A CAUCHY CRITERION AND A CONVERGENCE THEOREM FOR RIEMANN-COMPLETE INTEGRAL

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In 1957 Kurzweil [1] proved some theorems concerning a generalized type of differential equations by defining a Riemann-type integral, but he did not study its properties beyond the needs of that research. This was done by R. Henstock [2, 3], who named it a Riemann-complete integral. He showed that the Riemann-complete integral includes the Lebesgue integral and that the Levi monotone convergence theorem holds. The purpose of the present paper is to give a necessary and sufficient condition for a function to be Riemann-complete integrable and to establish a termwise integration theorem for a uniformly convergent sequence of Riemann-complete integrable functions.

Throughout this paper, all functions considered are real-valued and defined in a closed interval $[a, b]$.

DEFINITION 1. A *division* $\mathfrak{D}$ of $[a, b]$ consists of two finite sequences $\{x_j\}_{j=0}^n$ and $\{z_j\}_{j=1}^n$ with conditions:

$$a = x_0 < x_1 < \cdots < x_n = b$$

and

$$x_{j-1} \leqq z_j \leqq x_j \quad (j = 1, \cdots, n).$$

DEFINITION 2. A division $\mathfrak{D}$ of $[a, b]$ is said to be *compatible with* $\delta(z) > 0$ defined in $[a, b]$ if, for each $j = 1, \cdots, n$, $|x_j - z_j| < \delta(z_j)$ and $|z_j - x_{j-1}| < \delta(z_j)$.

It should be noted that there is at least a division $\mathfrak{D}$ of $[a, b]$ which is compatible with a given function $\delta(z) > 0$ defined in $[a, b)]$ [3, p. 83].

DEFINITION 3. A function f is *Riemann-complete integrable* in $[a, b]$ with integral $I(f)$ if there is a real number $I(f)$ such that to each $\varepsilon > 0$ there corresponds a function $\delta(z) > 0$ defined in $[a, b]$ with

$$\left|\sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - I(f)\right| < \varepsilon$$

for all sums over divisions $\mathfrak{D}$ of $[a, b]$ compatible with $\delta(z)$.

In the sequel, we shall simply use the terms 'integrable' and 'integral' for

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'Riemann-complete integrable in $[a, b]$' and 'Riemann-complete integral in $[a, b]$' respectively. Also, for simplicity, we shall replace the words 'sum over a division of $[a, b]$ compatible with $\delta(z)$' by 'sum over $(\mathfrak{D}, \delta)$'.

LEMMA 4. *If f and g are integrable and α, β are real numbers, then $\alpha f+\beta g$ is also integrable and $I(\alpha f+\beta g) = \alpha I(f)+\beta I(g)$.*

PROOF. Let $\varepsilon > 0$ be given. If $\alpha \neq 0$, since f is integrable, there corresponds a function $\delta_f(z) > 0$ defined in $[a, b]$ such that

$$|S_f - I(f)| < \varepsilon/(2|\alpha|)$$

for all sums for f over $(\mathfrak{D}, \delta_f)$. Clearly we have, for any α,

$$|S_{\alpha f} - \alpha I(f)| = |\alpha|\,|S_f - I(f)| < \varepsilon/2$$

for all sums for αf over $(\mathfrak{D}, \delta_f)$. Similarly, there also corresponds a function $\delta_g(z) > 0$ defined in $[a, b]$ such that

$$|s_{\beta g} - \beta I(g)| < \varepsilon/2$$

for all sums for βg over $(\mathfrak{D}, \delta_g)$.

Let $\delta(z) = \min\{\delta_f(z), \delta_g(z)\}$ for all z in $[a, b]$. Thus

$$|S_{\alpha f+\beta g} - (\alpha I(f)+\beta I(g))| \leqq |S_{\alpha f} - \alpha I(f)| + |S_{\beta g} - \beta I(g)| < \varepsilon$$

for all sums for $\alpha f+\beta g$ over $(\mathfrak{D}, \delta)$. The proof is completed.

DEFINITION 5. Let $\mathscr{D}$ be the set of all pairs $(\mathfrak{D}, \delta)$, where δ is a positive function defined in $[a, b]$, and $\mathfrak{D}$ is a division of $[a, b]$ compatible with δ. For each function f, define $S_f : \mathscr{D} \to R$ by setting $S_f(\mathfrak{D}, \delta)$ to be the sum for f over $(\mathfrak{D}, \delta)$.

DEFINITION 6. Let $(\mathfrak{D}_i, \delta_i) \in \mathscr{D}$, $i = 1, 2$. We shall say $(\mathfrak{D}_1, \delta_1) < (\mathfrak{D}_2, \delta_2)$ if $\delta_2 \leqq \delta_1$. Clearly, this is a partial ordering of $\mathscr{D}$.

LEMMA 7. *For any f, $S_f : \mathscr{D} \to R$ is a net.*

PROOF. We need only show that $\{\mathscr{D}, <\}$ is a directed set. Let $(\mathfrak{D}_i, \delta_i) \in \mathscr{D}$, $i = 1, 2$, be given. Define δ_0 by $\delta_0(z) = \min\{\delta_1(z), \delta_2(z)\}$ for all $z \in [a, b]$. Then for any division $\mathfrak{D}_0$ compatible with δ_0, $(\mathfrak{D}_0, \delta_0) \in \mathscr{D}$ and $(\mathfrak{D}_i, \delta_i) < (\mathfrak{D}_0, \delta_0)$, $i = 1, 2$.

For a function f, we consider the following

CONDITION 8. To each $\varepsilon > 0$ there corresponds a function $\delta(z) > 0$ defined in $[a, b]$ with $|S' - S''| < \varepsilon$ whenever S' and S'' are sums for f over divisions compatible with $\delta(z)$.

LEMMA 9. *Condition 8 is necessary and sufficient for the net $S_f : \mathscr{D} \to R$ to be Cauchy.*

PROOF. Sufficiency: Let $\varepsilon > 0$ be given. Consider the function $\delta(z) > 0$

stated in condition 8, it is immediate that $|S_f(\mathfrak{D}_1, \delta_1) - S_f(\mathfrak{D}_2, \delta_2)| < \varepsilon$ for any $(\mathfrak{D}_i, \delta_i) \in \mathscr{D}$ such that $(\mathfrak{D}, \delta) < (\mathfrak{D}_i, \delta_i)$, $i = 1, 2$, where $\mathfrak{D}$ is an arbitrary division compatible with $\delta(z) > 0$. The proof for necessity is similar.

Since R is a complete uniform space, a Cauchy net $S_f : \mathscr{D} \to R$ has a limit in R [4, pp. 193–194]. Thus we obtain the following Cauchy criterion:

THEOREM 10. *A function f is integrable if and only if it satisfies condition* 8, *or equivalently, if and only if the net $S_f : \mathscr{D} \to R$ is Cauchy.*

THEOREM 11. *If $\{f_n\}$ is a sequence of integrable functions and converges uniformly to f in $[a, b]$, then f is integrable with integral $I(f) = \lim_{n\to\infty} I(f_n)$.*

PROOF. Let $\varepsilon > 0$ be given, choose a positive number $\eta < \varepsilon/(4(b-a))$. By hypothesis, there is an n_0 such that

$$|f_n(z) - f(z)| < \eta \text{ for every } z \in [a, b] \text{ and every } n \geqq n_0.$$

Since f_{n_0} is integrable, there exists $(\mathfrak{D}^*, \delta^*) \in \mathscr{D}$ such that

$$|Sf_{n_0}(\mathfrak{D}_1, \delta_1) - Sf_{n_0}(\mathfrak{D}_2, \delta_2)| < \varepsilon/2 \quad \text{whenever } (\mathfrak{D}^*, \delta^*) < (\mathfrak{D}_i, \delta_i),$$

$i = 1, 2$. Thus we have

$$\begin{aligned} &|S_f(\mathfrak{D}_1, \delta_1) - S_f(\mathfrak{D}_2, \delta_2)| \\ &\quad \leqq |S_f(\mathfrak{D}_1, \delta_1) - Sf_{n_0}(\mathfrak{D}_1, \delta_1)| + |Sf_{n_0}(\mathfrak{D}_1, \delta_1) - Sf_{n_0}(\mathfrak{D}_2, \delta_2)| \\ &\quad + |Sf_{n_0}(\mathfrak{D}_2, \delta_2) - S_f(\mathfrak{D}_2, \delta_2)| < \eta(b-a) + \varepsilon/2 + \eta(b-a) < \varepsilon \\ &\quad \text{whenever } (\mathfrak{D}^*, \delta^*) < (\mathfrak{D}_i, \delta_i), \qquad i = 1, 2. \end{aligned}$$

It follows from theorem 10 that f is integrable.

It remains to show that $\lim_{n\to\infty} I(f_n) = I(f)$. For this purpose, let $\varepsilon > 0$ be given and f_{n_0} be the same as above. Since f and all f_n are integrable, there exist $(\mathfrak{D}_0, \delta_0)$ and $(\mathfrak{D}_n, \delta_n)$ for each n such that

$$|S_f(\mathfrak{D}, \delta) - I(f)| < \varepsilon/4 \text{ whenever } (\mathfrak{D}_0, \delta_0) < (\mathfrak{D}, \delta)$$

and

$$|S_{f_n}(\mathfrak{D}, \delta) - I(f_n)| < \varepsilon/4 \text{ whenever } (\mathfrak{D}_n, \delta_n) < (\mathfrak{D}, \delta), \text{ for each } n.$$

Evidently, for $n \geqq n_0$ and $(\mathfrak{D}, \delta) \in \mathscr{D}$, we have

$$\begin{aligned} |I(f_n) - I(f)| &\leqq |I(f_n) - S_{f_n}(\mathfrak{D}, \delta)| + |S_{f_n}(\mathfrak{D}, \delta) - S_f(\mathfrak{D}, \delta)| + |S_f(\mathfrak{D}, \delta) - I(f)| \\ &< \varepsilon/4 + |S_f(\mathfrak{D}, \delta) - I(f)| + |S_{f_n}(\mathfrak{D}, \delta) - I(f_n)| \end{aligned}$$

and the last two terms can be made less than $\varepsilon/2$ by choosing $(\mathfrak{D}, \delta)$ in $\mathscr{D}$ with $(\mathfrak{D}_0, \delta_0) < (\mathfrak{D}, \delta)$ and $(\mathfrak{D}_n, \delta_n) < (\mathfrak{D}, \delta)$. The proof is completed.

COROLLARY 12. *If $E \subset [a, b]$ is a Lebesgue null set and $\{f_n\}$ a sequence of integrable functions which converges to f uniformly on $[a, b] - E$, then f is integrable with integral $I(f) = \lim_{n\to\infty} I(f_n)$.*

PROOF. For each n let g_n and h_n be functions defined by

$$g_n(z) = f_n(z) \text{ for } z \in [a, b]-E,$$
$$= 0 \qquad \text{for } z \in E,$$

and

$$h_n = f_n - g_n.$$

Similarly, we define g and h by

$$g(z) = f(z) \text{ for } z \in [a, b]-E,$$
$$= 0 \qquad \text{for } z \in E$$

and

$$h = f - g.$$

It is trivial from the hypothesis that the sequence $\{g_n\}$ converges uniformly to g on $[a, b]$ and that the functions h and all h_n are Lebesgue null. Since the integral considered here includes the Lebesgue integral, h and all h_n are integrable with $I(h) = I(h_n) = 0$ for all n. Since for each n $g_n = f_n - h_n$ and by hypothesis f_n is integrable, in view of lemma 4, each g_n is integrable and $I(g_n) = I(f_n)$. By theorem 11, g is integrable with $I(g) = \lim_{n\to\infty} I(g_n)$. By lemma 4 again, $f = h + g$ is integrable and $I(f) = I(g) = \lim_{n\to\infty} I(g_n) = \lim_{n\to\infty} I(f_n)$.

It is worth while noting that the above corollary is not true for Riemann integrals.

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References

[1] J. Kurzweil, 'Generalized ordinary differential equations and continuous dependence on a parameter', *Czechoslovak Math. Jour.* 7 (82) (1957), 418–446.
[2] R. Henstock, *Theory of Integration* (Butterworths, London, 1963).
[3] R. Henstock, 'A Riemann-type integral of Lebesgue power', *Canad. J. Math.* 20 (1968), 79–87.
[4] J. L. Kelley, *General Topology* (Van Nostrand, Princeton, 1965).

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