SOME LINEAR OPERATORS IN THE L^p SPACES

LEE PENG-YEE

1. Introduction. Let L^p, $1 \leqq p < \infty$, denote the space of all functions f (real or complex) such that f and $|f|^p$ are variationally integrable (see **2**, p. 40, for definition) with respect to a pair $\mathbf{h}$ of interval functions in an elementary set E. In what follows, we fix both $\mathbf{h}$ and E and assume that $\mathbf{h} = \{h_l, h_r\}$ is variationally integrable and $h_s \geqq 0$ $(s = l, r)$ in E. Further, let L^∞ denote the space of all functions f (real or complex) which are $\mathbf{h}$-measurable (cf. **2**, p. 95) and bounded almost everywhere, i.e. except in a set X satisfying (cf. **2**, p. 47) $V(\mathbf{h}; E; X) = 0$. If we regard f and g as identical when $f = g$ almost everywhere, then it has been pointed out in (**3**, § 1) that L^p, $1 \leqq p \leqq \infty$, with

$$||f||_p = [(V)\textstyle\int_E |f|^p\, d\mathbf{h}]^{1/p} \qquad (1 \leqq p < \infty),$$

$$||f||_\infty = \inf\{N; |f| \leqq N \text{ almost everywhere}\},$$

are Banach spaces. Note that with the help of (**3**, p. 315, Lemma 3), the continuity of $\mathbf{h}$ in (**2**, pp. 133–134, Theorems 53.4 and 53.5) can be dropped, and hence is not required in proving that L^p are Banach spaces.

In (**3**, Theorem 1), we gave necessary and sufficient conditions for the weak convergence of the sequence

$$F(f; y) = (V)\textstyle\int_E fg(\cdot, y)\, d\mathbf{h}$$

of linear *functionals* in L^p. In this note, we consider linear *operators* from L^p into the spaces of bounded functions and of continuous functions. Furthermore, we define two new function spaces, namely Λ and Ω, which are akin to the L^p spaces, and consider linear *operators* from L^p into Λ and Ω and conversely.

2. Linear operators from L^p into B or C. Let p, q denote a pair of numbers having the following relation: when $1 < p < \infty$, $1/p + 1/q = 1$; when $p = 1$, $q = \infty$; and when $p = \infty$, $q = 1$. Then, as in Lebesgue theory, we have the following result.

LEMMA 1. *If $f \in L^p$ and $g \in L^q$, then both fg and $|fg|$ are variationally integrable with respect to $\mathbf{h}$ in E and*

$$|(V)\textstyle\int_E fg\, d\mathbf{h}| \leqq (V)\textstyle\int_E |fg|\, d\mathbf{h} \leqq ||f||_p ||g||_q.$$

If $f \geqq 0$, $g \geqq 0$, the integrability of fg and $|fg|$ follows from (**2**, p. 133, Theorem 53.4; **3**, p. 315, Lemma 3). In general, if f and g are real, we divide

Received December 21, 1967, and in revised form, May 28, 1968.

https://doi.org/10.4153/CJM-1969-073-0 Published online by Cambridge University Press

E into four subsets, namely $X(f \geqq 0, g \geqq 0)$, $X(f < 0, g < 0)$, $X(f \geqq 0, g < 0)$, $X(f < 0, g \geqq 0)$. Then fg and $|fg|$ are integrable in each subset and therefore in E. When f and g are complex, we split them into real and imaginary parts, and hence the result. The remainder of the lemma follows from (**2**, pp. 61, 133, Theorems 31.2 (31.12) and 53.3 (53.10)).

For convenience, we restate here two other lemmas given in (**3**).

LEMMA 2. *If $g \in L^q$, and*

$$F(f) \equiv (V)\int_E fg\, d\mathbf{h}$$

is defined for all $f \in L^p$, then F is a continuous linear functional in L^p. Moreover, $||F|| = ||g||_q$.

LEMMA 3. *In order that $F(f)$ should be defined for all $f \in L^p$, it is necessary and sufficient that $g \in L^q$.*

Let B and C be the Banach spaces of bounded functions and of continuous functions, respectively, all in E, with norms defined to be $\sup\{|f(x)|; x \in E\}$.

THEOREM 1. *In order that for every $y \in E$ and for every $f \in L^p$*

$$F(f;y) \equiv (V)\int_E fg(\cdot, y)\, d\mathbf{h}$$

should be defined and $F(f;\cdot) \in B$, it is necessary and sufficient that
(i) *for every $y \in E$, $g(\cdot, y) \in L^q$;*
(ii) $\sup\{||g(\cdot, y)||_q; y \in E\} < \infty$.

The necessity of (i) follows from Lemma 3.

To prove (ii), we first show that F is a continuous linear operator from L^p into B. By Lemma 2, it is linear. In view of (**1**, p. 79, Theorem 2, or **7**, p. 265, Lemma 6), to prove its continuity it suffices to show that $||F(f;\cdot)||_B$ is lower semi-continuous in L^p.

Let $f_j \to f$ in L^p. Then, in view of (**2**, p. 137, Theorem 54.2),

$$|F(f;y)| = \lim_{j\to\infty} |F(f_j;y)| \leqq \liminf_{j\to\infty} ||F(f_j;\cdot)||_B$$

for every $y \in E$, so that $||F(f;\cdot)||_B \leqq \liminf ||F(f_j;\cdot)||_B$.

Since we have proved that F is a continuous linear operator from L^p into B, there is an $N > 0$ such that

$$||F(f;\cdot)||_B \leqq N||f||_p,$$

i.e.,

$$\sup\{|F(f;y)|; y \in E\} \leqq N||f||_p$$

for all $f \in L^p$. Then by Lemma 2

$$\sup\{||g(\cdot, y)||_q; y \in E\} \leqq N.$$

Hence, we obtain (ii).

The sufficiency of the conditions (i) and (ii) follows from Lemmas 1 and 3.

THEOREM 2. *In order that for every* $y \in E$ *and for every* $f \in L^p$

$$F(f;y) \equiv (V)\int_E fg(\cdot, y)\,d\mathbf{h}$$

should be defined and $F(f;\cdot) \in C$, *it is necessary and sufficient that* (i)–(iii) *of the following be satisfied when* $1 \leqq p < \infty$, *and* (i)–(iv) *when* $p = \infty$:

(i) *For every* $y \in E$, $g(\cdot, y) \in L^q$;

(ii) $\sup\{\|g(\cdot, y)\|_q;\ y \in E\} < \infty$;

(iii) *For each elementary set* $E_1 \subseteq E$ *and for each* $z \in E$,

$$\lim_{y \to z} F(\mathrm{ch}(E_1); y) = F(\mathrm{ch}(E_1); z),$$

where $\mathrm{ch}(E_1)$ *denotes the characteristic function of* E_1;

(iv) *For every* $\epsilon > 0$ *there are* $\eta > 0$ *and* $\delta > 0$ *such that whenever* $\mathrm{ch}(X)$ *is* $\mathbf{h}$-*measurable and* $V(\mathbf{h}; E; X) < \eta$ *and* $|y - z| < \delta$,

$$|F(\mathrm{ch}(X); y) - F(\mathrm{ch}(X); z)| < \epsilon.$$

The necessity of (i) and (ii) follows as in Theorem 1, and that of (iii) and (iv) by setting $f = \mathrm{ch}(E_1)$ and $\mathrm{ch}(X)$, respectively. By using (**3**, Theorem 1), we obtain the sufficiency.

We remark that conditions (iii) and (iv) can be replaced by the following condition:

(v) *For each* $X \subseteq E$ *such that* $\mathrm{ch}(X)$ *is* $\mathbf{h}$-*measurable and for each* $z \in E$,

$$\lim_{y \to z} F(\mathrm{ch}(X); y) = F(\mathrm{ch}(X); z).$$

The necessity of (v) follows by setting $f = \mathrm{ch}(X)$. Since (v) implies (iii) and (iv), hence the sufficiency.

3. The spaces Λ and Ω. Let $\mathbf{h}$ be defined as in § 1, and let ϕ be a real-valued monotone decreasing function variationally integrable with respect to $\mathbf{h}$, and with $\inf\{\phi(x);\ x \in E\} > 0$. We denote by Λ the space of all functions f in L^1 such that

$$\|f\|_\Lambda = (V)\int_E |f|\phi\,d\mathbf{h} < \infty.$$

It was pointed out by the referee that when $E = [0, 1]$, $\phi(x) = \alpha x^{\alpha-1}$ $(0 < \alpha < 1)$, and $h_s(u, v) = v - u$ $(s = l, r)$, the space Λ is different from the space $\Lambda(\alpha)$ defined by Lorentz (**4**). For example, if $\alpha = \frac{1}{2}$ and $f(x) = (1 - x)^{-\frac{1}{2}}$, then $f \in \Lambda$ but $f \notin \Lambda(\frac{1}{2})$.

Further, we denote by Ω the space of all functions f in L^1 such that

$$\|f\|_\Omega = \sup\left\{\frac{(V)\int_E |f|\,\mathrm{ch}(E_1)\,d\mathbf{h}}{(V)\int_E \phi\,\mathrm{ch}(E_1)\,d\mathbf{h}};\ E_1 \subseteq E\right\} < \infty,$$

where E_1 is an arbitrary elementary set in E. Then $f \in \Omega$ if and only if there is an $N > 0$ such that $|f| \leqq N\phi$ almost everywhere in E. The sufficiency is easy. To prove the necessity, since $f \in \Omega$, there is an $N > 0$ such that

$$(V)\int_E |f|\operatorname{ch}(I)\, d\mathbf{h} \leqq N(V)\int_E \phi \operatorname{ch}(I)\, d\mathbf{h},$$

where I is an arbitrary interval open on the right. By (**2**, p. 63, Theorem 31.3 (31.20)),

$$\begin{aligned}(V)\int_E \{|f| \operatorname{ch}(I)/\phi\}\, d\mathbf{h}^* &= (V)\int_E |f| \operatorname{ch}(I)\, d\mathbf{h}\\ &\leqq N(V)\int_E \phi \operatorname{ch}(I)\, d\mathbf{h}\\ &= N(V)\int_E \operatorname{ch}(I)\, d\mathbf{h}^*,\end{aligned}$$

where $\mathbf{h}^*$ is variationally equivalent to $\{\phi h_l, \phi h_r\}$ in E. By (**2**, p. 78, Exercise 34.2 and Theorem 35.1 (35.3)), it follows that $|f|/\phi \leqq N$ except possibly in a set X satisfying $V(\mathbf{h}; E; X) = 0$. We have also used (**2**, p. 49, Theorem 28.1).

We remark that when ϕ is bounded, Λ reduces to the L^1 space and Ω to L^∞. As usual, two functions which are equal almost everywhere are regarded as identical.

LEMMA 4. *Λ and Ω are Banach spaces.*

We shall only prove that Λ is complete. The proof for Ω is similar. We may assume without loss of generality that $E = [a, b]$. First, we observe that

$$(V)\int_{a+\delta}^{b} |f|\phi\, d\mathbf{h}$$

is continuous in L^1 and monotone increasing as $\delta \to 0+$. Then

$$||f||_\Lambda = \lim_{\delta\to 0+} (V)\int_{a+\delta}^{b} |f|\phi\, d\mathbf{h}$$

is lower semicontinuous in L^1. Note that this includes both cases when $\phi(a) < \infty$ and when $\phi(a) = \infty$.

Let $\{f_j\}$ be a Cauchy sequence in Λ. Then

$$\begin{aligned}||f_j - f_k||_1 &= (V)\int_a^b |f_j - f_k|\, d\mathbf{h}\\ &\leqq \phi(b)^{-1}(V)\int_a^b |f_j - f_k|\phi\, d\mathbf{h}\\ &= \phi(b)^{-1}||f_j - f_k||_\Lambda \to 0 \quad \text{as } j, k \to \infty.\end{aligned}$$

Since L^1 is complete, there is $f \in L^1$ such that $||f_j - f||_1 \to 0$. By the lower semicontinuity of $||f||_\Lambda$ in L^1, it follows that $f \in \Lambda$, and that for every $\epsilon > 0$, there is a j_0 such that whenever $j \geqq j_0$

$$||f_j - f||_\Lambda \leqq \liminf_{k\to\infty} ||f_j - f_k||_\Lambda \leqq \epsilon.$$

Hence, Λ is complete.

LEMMA 5. *If $f \in \Lambda$ and $g \in \Omega$, then both fg and $|fg|$ are variationally integrable with respect to* $\mathbf{h}$ *in E and*

$$|(V)\int_E fg\,d\mathbf{h}| \leqq (V)\int_E |fg|\,d\mathbf{h} \leqq ||f||_\Lambda ||g||_\Omega.$$

It is evident that for every elementary set $E_1 \subseteq E$,

$$(V)\int_E |g|\,\mathrm{ch}(E_1)\,d\mathbf{h} \leqq ||g||_\Omega (V)\int_E \phi\,\mathrm{ch}(E_1)\,d\mathbf{h}.$$

Then the required inequality is true for any step function f. Let $f \in \Lambda$. In view of (**3**, Lemma 4), there is a sequence $\{f_j\}$ of step functions such that $||f_j - f||_1 \to 0$. By (**2**, pp. 135, 86, Theorems 54.1 and 37.1 (37.10)), there is a subsequence $\{f_{j(n)}\}$ such that

$$\lim_{n\to\infty} f_{j(n)} = f$$

for almost all x in E, and both fg and $|fg|$, as limits of $f_{j(n)}g$ and $|f_{j(n)}g|$, respectively, are integrable with respect to $\mathbf{h}$ in E. It follows that

$$(V)\int_E |fg|\,d\mathbf{h} \leqq ||g||_\Omega (V)\int_E \phi|f|\,d\mathbf{h} = ||g||_\Omega ||f||_\Lambda.$$

LEMMA 6. *In order that $F(f)$ in Lemma 2 should be defined for all $f \in \Lambda$, it is necessary and sufficient that $g \in \Omega$.*

The sufficiency follows from Lemma 5. To prove the necessity, let X_j be the set of x for which $|g| \leqq j\phi$ almost everywhere in E. Then by (**2**, p. 82, Theorem 36.1)

$$\lim_{j\to\infty} (V)\int_E fg\,\mathrm{ch}(X_j)\,d\mathbf{h} = (V)\int_E fg\,d\mathbf{h}.$$

Since Λ is complete and $F_j(f) = F(f\,\mathrm{ch}(X_j))$, $j \geqq 1$, are continuous linear functionals in Λ with $||F_j|| = ||g\,\mathrm{ch}(X_j)||_\Omega$, it follows from the Banach-Steinhaus theorem (see, for example, **1**, p. 80, Theorem 5) that

$$\sup\{||g\,\mathrm{ch}(X_j)||_\Omega;\, j \geqq 1\} < \infty.$$

Hence $g \in \Omega$.

4. Linear operators from L^p into Λ or Ω.

We write

$$F_1(f;y) = (V)\int_E fg(\cdot, y)\,d\mathbf{h}, \qquad F_2(f;x) = (V)\int_E fg(x,\cdot)\,d\mathbf{h},$$

when the integrals exist.

THEOREM 3. *Let $g(x, y) \geqq 0$ be variationally integrable* (see **2**, p. 104, for definition) *with respect to $\mathbf{h} \times \mathbf{h} = \{h_r h_l,\, h_l h_l,\, h_l h_r,\, h_r h_r\}$ in (E, E). Then the following statements are equivalent:*

(i) *For every $f \in L^p$, $F_1(f;y)$ as a function of y is defined almost everywhere in E, and $F_1(f;\cdot) \in \Omega$;*

(ii) *For every $f \in \Lambda$, $F_2(f;x)$ as a function of x is defined almost everywhere in E, and $F_2(f;\cdot) \in L^q$;*

(iii) *$g(x,\cdot) \in \Omega$ for almost all $x \in E$, $g(\cdot, y) \in L^q$ for almost all $y \in E$, and*

$$\sup\left\{\frac{||F_2(\mathrm{ch}(E_1);\cdot)||_q}{(V)\int_E \phi\,\mathrm{ch}(E_1)\,d\mathbf{h}};\, E_1 \subseteq E\right\} < \infty.$$

https://doi.org/10.4153/CJM-1969-073-0 Published online by Cambridge University Press

Assume that (i) holds. Let $f \in L^p$, $f^* \in \Lambda$, and let

$$f_j = \begin{cases} f & (|f| < j), \\ j & (|f| \geqq j); \end{cases} \qquad f_j^* = \begin{cases} f^* & (|f^*| < j), \\ j & (|f^*| \geqq j). \end{cases}$$

Since the product $f_j f_j^*$ is bounded and $(\mathbf{h} \times \mathbf{h})$-measurable in (E, E), in view of the two-dimensional analogue of (**2**, p. 99, Theorem 40.2, second part), we find that the double integral

$$(V) \int_{(E,E)} f_j(x) f_j^*(y) g(x, y) \, d\mathbf{h} \times \mathbf{h}$$

exists. Then by (**2**, p. 106, Theorem 44.1 (Fubini's theorem)), $F_2(f_j^*; x)$ exists for almost all x in E and also

$$\begin{aligned} |(V) \textstyle\int_E f_j F_2(f_j^*; \cdot) \, d\mathbf{h}| &= |(V) \textstyle\int_{(E,E)} f_j f_j^* g \, d\mathbf{h} \times \mathbf{h}| \\ &= |(V) \textstyle\int_E f_j^* F_1(f_j; \cdot) \, d\mathbf{h}| \\ &\leqq (V) \textstyle\int_E |f^* F_1(|f|; \cdot)| \, d\mathbf{h} \\ &\leqq ||f^*||_\Lambda ||F_1(|f|; \cdot)||_\Omega < \infty. \end{aligned}$$

If $f \geqq 0$, $f^* \geqq 0$, then $f_j f_j^*$ is monotone increasing in j, in view of the above inequalities and the two-dimensional analogue of (**2**, p. 82, Theorem 36.1), we find that the double integral

$$(V) \int_{(E,E)} f(x) f^*(y) g(x, y) \, d\mathbf{h} \times \mathbf{h}$$

exists. For general f and f^*, we split them as in the proof of Lemma 1; hence, the double integral exists for $f \in L^p$ and $f^* \in \Lambda$. Then, again, by Fubini's theorem, $F_2(f^*; x)$ exists for almost all x in E and

$$(V) \int_E f F_2(f^*; \cdot) \, d\mathbf{h}$$

exists for all $f \in L^p$. It follows from Lemma 3 that $F_2(f^*; \cdot) \in L^q$ for all $f^* \in \Lambda$. Hence, (i) implies (ii).

Assume that (ii) holds. By Lemma 6, the condition that $g(x, \cdot) \in \Omega$ for almost all x in E is necessary. From an argument similar to the one above, it follows that $F_1(f; y)$ exists for all $f \in L^p$ and $g(\cdot, y) \in L^q$ for almost all y.

Further, let $f_j \to f$ in Λ. Following the proof of (**2**, p. 137, Theorem 54.2), with $||\cdot||_p$ and $||\cdot||_q$ replaced by $||\cdot||_\Lambda$ and $||\cdot||_\Omega$, we find that

$$\lim_{j\to\infty} (V) \int_E f_j g(x, \cdot) \, d\mathbf{h} = (V) \int_E f g(x, \cdot) \, d\mathbf{h}$$

for almost all x. Then

$$\begin{aligned} (V) \textstyle\int_E |F_2(f; \cdot)|^q \, d\mathbf{h} &= (V) \textstyle\int_E \lim_{j\to\infty} |F_2(f_j; \cdot)|^q \, d\mathbf{h} \\ &\leqq \liminf_{j\to\infty} (V) \textstyle\int_E |F_2(f_j; \cdot)|^q d\mathbf{h}, \end{aligned}$$

so that $||F_2(f; \cdot)||_q \leqq \liminf ||F_2(f_j; \cdot)||_q$. Hence, we have proved that F_2 is a continuous linear operator from Λ into L^q.

Now put $f = \mathrm{ch}(E_1) \in \Lambda$. It follows that there is an $N > 0$ such that

$$||F_2(\mathrm{ch}(E_1); \cdot)||_q \leqq N ||\mathrm{ch}(E_1)||_\Lambda$$

for every $E_1 \subseteq E$. Hence, (ii) implies (iii).

https://doi.org/10.4153/CJM-1969-073-0 Published online by Cambridge University Press

Assume that (iii) holds. The first part of (i) follows from Lemma 1. Again, we can show that for every $f \in L^p$ and for every $E_1 \subseteq E$, the double integral

$$(V)\int_{(E,E)} f(x)\,\mathrm{ch}\,(E_1; y) g(x, y)\, d\mathbf{h} \times \mathbf{h}$$

exists. Then, by Fubini's theorem,

$$\begin{aligned}(V)\int_E |F_1(f;\cdot)|\,\mathrm{ch}\,(E_1)\,d\mathbf{h} &\leqq (V)\int_E F_1(|f|;\cdot)\,\mathrm{ch}\,(E_1)\,d\mathbf{h}\\ &= (V)\int_E |f| F_2(\mathrm{ch}\,(E_1);\cdot)\,d\mathbf{h}\\ &\leqq ||f||_p ||F_2(\mathrm{ch}\,(E_1);\cdot)||_q\\ &\leqq N||f||_p (V)\int_E \phi\,\mathrm{ch}\,(E_1)\,d\mathbf{h}\end{aligned}$$

for some $N > 0$. Since E_1 is arbitrary, $F_1(f;\cdot) \in \Omega$. Hence, (iii) implies (i).

THEOREM 4. *Let g be defined as in Theorem 3. Then the following three statements are equivalent:*

(i) *For every* $f \in L^q$, $F_1(f; y)$ *as a function of y is defined almost everywhere in E, and* $F_1(f;\cdot) \in \Lambda$;

(ii) *For every* $f \in \Omega$, $F_2(f; x)$ *as a function of x is defined almost everywhere, in E, and* $F_2(f;\cdot) \in L^p$;

(iii) $g(x,\cdot) \in \Lambda$ *for almost all x in E,* $g(\cdot, y) \in L^p$ *for almost all y in E, and*

$$||F_2(\phi;\cdot)||_p < \infty.$$

The proof is similar to that of Theorem 3.

We remark that the conditions imposed on g in Theorems 3 and 4 are essential for proving the existence of the double integrals involved and so for the application of Fubini's theorem. We also remark that special cases of Theorems 3 and 4 when $q = 1$ and $\phi = 1$ have been investigated by Tatchell (**7**) and by Sunouchi and Tsuchikura (**6**) for the Lebesgue integral. Sunouchi and Tsuchikura assumed that g is measurable in the plane, but Tatchell did not. The analogue of Theorem 3 for sequence spaces has been proved by Sargent (**5**).

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Monogr. Mat., Vol. 1, 1932, Warsaw).

2. R. Henstock, *Theory of integration* (Butterworths, London, 1963).

3. Lee Peng-Yee, *A note on some generalizations of the Riemann-Lebesgue theorem*, J. London Math. Soc. *41* (1966), 313–317.

4. G. G. Lorentz, *Some new functional spaces*, Ann. of Math. (2) *51* (1950), 37–55.

5. W. L. C. Sargent, *Some sequence spaces related to the l^p spaces*, J. London Math. Soc. *35* (1960), 116–171.

6. G. Sunouchi and T. Tsuchikura, *Absolute regularity for convergent integrals*, Tôhoku Math. J. (2) *4* (1952), 153–156.

7. J. B. Tatchell, *On some integral transformations*, Proc. London Math. Soc. (3) *3* (1953), 257–266.

University of Auckland,
Auckland, New Zealand