^{-1}$ and $\tilde{p}_2^{-1}$ map the tube $\mathcal{T}$ onto the tube of radius r about the focal variety $\mathcal{F}(v)$ defined by equation (27) and map the points $\tilde{p}_1$ and $\tilde{p}_2$ to e. We can write the points $\tilde{p}_1^{-1}p_1$ and $\tilde{p}_2^{-1}p_2$ as $\exp_e(\xi_1)$ and $\exp_e(\xi_2)$ respectively with some uniquely defined vectors $\xi_1,\xi_2 \in (T_e\mathcal{F}(v))^{\perp}$ of length r.

Thus, to prove that $\mathcal{T}$ is homogeneous, it is enough to show that there is an isometry I of the space such that $I(\mathcal{F}(v)) = \mathcal{F}(v)$, $I(e) = e$, and $T_eI(\xi_1) = \xi_2$. In the case $\|\xi_1\| = \|\xi_2\| = 0$, we can choose the identity map for I, so assume $\|\xi_1\| = \|\xi_2\| \neq 0$.

Let $\mathrm{Spin}(m) \subset \mathrm{Cl}(\mathfrak{z},q)$ be the spin group and $\rho\colon \mathrm{Spin}(m) \to \mathrm{SO}(\mathfrak{z},q)$ its canonical representation. Recall that $\mathfrak{z}$ is embedded into $\mathrm{Cl}(\mathfrak{z},q)$ as a subspace, and for $\sigma \in \mathrm{Spin}(m)$, we have $(\rho(\sigma))(z) = \sigma z \sigma^{-1}$. As the vector v satisfies the J^2-condition, it generates an irreducible Clifford module $J_{\mathfrak{z}}v \oplus \mathbb{R}v = (T_e\mathcal{F}(v))^{\perp}$, which contains both ξ_1 and ξ_2. It is known that for $m \in \{3,7\}$, the group $\mathrm{Spin}(m)$ acts transitively on the unit sphere of any irreducible $\mathrm{Cl}(\mathfrak{z},q)$-module. For $m = 3$, this follows from the fact that the group $\mathrm{Spin}(3)$ is isomorphic to the group of unit quaternions, see [29, Ch. I., Thm. 8.1], and this group acts on itself transitively both by left and right translations. The case $m = 7$ is a theorem of A. Borel, see [29, Ch. I., Thm. 8.2]. As a consequence, there is an element $\sigma \in \mathrm{Spin}(m)$, such that $(J(\sigma))(\xi_1) = \xi_2$. The orthogonal transformation $\iota\colon \mathfrak{v}\oplus\mathfrak{z}\oplus\mathfrak{a} \to \mathfrak{v}\oplus\mathfrak{z}\oplus\mathfrak{a}$,

$$\iota((v,z,a)) = ((J(\sigma))(v),(\rho(\sigma))(z),a)$$

is an automorphism of the Lie algebra $\mathfrak{s} = \mathfrak{v}\oplus\mathfrak{z}\oplus\mathfrak{a}$. This is an easy consequence of the identity

$$J_{\iota(z)}\iota(v) = \big(J(\sigma)\circ J(z)\circ J(\sigma)^{-1}\circ J(\sigma)\big)(v) = \big(J(\sigma)\circ J(z)\big)(v) = \iota(J_z v) \quad \forall v \in \mathfrak{v}, z \in \mathfrak{z}.$$

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Thus, ι is the derivative of an isometric automorphism I of the group S at e. It is clear from the construction of I that $I(e)=e$ and $T_eI(\xi_1)=\iota(\xi_1)=\xi_2$. This implies that

$$\iota(J_{\mathfrak{z}}\xi_1\oplus\mathbb{R}\xi_1)=J_{\iota(\mathfrak{z})}\iota(\xi_1)\oplus\mathbb{R}\iota(\xi_1)=J_{\mathfrak{z}}\xi_2\oplus\mathbb{R}\xi_2. \tag{28}$$

Since ξ_1 and ξ_2 are in the $(m+1)$-dimensional Clifford module $J_{\mathfrak{z}}v\oplus\mathbb{R}v$, the $(m+1)$-dimensional linear spaces $J_{\mathfrak{z}}\xi_1\oplus\mathbb{R}\xi_1$ and $J_{\mathfrak{z}}\xi_2\oplus\mathbb{R}\xi_2$ are also contained in $J_{\mathfrak{z}}v\oplus\mathbb{R}v$, which implies

$$J_{\mathfrak{z}}\xi_1\oplus\mathbb{R}\xi_1=J_{\mathfrak{z}}v\oplus\mathbb{R}v=J_{\mathfrak{z}}\xi_2\oplus\mathbb{R}\xi_2. \tag{29}$$

Equations (28) and (29) give $\iota(J_{\mathfrak{z}}v\oplus\mathbb{R}v)=J_{\mathfrak{z}}v\oplus\mathbb{R}v$, and since ι is orthogonal,

$$\iota(T_e\mathcal{F}(v))=\iota\left((J_{\mathfrak{z}}v\oplus\mathbb{R}v)^{\perp}\right)=(J_{\mathfrak{z}}v\oplus\mathbb{R}v)^{\perp}=T_e\mathcal{F}(v).$$

By Theorem 10.3, the focal variety $\mathcal{F}(v)$ is a totally geodesic submanifold, hence it is the Riemannian exponential image of $T_e\mathcal{F}(v)$. Since the derivative ι of the isometry I at e maps $T_e\mathcal{F}(v)$ into itself, $I(\mathcal{F}(v))=\mathcal{F}(v)$. □

12. Mean curvature of the tubes about the focal varieties

Theorem 12.1. *Let h be the trace of the shape operator of a regular hypersurface Σ of one of the isoparametric functions D_{x_0} or $D^{\eta}_{\circledast}$ on the Damek–Ricci space. Then*

$$h=\begin{cases}-\frac{m+n}{2}\coth(r/2)-\frac{m}{2}\tanh(r/2) & \text{if } \Sigma \text{ is a sphere of radius } r,\\ -(m+\frac{n}{2}) & \text{if } \Sigma \text{ is a horosphere,}\\ -\frac{m+n}{2}\tanh(r/2)-\frac{m}{2}\coth(r/2) & \text{if } \Sigma \text{ is a tube of radius } r \text{ about } \mathcal{F}_{x_0}\\ & \text{with } x_0\in\mathfrak{n}\times\mathbb{R}_- \text{ or } \mathcal{F}^{\eta}_{\circledast}.\end{cases}$$

Proof. The case of spheres and horospheres is well known. For a complete harmonic manifold (M,g), there is a smooth function $\omega\colon\mathbb{R}\to\mathbb{R}$, called the volume density function, defined by the identity $\omega(\|\xi\|)=\sqrt{\det(G(\xi))}$, where $\xi\in T_pM$ is an arbitrary tangent vector of M, and $G(\xi)$ is the matrix of the pull-back form $(T_\xi\exp_p)^*(g_{\exp_p(\xi)})$ with respect to a g_p-orthonormal basis of $T_\xi(T_pM)\cong T_pM$. It is known [15] that the volume density function ω of the Damek–Ricci space is

$$\omega(r)=\cosh^m(r/2)\left(\frac{\sinh(r/2)}{r/2}\right)^{m+n}.$$

The function h for a sphere of radius r can be expressed with the help of the volume density function

$$h=-\partial_r(\ln r^{m+n}\omega)$$

(see [41]), from which we obtain the formula for the mean curvature of spheres, and taking the limit $r\to\infty$ gives the formula for the horospheres.

Consider now a regular level set $\Sigma=D^{-1}_{x_0}(c)$ of a function D_{x_0} with $x_0=(V_0,Z_0,t_0)$, $t_0<0$. Then, according to the proof of Theorem 6.1, the functions a and b certifying that D_{x_0} is isoparametric are $a(x)=(m+\frac{n}{2}+1)x-2(m+1)t_0$ and $b(x)=x^2-4t_0x$.

The minimal value of D_{x_0} is 0 and by Proposition 2.5, the hypersurface Σ is a tube of radius

$$r = \int_0^c \frac{\mathrm{d}x}{\sqrt{x^2 - 4t_0 x}} = 2\ln(\sqrt{c-4t_0} + \sqrt{c}) - 2\ln(\sqrt{-4t_0})$$

about $\mathcal{F}_{x_0}$. Expressing c as a function of r from this equation, we get

$$c = -4t_0 \sinh^2(r/2).$$

By Proposition 2.2, the trace of the shape operator of Σ is

$$\begin{aligned} h &= \frac{-2a(c) + b'(c)}{2\sqrt{b(c)}} = -\frac{(m+n/2)c - 2mt_0}{\sqrt{c^2 - 4t_0 c}} = -\frac{m+n}{2}\sqrt{\frac{c}{c-4t_0}} - \frac{m}{2}\sqrt{\frac{c-4t_0}{c}} \\ &= -\frac{m+n}{2}\tanh(r/2) - \frac{m}{2}\coth(r/2), \end{aligned}$$

as claimed.

By equation (17), the functions a and b corresponding to the isoparametric functions $D^{\eta}_{\circledast}$ are $a(x) = (m + \frac{n}{2} + 1)x + 2(m+1)$ and $b(x) = x^2 + 4x$. Hence, substituting $t_0 = -1$ into the above formulae obtained for the level sets of D_{x_0}, we get the corresponding formulae for $D^{\eta}_{\circledast}$, which completes the proof. □

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References

[1] Berndt J and Brück M (2001) Cohomogeneity one actions on hyperbolic spaces. *J. Reine Angew. Math.* **541**, 209–235.

[2] Berndt J and Tamaru H (2007) Cohomogeneity one actions on noncompact symmetric spaces of rank one. *Trans. Amer. Math. Soc.* **359**(7), 3425–3438.

[3] Berndt J, Tricerri F and Vanhecke L (1995) *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, vol. 1598 of Lecture Notes in Mathematics. Berlin: Springer-Verlag.

[4] Cartan É (1938) Familles de surfaces isoparamétriques dans les espaces à courbure constante. *Ann. Mat. Pura Appl.* **17**(1), 177–191.

[5] Cecil TE, Chi Q-S and Jensen GR (2007) Isoparametric hypersurfaces with four principal curvatures. *Ann. Math. (2)* **166**(1), 1–76.

[6] Chi Q-S (2009) Isoparametric hypersurfaces with four principal curvatures revisited. *Nagoya Math. J.* **193**, 129–154.

[7] Chi Q-S (2011) Isoparametric hypersurfaces with four principal curvatures, II. *Nagoya Math. J.* **204**, 1–18.

[8] Chi Q-S (2013) Isoparametric hypersurfaces with four principal curvatures, III. *J. Differential Geom.* **94**(3), 469–504.
[9] Chi Q-S (2020) Isoparametric hypersurfaces with four principal curvatures, IV. *J. Differential Geom.* **115**(2), 225–301.
[10] Chi Q-S (2020) The isoparametric story, a heritage of Élie Cartan. In *Proceedings of the International Consortium of Chinese Mathematicians 2018*. Boston, MA: International Press, 197–260.
[11] Copson ET and Ruse HS (1940) Harmonic Riemannian spaces. *Proc. Roy. Soc. Edinburgh* **60**, 117–133.
[12] Cowling M, Dooley AH, Korányi A and Ricci F (1991) H-type groups and Iwasawa decompositions. *Adv. Math.* **87**(1), 1–41.
[13] Csikós B and Horváth M (2011) On the volume of the intersection of two geodesic balls. *Differ. Geom. Appl.* **29**(4), 567–576.
[14] Csikós B and Horváth M (2012) A characterization of harmonic spaces. *J. Differential Geom.* **90**(3), 383–389.
[15] Damek E and Ricci F (1992) A class of nonsymmetric harmonic Riemannian spaces. *Bull. Amer. Math. Soc. (N.S.)* **27**(1), 139–142.
[16] Díaz-Ramos JC and Domínguez-Vázquez M (2012) Inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces. *Math. Z.* **271**(3–4), 1037–1042.
[17] Díaz-Ramos JC and Domínguez-Vázquez M (2013) Isoparametric hypersurfaces in Damek-Ricci spaces. *Adv. Math.* **239**, 1–17.
[18] Díaz-Ramos JC, Domínguez-Vázquez M and Rodríguez-Vázquez A (2021) Homogeneous and inhomogeneous isoparametric hypersurfaces in rank one symmetric spaces. *J. Reine Angew. Math.* **779**, 189–222.
[19] Díaz-Ramos JC, Domínguez-Vázquez M and Sanmartín-López V (2017) Isoparametric hypersurfaces in complex hyperbolic spaces. *Adv. Math.* **314**, 756–805.
[20] Dorfmeister J and Neher E (1985) Isoparametric hypersurfaces, case $g = 6$, $m = 1$. *Comm. Algebra* **13**(11), 2299–2368.
[21] Eschenburg J-H (1977) Horospheres and the stable part of the geodesic flow. *Math. Z.* **153**(3), 237–251.
[22] Eschenburg J-H (1989) Maximum principle for hypersurfaces. *Manuscripta Math.* **64**(1), 55–75.
[23] Ferus D, Karcher H and Münzner H-F (1981) Cliffordalgebren und neue isoparametrische Hyperflächen. *Math. Z.* **177**, 479–502.
[24] Ge J and Tang Z (2014) Geometry of isoparametric hypersurfaces in Riemannian manifolds. *Asian J. Math.* **18**(1), 117–125.
[25] Heber J (2006) On harmonic and asymptotically harmonic homogeneous spaces. *Geom. Funct. Anal.* **16**(4), 869–890.
[26] Kim S, Nikolayevsky Y and Park JH (2021) Totally geodesic submanifolds of Damek-Ricci spaces. *Rev. Mat. Iberoam.* **37**(4), 1321–1332.
[27] Kim S and Park JH (2023) Two theorems on the intersections of horospheres in asymptotically harmonic spaces. *Rev. Mat. Complut.*
[28] Knieper G (2012) New results on noncompact harmonic manifolds. *Comment. Math. Helv.* **87**(3), 669–703.
[29] Lawson HB, Jr and Michelsohn M-L (1989) *Spin Geometry*, vol. 38 of Princeton Mathematical Series. Princeton, NJ: Princeton University Press.
[30] Ledger AJ (1957) Symmetric harmonic spaces. *J. London Math. Soc.* **32**, 53–56.
[31] Lichnerowicz A (1944) Sur les espaces riemanniens complètement harmoniques. *Bull. Soc. Math. France* **72**, 146–168.

[32] Miyaoka R (2013) Isoparametric hypersurfaces with $(g,m) = (6,2)$. *Ann. Math. (2)* **177**(1), 53–110.

[33] Miyaoka R (2016) Errata to: "Isoparametric hypersurfaces with $(g,m) = (6,2)$". *Ann. Math. (2)* **183**(3), 1057–1071.

[34] Nikolayevsky Y (2005) Two theorems on harmonic manifolds. *Comment. Math. Helv.* **80**(1), 29–50.

[35] Ozeki H and Takeuchi M (1975) On some types of isoparametric hypersurfaces in spheres. I. *Tôhoku Math. J. (2)* **27**, 515–559.

[36] Ozeki H and Takeuchi M (1976) On some types of isoparametric hypersurfaces in spheres. II. *Tôhoku Math. J. (2)* **28**, 7–55.

[37] Qian C and Tang Z (2015) Isoparametric functions on exotic spheres. *Adv. Math.* **272**, 611–629.

[38] Ranjan A and Shah H (2003) Busemann functions in a harmonic manifold. *Geom. Dedicata* **101**, 167–183.

[39] Rouvière F and de Damek-Ricci Espaces (2003) géométrie et analyse. In *Analyse sur les groupes de Lie et théorie des représentations (Kénitra, 1999), vol. 7 of Sémin. Congr. Soc. Math. France, Paris*, 45–100.

[40] Segre B (1938) Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni. *Atti Accad. Naz. Lincei, Rend., VI. Ser* **27**, 203–207.

[41] Szabó ZI (1990) The Lichnerowicz conjecture on harmonic manifolds. *J. Differential Geom.* **31**(1), 1–28.

[42] Walker AG (1949) On Lichnerowicz's conjecture for harmonic 4-spaces. *J. London Math. Soc.* **24**, 21–28.

[43] Wang QM (1987) Isoparametric functions on Riemannian manifolds. I. *Math. Ann.* **277**(4), 639–646.