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FULLNESS OF MAPS

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ABSTRACT. An example is given of a surjective map $\tau:[0,1]\to[0,1]$ which takes every interval of $[0,1]$ onto $[0,1]$ eventually, but does not do so for certain other sets of positive measure.

1. **Introduction.** Let $I=[0,1]$, $\mathcal{B}=\{A:A\subset I,\ A$ Lebesgue measurable$\}$ and let λ denote the Lebesgue measure on $(I,\mathcal{B})$.

DEFINITION. Let $\tau:I\to I$ be measurable and surjective. We say τ is *full* if for all $A\in\mathcal{B}$, $\lambda(A)>0$, and $\tau(A),\tau^2(A),\ldots,$ measurable,

$$\lim_{n\to\infty}\lambda(\tau^n(A))=1 \tag{1}$$

holds. If (1) is true for any interval $A\subset I$, we say τ is *interval full.*

In this note we prove the existence of a surjective map that is interval full but not full. The key to the construction lies in the observation that while topological conjugation preserves topological properties it does not preserve measure-theoretic properties.

2. **Main Results**. Define the continuous surjective map $\tau:I\to I$ as follows:

$$\tau(x)=\begin{cases}3x, & x\in I_1=[0,\frac{1}{3}]\\ 2-3x, & x\in I_2=[\frac{1}{3},\frac{2}{3}]\\ 3x-2, & x\in I_3=[\frac{2}{3},1]\end{cases} \tag{2}$$

LEMMA 1. *τ is interval full.*

Proof. Let $J=[\alpha,\mathcal{B}]$ be any subinterval of I. If $\frac{2}{3}\in J$, then since $\tau(\frac{2}{3})=0$ and $\tau(0)=0$, $\tau^n(J)$ is an interval about 0 for all $n=1,2,\ldots.$ If $\tau^k(j)\subseteq[0,\frac{1}{3}]$, $k=1,\ldots,n-1$, then the length of $\tau^n(J)$ is 3^{n-1} times the length of $\tau(J)$ since $\tau\,|\,[0,\frac{1}{3}]$ is given by $\tau(x)=3x$. Thus for some n we must have $\frac{1}{3}\in\tau^n(J)$. Then $\tau^{n+1}(J)$ is an interval containing 0 and $\tau(\frac{1}{3})=1$ and $\tau^{n+1}(J)=[0,1]$. On the other hand, if $\frac{1}{3}\in J$ then $\tau^n(J)$ is an interval about 1 since $\tau(\frac{1}{3})=1$ and $\tau(1)=1$. Reasoning as above $\tau^n(J)$ must contain $\frac{2}{3}$ for some n and then $\tau^{n+1}(J)=[0,1]$.

If now $J\subset I_i$, $i=1,2$, or 3, then $\lambda(\tau(J))=3\lambda(J)$, since $|d\tau/dx|=3$ on each of the subintervals I_1, I_2, I_3. If $\frac{1}{3}$ or $\frac{2}{3}\in\tau(J)$, we proceed as above to obtain the

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result. If not, then we get $\lambda(\tau^2(J)) = 9\lambda(J)$. More generally,

$$\lambda(\tau^k(J)) = 3^k\lambda(J),$$

where $J, \tau(J), \ldots, \tau^k(J)$ are all in one of I_1, I_2, I_3. The expansion, however, forces $\tau^l(J)$ to contain $\frac{1}{3}$ or $\frac{2}{3}$ for some l. Then we proceed as above.

Q.E.D.

Remark. The τ defined above is an example of a piecewise linear map Markov map. In [1] it is shown that a class of non-linear Markov maps are interval full.

Now, the standard ternary representation of the elements of the Cantor set $\mathscr{C}$ leads directly to the conclusion : $\tau(\mathscr{C}) \subseteq \mathscr{C}$. Recall $\mathscr{C}$ has Lebesgue measure 0. Let $\mathscr{A}$ be any Cantor set in I that has positive Lebesgue measure.

LEMMA 2. *There exists a homeomorphism ϕ of I onto itself such that $\phi(\mathscr{C}) = \mathscr{A}$.*

Proof. [2, p. 101].

PROPOSITION. *Let $\sigma = \phi \circ \tau \circ \phi^{-1}$, where τ is defined by (2) and ϕ is the homeomorphism of Lemma 2. Then $\sigma : I \to I$ is interval full but not full.*

Proof. Let J be an interval. Then $\phi^{-1}(J)$ is an interval, and it follows that there exists an integer n such that $\tau^n(\phi^{-1}(J)) = I$, since τ is interval full. Noting that $\sigma^n = \phi \circ \tau^n \circ \phi^{-1}$, we have

$$\begin{aligned} \sigma^n(J) &= \phi(\tau^n(\phi^{-1}(J))) \\ &= \phi(I) = I, \end{aligned}$$

since ϕ is a homeomorphism. Thus σ is interval full. It is, however, not full, since for any integer n

$$\begin{aligned} \sigma^n(\mathscr{A}) &= \phi(\tau^n(\phi^{-1}(\mathscr{A}))) \\ &= \phi(\tau^n(\mathscr{C})) \subseteq \phi(\mathscr{C}), \end{aligned}$$

since $\tau(\mathscr{C}) \subset \mathscr{C}$. But $\phi(\mathscr{C}) = \mathscr{A}$. Thus,

$$\sigma^n(\mathscr{A}) \subseteq \mathscr{A}.$$

Since $\mathscr{A}$ has Lebesgue measure strictly less than 1, the conclusion follows.

Q.E.D.

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