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QUANTISATION SPACES OF CLUSTER ALGEBRAS

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Abstract. The article concerns the existence and uniqueness of quantisations of cluster algebras. We prove that cluster algebras with an initial exchange matrix of full rank admit a quantisation in the sense of Berenstein–Zelevinsky and give an explicit generating set to construct all quantisations.

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1. Introduction. Sergey Fomin and Andrei Zelevinsky have introduced and studied cluster algebras in a series of four articles [**1, 9–11**] (one of which is coauthored by Arkady Berenstein) in order to study Lusztig's canonical basis and total positivity. Cluster algebras are commutative algebras which are constructed by generators and relations. The generators are called cluster variables and they are grouped into several overlapping sets, so-called clusters. A combinatorial mutation process relates the clusters and provides the defining relations of the algebra. The rules for this mutation process are encoded in a rectangular exchange matrix usually denoted by the symbol $\tilde{B}$. Surprisingly, Fomin–Zelevinsky's Laurent phenomenon [**9**, Theorem 3.1] asserts that every cluster variable can be expressed as a Laurent polynomial in an arbitrarily chosen cluster. The second main theorem of cluster theory is the classification of cluster algebras with only finitely many cluster variables by finite type root systems, see Fomin–Zelevinsky [**10**].

The connections of cluster algebras to other areas of mathematics are manifold. A major contribution is Caldero–Chapoton's map [**4**] which relates cluster algebras and representation theory of quivers. Another contribution is the construction of cluster algebras from oriented surfaces which relates cluster algebras to differential geometry, see Fomin–Shapiro–Thurston [**7**] and Fomin–Thurston [**8**].

Arkady Berenstein and Andrei Zelevinsky [**2**] have introduced the concept of quantum cluster algebras. Quantum cluster algebras are q-deformations which specialise to the ordinary cluster algebras in the classical limit $q = 1$. Such quantisations play an important role in cluster theory: On the one hand, quantisations are essential when trying to link cluster algebras to Lusztig's canonical bases, see for example, [**12, 18, 20–25**]. On the other hand, Goodearl–Yakimov [**15**] use quantisations to approximate cluster algebras by their upper bounds. The latter result is particularly important since it enables us to study cluster algebras as unions of Laurent polynomial rings. A different but closely related quantisation of cluster algebras can be found in Fock–Goncharov's work [**6**].

In general, the notion of q-deformation turns former commutative structures into non-commutative ones. In the case of quantum cluster algebras, this yields q-commutativity between variables within the same quantum cluster which is stored in an additional matrix usually denoted by the symbol Λ. In order to keep the q-commutativity intact under mutation, Berenstein and Zelevinsky require some compatibility relation between the matrices $\tilde{B}$ and Λ. The very same compatibility condition also parametrises compatible Poisson structures for cluster algebras, see Gekhtman–Shapiro–Vainshtein [**13**]. Solutions to a homogeneous version of the compatibility equation are used in recent works by Grabowski and Launois, see [**17**] and [**16**], to construct gradings of cluster algebras.

Unfortunately, not every cluster algebra admits a quantisation, because not every exchange matrix admits a compatible Λ. But in the case where there exists such a quantisation, Berenstein–Zelevinsky have shown that $\tilde{B}$ is of full rank.

This paper has several aims. First, we reinterpret what it means for exchange matrices without frozen indices to be of full rank via Pfaffians and perfect matchings. Second, we show the converse of the above statement posted by Zelevinsky [**26**]: Assuming $\tilde{B}$ is of full rank, there always exists a quantisation. This result is shown by using concise linear algebra arguments. It should be noted that Gekhtman–Shapiro–Vainshtein in [**13**, Theorem 4.5] prove a similar statement in the language of Poisson structures. Third, when a quantisation exists, it is not necessarily unique. This ambiguity we make more precise by relating all such quantisations via matrices we construct from a given $\tilde{B}$ using particular minors.

2. Berenstein–Zelevinsky's quantum cluster algebras.

2.1. Notation. Let m, n be integers with $1 \leq n \leq m$ and A an $m \times n$ matrix with integer entries. For $n < m$, we use the notation $[n, m] = \{n+1, \ldots, m\}$ and in the case $n = 1$ the shorthand $[m] = [1, m]$. For a subset $J \subseteq [m]$, we denote by A_J the submatrix of A with rows indexed by J and all columns. By q, we denote throughout the paper a formal indeterminate.

2.2. The definition of quantum cluster algebras. Let $\tilde{B} = (b_{i,j})$ be an $m \times n$ matrix with integer entries. For further use, we write $\tilde{B} = \left[\begin{smallmatrix} B \\ C \end{smallmatrix}\right]$ in block form with an $n \times n$ matrix B and an $(m-n) \times n$ matrix C. The matrix B is called the *principal part* of $\tilde{B}$. We call indices $i \in [n]$ *mutable* and the indices $j \in [n+1, m]$ *frozen*.

We say that the principal part B is *skew-symmetrisable* if there exists a diagonal $n \times n$ matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with positive integer diagonal entries such that the matrix DB is skew-symmetric, i.e. $d_i b_{i,j} = -d_j b_{j,i}$ for all $1 \leq i, j \leq n$. The matrix D is then called a *skew-symmetriser* for B and $\tilde{B}$ is called an *exchange matrix*. Note that skew-symmetrising from the right yields identical restraints and $b_{i,j} \neq 0$ if and only if $b_{j,i} \neq 0$.

The skew-symmetriser is essentially unique by the following discussion: Consider the unoriented simple graph $\Delta(B)$ with vertex set $\{1, 2, \ldots, n\}$ such that there is an edge between two vertices i and j if and only if $b_{i,j} \neq 0$. We say that the principal part B is *connected* if $\Delta(B)$ is connected.

Assume now that B is connected. Suppose there exist two diagonal $n \times n$-matrices D and D' with positive integer diagonal entries such that both DB and $D'B$ are skew-

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symmetric. Then there exists a rational number λ with $D = \lambda D'$, as for all indices i, j with $b_{i,j} \neq 0$ the equality $d_i/d_j = d'_i/d'_j$ holds true. We refer to the smallest such D as the *fundamental skew-symmetriser*. If B is not connected, then every skew-symmetriser D is an $\mathbb{N}^+$-linear combination of the fundamental skew-symmetrisers of the connected components of B.

This concludes the discussion of the first datum to construct quantum cluster algebras. The next piece of data is the notion of compatible matrix pairs.

From now on, assume that $\tilde{B} = \left[\begin{smallmatrix} B \\ C \end{smallmatrix}\right]$ is a not necessarily connected matrix with skew-symmetrisable principal part B. A skew-symmetric $m \times m$ integer matrix $\Lambda = (\lambda_{i,j})$ is called *compatible with* $\tilde{B}$ if there exists a diagonal $n \times n$ matrix $D' = \mathrm{diag}(d'_1, d'_2, \dots, d'_n)$ with positive integers $d'_1, d'_2, \dots, d'_n$ such that

$$\tilde{B}^T \Lambda = \begin{bmatrix} D' & 0 \end{bmatrix} \tag{1}$$

as a block matrix with a first block of size $n \times n$ and a second block of size $n \times (m-n)$. In this case, we call $(\tilde{B}, \Lambda)$ a *compatible* pair. To any $m \times n$ matrix $\tilde{B}$, there need not exist a compatible Λ. As a necessary condition Berenstein–Zelevinsky [**2**, Prop. 3.3] note that if a matrix $\tilde{B}$ belongs to a compatible pair $(\tilde{B}, \Lambda)$, then its principal part B is skew-symmetrisable, D' itself is a skew-symmetriser and $\tilde{B}$ itself is of full rank, i. e. $\mathrm{rank}(\tilde{B}) = n$.

Let us fix a compatible pair $(\tilde{B}, \Lambda)$. We are now ready to complete the necessary data to define quantum cluster algebras. First of all, let $\{e_i \colon 1 \leq 1 \leq m\}$ be the standard basis of $\mathbb{Q}^m$. With respect to this standard basis, the skew-symmetric matrix Λ defines a skew-symmetric bilinear form $\beta \colon \mathbb{Q}^m \times \mathbb{Q}^m \to \mathbb{Q}$. The *based quantum torus* $\mathcal{T}_\Lambda$ associated with Λ is the $\mathbb{Z}[q^{\pm 1}]$-algebra with $\mathbb{Z}[q^{\pm 1}]$-basis $\{X^a \colon a \in \mathbb{Z}^m\}$ where we define the multiplication of basis elements by the formula $X^a X^b = q^{\beta(a,b)} X^{a+b}$ for all elements $a, b \in \mathbb{Z}^m$. It is an associative algebra with unit $1 = X^0$ and every basis element X^a has an inverse $(X^a)^{-1} = X^{-a}$. The based quantum torus is commutative if and only if Λ is the zero matrix, in which case $\mathcal{T}_\Lambda$ is a Laurent polynomial algebra. In general, it is an Ore domain, see [**2**, Appendix] for further details. We embed $\mathcal{T}_\Lambda \subseteq \mathcal{F}$ into an ambient skew field.

Although $\mathcal{T}_\Lambda$ is not commutative in general, the relation

$$X^a X^b = q^{2\beta(a,b)} X^b X^a$$

holds for all elements $a, b \in \mathbb{Z}^m$. Because of this relation, we say that the basis elements are *q-commutative*. Put $X_i = X^{e_i}$ for all $i \in [m]$. The definition implies $X_i X_j = q^{\lambda_{i,j}} X_j X_i$ for all $i, j \in [m]$. Then we may write $\mathcal{T}_\Lambda = \mathbb{Z}[q^{\pm 1}][X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}]$ and the basis vectors satisfy the relation

$$X^a = q^{\sum_{i>j} \lambda_{i,j} a_i a_j} X_1^{a_1} X_2^{a_2} \cdot \ldots \cdot X_m^{a_m}$$

for all $a = (a_1, a_2, \dots, a_m) \in \mathbb{Z}^m$.

We call a sequence of pairwise q-commutative and algebraically independent elements such as $X = (X_1, X_2, \dots, X_m)$ in $\mathcal{F}$ an *extended quantum cluster*, the elements $X_1, X_2, \dots, X_n$ of an extended quantum cluster *quantum cluster variables*, the elements $X_{n+1}, X_{n+2}, \dots, X_m$ *frozen variables* and the triple $(\tilde{B}, X, \Lambda)$ a *quantum seed*.

Let k be a mutable index. Define mutation map $\mu_k \colon (\tilde{B}, X, \Lambda) \mapsto (\tilde{B}', X', \Lambda')$ as follows:

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(M_1) The matrix $\tilde{B}' = \mu_k(\tilde{B})$ is the $m \times n$ matrix with entries

$$b'_{i,j} = -b_{i,j} \text{ if } i = k \text{ or } j = k,$$
$$b'_{i,j} = b_{i,j} + \operatorname{sgn}(b_{i,k}) \max(0, b_{i,k} b_{k,j}) \text{ if } i \neq k \neq j.$$

(M_2) The matrix $\Lambda' = (\lambda'_{i,j})$ is the $m \times m$ matrix with entries $\lambda'_{i,j} = \lambda_{i,j}$ except for

$$\lambda'_{i,k} = -\lambda_{i,k} + \sum_{r \neq k} \lambda_{i,r} \max(0, -b_{r,k}) \text{ for all } i \in [m] \backslash \{k\},$$
$$\lambda'_{k,j} = -\lambda_{k,j} - \sum_{r \neq k} \lambda_{j,r} \max(0, -b_{r,k}) \text{ for all } j \in [m] \backslash \{k\}.$$

(M_3) To obtain the quantum cluster X', we replace the element X_k with the element

$$X'_k = X^{-e_k + \sum_{i=1}^{m} \max(0, b_{i,k}) e_i} + X^{-e_k + \sum_{i=1}^{m} \max(0, -b_{i,k}) e_i} \in \mathcal{F}.$$

The variables $X' = (X'_1, X'_2, \ldots, X'_m)$ are pairwise q-commutative: for all $j \in [m]$ with $j \neq k$, the integers

$$\beta\left(-e_k + \sum_{i=1}^{m} \max(0, b_{i,k}) e_i, e_j\right) = -\lambda_{k,j} + \sum_{i=1}^{m} \max(0, b_{i,k}) \lambda_{i,j},$$
$$\beta\left(-e_k + \sum_{i=1}^{m} \max(0, -b_{i,k}) e_i, e_j\right) = -\lambda_{k,j} + \sum_{i=1}^{m} \max(0, -b_{i,k}) \lambda_{i,j}$$

are equal, because their difference is equal to the sum $\sum_{i=1}^{m} b_{i,k} \lambda_{i,j}$ which is the zero entry indexed by (k, j) in the matrix $\tilde{B}^T \Lambda$. So the compatibility condition implies that the variable X'_k q-commutes with all X_j. Hence, the variables $X' = (X'_1, X'_2, \ldots, X'_m)$ generate again a based quantum torus whose q-commutativity relations are given by the skew-symmetric matrix Λ'. Moreover, the pair $(\tilde{B}', \Lambda')$ is compatible by [**2**, Prop. 3.4] so that the matrix $\tilde{B}'$ has a skew-symmetrisable principle part. We conclude that the mutation $\mu_k(\tilde{B}', X', \Lambda') = (\tilde{B}', X', \Lambda')$ is again an extended quantum seed. Note that the mutation map is involutive, i. e. $(\mu_k \circ \mu_k)(\tilde{B}, X, \Lambda) = (\tilde{B}, X, \Lambda)$.

A main property of classical cluster algebras are the binomial exchange relations. For the quantised version, we require pairwise q-commutativity for the quantum cluster variables in a single cluster. This implies that a monomial $X_1^{a_1} X_2^{a_2} \cdot \ldots \cdot X_m^{a_m}$ with $a \in \mathbb{Z}^m$ remains (up to a power of q) a monomial under reordering the quantum cluster variables.

We call two quantum seeds $(\tilde{B}, X, \Lambda)$ and $(\tilde{B}', X', \Lambda')$ *mutation equivalent* if one can relate them by a sequence of mutations. This defines an equivalence relation on quantum seeds, denoted by $(\tilde{B}, X, \Lambda) \sim (\tilde{B}', X', \Lambda')$. The *quantum cluster algebra* $\mathcal{A}_q(\tilde{B}, X, \Lambda)$ associated to a given quantum seed $(\tilde{B}, X, \Lambda)$ is the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by the set

$$\chi(\tilde{B}, X, \Lambda) = \left\{X_i^{\pm 1} \mid i \in [n+1, m]\right\} \cup \bigcup_{(\tilde{B}', X', \Lambda') \sim (\tilde{B}, X, \Lambda)} \left\{X'_i \mid i \in [n]\right\}.$$

Specialising at $q = 1$ identifies the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, X, \Lambda)$ with the classical cluster algebra $\mathcal{A}(\tilde{B}, X)$. Generally, the definitions of classical and quantum cluster algebras admit additional analogies. One such analogy is the quantum Laurent phenomenon, as proven in [**2**, Cor. 5.2]: we have $\mathcal{A}_q(\tilde{B}, X, \Lambda) \subseteq \mathcal{T}_\Lambda$. Remarkably, $\mathcal{A}_q(\tilde{B}, X, \Lambda)$ and $\mathcal{A}(\tilde{B}, X)$ also possess the same exchange graph by [**2**, Theorem 6.1]. In particular, quantum cluster algebras of finite type are also classified by Dynkin diagrams.

3. The quantisation space.

3.1. Remarks on skew-symmetric matrices of full rank. The condition for an exchange matrix $\tilde{B}$ to be of full rank plays a crucial part in quantisations and gradings of cluster algebras: we show in this section that the cluster algebra $\mathcal{A}(\tilde{B}, X)$ associated to $\tilde{B}$ allows a quantisation if and only if $\tilde{B}$ has full rank. On the other hand, Grabowski–Launois [**17**] show in the case $n = m$ that $\mathcal{A}(\tilde{B}, X)$ possesses a non-trivial grading if and only if $\tilde{B}$ is of smaller rank.

How can we decide whether an exchange matrix $\tilde{B}$ has full rank? Let us consider the case $n = m$. Multiplication with a skew-symmetriser D does not change the rank, so without loss of generality, we may assume that $\tilde{B} = B$ is skew-symmetric. In this case, $B = B(Q)$ is the signed adjacency matrix of some quiver Q with n vertices.

First of all, if n is odd, then B cannot be of full rank, because $\det(B) = (-1)^n \det(B)$ implies $\det(B) = 0$. Especially, no (coefficient-free) cluster algebra attached to a quiver Q with an odd number of vertices admits a quantisation.

Now suppose that $n = m$ is even. In this case, a theorem of Cayley [**5**] asserts that there exists a polynomial $\mathrm{Pf}(B)$ in the entries of B such that $\det(B) = \mathrm{Pf}(B)^2$. The polynomial is called the *Pfaffian*. Hence, the cluster algebra $\mathcal{A}(\tilde{B}, X)$ admits a quantisation if and only if the Pfaffian does not vanish.

The formula for the Pfaffian in the entries of B is given by

$$\mathrm{Pf}(B) = \sum \mathrm{sgn}(i_1, \ldots, i_{n/2}, j_1, \ldots, j_{n/2}) b_{i_1 j_1} b_{i_2 j_2} \cdots b_{i_{n/2} j_{n/2}}, \tag{2}$$

where the sum is taken over all $(n-1)(n-3)\cdots 1$ possibilities of writing the set $\{1, 2, \ldots, n\}$ as a union $\{i_1, j_1\} \cup \{i_2, j_2\} \cup \ldots \cup \{i_{n/2}, j_{n/2}\}$ of $\frac{n}{2}$ sets of cardinality 2 and $\mathrm{sgn}(i_1, \ldots, i_{n/2}, j_1, \ldots, j_{n/2}) \in \{\pm 1\}$ is the sign of the permutation $\sigma \in S_n$ with $\sigma(2k-1) = i_k$ and $\sigma(2k) = j_k$ for all $k \in \{1, 2, \ldots, \frac{n}{2}\}$, cf. Knuth [**19**, Equation (0.1)]. It is easy to see that the sum is well-defined. For example, if $n = 4$, then $\mathrm{Pf}(B) = b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}$.

Definition 3.1. Let $G = \{V, E\}$ be an undirected graph with a set of vertices V and a set of edges $E \subseteq \binom{V}{2}$. A *perfect matching* of G is a collection of pairwise distinct edges $\{i, j\}$ with $i, j \in V$ such that every vertex of V is part of precisely one edge.

In this notation, a summand in the sum (2) vanishes unless the collection of pairs $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{n/2}, j_{n/2}\}$ is a perfect matching of the underlying undirected graph of Q.

In particular, if there exists no perfect matching of the underlying undirected graph of Q, then the Pfaffian $\mathrm{Pf}(B)$ vanishes, the determinant $\det(B)$ is zero, the matrix B does not have full rank and there exists no quantisation of $\mathcal{A}(\tilde{B}, X)$.

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For example, let $Q = \overrightarrow{A_n}$ be an orientation of a Dynkin diagram of type A_n with an even number n. Then Q admits exactly one perfect matching $\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}$. Hence, $\det(B(Q)) = \pm 1$ so that $B(Q)$ has full rank. The same is true for all quivers Q of type $\overrightarrow{E_6}$ or $\overrightarrow{E_8}$. On the other hand, there does not exist a perfect matching for a Dynkin diagram of type D_n. Hence, $\det(B(Q)) = 0$ for all quivers Q of type D_n.

To summarise, a (coefficient-free) cluster algebra of finite type has a quantisation if and only if it is of Dynkin type A_n with even n or of type E_6 or E_8.

REMARK 3.2. These are precisely the Dynkin diagrams for which the stable category $\underline{\mathrm{CM}}(R)$ of Cohen–Macaulay modules of the corresonding hypersurface singularity R of dimension 1 does not have an indecomposable rigid object, see Burban–Iyama–Keller–Reiten [**3**, Theorem 1.3].

3.2. Existence of quantisation. Suppose that $\mathrm{rank}(\tilde{B}) = n$. In this subsection, we prove that the cluster algebra $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ admits a quantisation.

The n column vectors of $\tilde{B}$ are linearly independent elements in $\mathbb{Q}^m$. We extend them to a basis of $\mathbb{Q}^m$ by adding $m-n$ appropriate column vectors. Hence, there is an invertible $m \times n$ plus $m \times (m-n)$ block matrix $\left[\tilde{B}\ \tilde{E}\right] \in \mathrm{GL}_m(\mathbb{Q})$ which we denote by M. We also write $\tilde{E}$ itself in block form as $\tilde{E} = \left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right]$ with an $n \times (m-n)$ matrix E and an $(m-n) \times (m-n)$ matrix F. Of course, the choice for the basis completion is not canonical. In particular, one can choose standard basis vectors for columns of $\tilde{E}$, making it sparse. After these preparations, we are ready to state the theorem about the existence of a quantisation.

THEOREM 3.3. *Let D be a skew-symmetriser of B. Then there exists a skew-symmetric $m \times m$-matrix Λ with integer entries and a multiple $D' = \lambda D$ with $\lambda \in \mathbb{Q}^+$ such that $\tilde{B}^T \Lambda = \left[D'\ 0\right]$.*

Proof. Put

$$\Lambda_0 = M^{-T} \begin{bmatrix} DB & DE \\ -E^T D & 0 \end{bmatrix} M^{-1} \in \mathrm{Mat}_{m\times m}(\mathbb{Q}),$$

and let Λ be an integer multiple of Λ_0 which lies in $\mathrm{Mat}_{m\times m}(\mathbb{Z})$. The matrix Λ is skew-symmetric by construction and the relation $M^T M^{-T} = I_{m,m}$ implies $\tilde{B}^T M^{-T} = \left[I_{n,n}\ 0_{n,m-n}\right]$. Thus,

$$\tilde{B}^T \Lambda_0 = \left[DB\ DE\right] M^{-1} = D\left[B\ E\right] M^{-1} = D\left[I_{n,n}\ 0_{n,m-n}\right] = \left[D\ 0\right].$$

Scaling the equation yields $\tilde{B}^T \Lambda = \left[D'\ 0\right]$ for some multiple D' of D. □

Together with Berenstein–Zelevinsky's initial result, this means that a cluster algebra $\mathcal{A}(\tilde{B})$ admits a quantisation if and only if $\tilde{B}$ has full rank. Since the rank of the exchange matrix is mutation invariant, one can use any seed to check whether a cluster algebra admits a quantisation.

Zelevinsky suggested in a private communication to reformulate the statement in terms of bilinear forms. With respect to the standard basis, the matrix Λ defines a skew-symmetric bilinear form.

Let us change the basis. The column vectors $b_1, b_2, \ldots, b_n$ of $\tilde{B}$ are linearly independent over $\mathbb{Q}$. Let $V' = \mathrm{span}_{\mathbb{Q}}(b_1, b_2, \ldots, b_n)$ be the column space of $\tilde{B}$. The column vectors $\tilde{e}_{n+1}, \tilde{e}_{n+2}, \ldots, \tilde{e}_m$ of $\tilde{E}$ extend to a basis of $V = \mathbb{Q}^m$. Let $V'' = \mathrm{span}_{\mathbb{Q}}(\tilde{e}_{n+1}, \tilde{e}_{n+2}, \ldots, \tilde{e}_m)$. The compatibility condition $\tilde{B}^T \Lambda = \begin{bmatrix} D' & 0 \end{bmatrix}$ says that for any given D', the skew-symmetric bilinear form $V \times V \to \mathbb{Q}$ is completely determined on $V' \times V$, hence also on $V \times V'$. Such a bilinear form can be chosen freely on $V'' \times V''$ giving a $\frac{1}{2}(m-n-1)(m-n)$-dimensional solution space. In particular, the quantisation is essentially unique (i.e. unique up to a scalar) when there are only 0 or 1 frozen vertices.

4. A minor generating set. In the previous section, we observed that any full-rank skew-symmetrisable matrix $\tilde{B}$ admits a quantisation. In the construction yielding Theorem 3.3, we chose some $m \times (m-n)$ integer matrix $\tilde{E}$ which completed a basis for $\mathbb{Q}^m$. This choice we now reformulate by giving a generating set of integer matrices for the equation

$$\tilde{B}^T \Lambda = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{3}$$

As previously remarked, this ambiguity does not occur for 0 or 1 frozen vertices, hence we may start with the case $m = n + 2$ in Section 4.1. From this result, we construct such a generating set for arbitrary m with $|m - n| > 2$ in the subsequent section. The construction below holds in more generality than what is naturally required in our setting. Thus, we now consider an arbitrary integer matrix A of dimension $m \times n$ instead of $\tilde{B}$ and obtain the generating set for equation (3) as a consequence.

4.1. Minor blocks. In this subsection, we assume $m = n + 2$.

For distinct $i, j \in [m]$, define a reduced indexing set $R(i, j)$ as the n-element subset of $[m]$ in which i and j do not occur. To an arbitrary $m \times n$ integer matrix $A = (a_{i,j})$, we associate the skew-symmetric $m \times m$ integer matrix $M = M(A) = \left(m_{i,j}\right)$ with entries

$$m_{i,j} = \begin{cases} (-1)^{i+j} \cdot \det(A_{R(i,j)}), & i < j, \\ 0, & i = j, \\ (-1)^{i+j+1} \cdot \det(A_{R(i,j)}), & j < i. \end{cases} \tag{4}$$

Then we first observe the following property of M, which carries some similarity to the well-known Plücker relations.

LEMMA 4.1. *For A, an $m \times n$ integer matrix, we obtain*

$$A^T \cdot M = [0\ 0].$$

Proof. By definition, we have

$$\left[A^T \cdot M\right]_{i,j} = \sum_{k=1}^{m} a_{k,i} m_{k,j} = \sum_{k \in [m] \setminus \{j\}} a_{k,i} m_{k,j}.$$

Now, let A_j be the matrix we obtain from A by removing the jth row and A_j^i the matrix that results from attaching the ith column of A_j to itself on the right. Then $\det(A_j^i) = 0$ and we observe that using the Laplace expansion along the last column, we obtain the right-hand side of the above equation up to sign change. The claim follows. □

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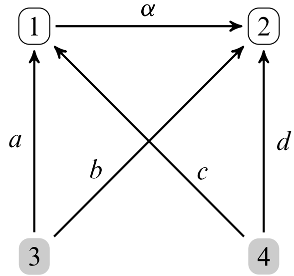
EXAMPLE 4.2. Let α, a, b, c and d be positive integers which we use below to indicate multiple arrows. Then consider the quiver Q given by



Thus, the matrices $\tilde{B}$ and M are

$$\tilde{B} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \\ a & b \\ c & d \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & -ad+bc & -\alpha d & \alpha b \\ ad-bc & 0 & \alpha c & -\alpha a \\ \alpha d & -\alpha c & 0 & -\alpha^2 \\ -\alpha b & \alpha a & \alpha^2 & 0 \end{bmatrix},$$

and we immediately see the result of the previous lemma, $\tilde{B}^T \cdot M = [0\ 0]$.

4.2. Composition of minor blocks. In this section, let $n+2 < m$ and as before let $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ be some rectangular integer matrix. Choose a subset $F \subset [m]$ of cardinality n and obtain a partition of the indexing set $[m]$ of the rows of A as $[m] = F \sqcup R$. Note that $|R| = m-n$. For distinct $i, j \in R$ set, the *extended indexing set associated to i, j* to be

$$E(i, j) := F \cup \{i, j\}.$$

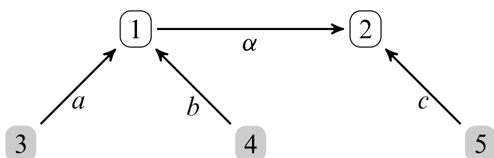
By Lemma 4.1 (after a reordering of rows) and slightly abusing the notation, there exists an $(n+2)\times(n+2)$ matrix $M_{E(i,j)} = (m_{r,s})$ such that

$$A^T_{E(i,j)} \cdot M_{E(i,j)} = [0\ 0]. \tag{5}$$

Now, let $\mathfrak{M}_{E(i,j)} = \mathfrak{M}_{E(i,j)}(A) = (\mathfrak{m}_{r,s})$ be the *enhanced solution matrix associated to i, j*, the $m\times m$ integer matrix we obtain from $M_{E(i,j)}$ by filling the entries labelled by $E(i, j)\times E(i, j)$ with $M_{E(i,j)}$ consecutively and setting all other entries to zero.

EXAMPLE 4.3. Consider the quiver Q with associated exchange matrix $\tilde{B}$ as shown below:

Q :



and $\tilde{B} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \\ a & 0 \\ b & 0 \\ 0 & c \end{bmatrix}$.

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We choose $F = \{1, 2\}$, assuming $\alpha \neq 0$ and get the following matrices $M_{E(i,j)}$ and their enhanced solution matrices for distinct $i, j \in \{3, 4, 5\}$:

$$M_{E(3,4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha b & -\alpha a \\ 0 & -\alpha b & 0 & -\alpha^2 \\ 0 & \alpha a & \alpha^2 & 0 \end{bmatrix}, \qquad \mathfrak{M}_{E(3,4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha b & -\alpha a & 0 \\ 0 & -\alpha b & 0 & -\alpha^2 & 0 \\ 0 & \alpha a & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{E(3,5)} = \begin{bmatrix} 0 & -ac & -\alpha c & 0 \\ ac & 0 & 0 & -\alpha a \\ \alpha c & 0 & 0 & -\alpha^2 \\ 0 & \alpha a & \alpha^2 & 0 \end{bmatrix}, \qquad \mathfrak{M}_{E(3,5)} = \begin{bmatrix} 0 & -ac & -\alpha c & 0 & 0 \\ ac & 0 & 0 & 0 & -\alpha a \\ \alpha c & 0 & 0 & 0 & -\alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha a & \alpha^2 & 0 & 0 \end{bmatrix},$$

$$M_{E(4,5)} = \begin{bmatrix} 0 & -bc & -\alpha c & 0 \\ bc & 0 & 0 & -\alpha b \\ \alpha c & 0 & 0 & -\alpha^2 \\ 0 & \alpha b & \alpha^2 & 0 \end{bmatrix}, \qquad \mathfrak{M}_{E(4,5)} = \begin{bmatrix} 0 & -bc & 0 & -\alpha c & 0 \\ bc & 0 & 0 & 0 & -\alpha b \\ 0 & 0 & 0 & 0 & 0 \\ \alpha c & 0 & 0 & 0 & -\alpha^2 \\ 0 & \alpha b & 0 & \alpha^2 & 0 \end{bmatrix}.$$

Here, we highlighted the added 0-rows/columns in gray. We observe by considering the lower right 3×3 matrices of $\mathfrak{M}_{E(3,4)}, \mathfrak{M}_{E(3,5)}, \mathfrak{M}_{E(4,5)}$ that these matrices are linearly independent. This we generalise in the theorem below.

THEOREM 4.4. *Let $A \in \mathbb{Z}^{m\times n}$ as above. Then for distinct $i, j \in R$, we have*

$$A^T \cdot \mathfrak{M}_{E(i,j)} = 0.$$

Furthermore, if A is of full rank and F is chosen such that the submatrix A_F yields the rank, then the matrices $\mathfrak{M}_{E(i,j)}$ are linearly independent.

Proof. By construction, for $s \in R\backslash\{i, j\}$ the sth column of $\mathfrak{M}_{E(i,j)}$ contains nothing but zeros. Hence, for arbitrary $r \in [m]$, we have

$$\left[A^T \cdot \mathfrak{M}_{E(i,j)}\right]_{r,s} = 0. \tag{6}$$

Now, let $s \in E(i, j)$. Then,

$$\sum_{k=1}^{m} a_{k,r}\mathfrak{m}_{k,s} = \sum_{k\in E(i,j)} a_{k,r}m_{k,s} = 0$$

by Lemma 4.1. Without loss of generality, assume $i < j$ and $F = [n]$. Then by assumption on the rank, $\beta := (-1)^{i+j} \det\left(A_{[n]}\right) \neq 0$ and by construction, $\mathfrak{M}_{E(i,j)}$ is of the form as in Figure 1. Then, $\pm\beta$ is the only entry of the submatrix of $\mathfrak{M}_{E(i,j)}$ indexed by $[n + 1, m] \times [n + 1, m]$. This immediately provides the linear independence. □

As an immediate consequence, we obtain that there are at least $\binom{m-n}{2}$ many $m \times m$ integer matrices M satisfying

$$A^T \cdot M = [0\ 0].$$

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$$
\begin{array}{c|c|ccccccc}
 & 1\cdots n & n+1 & \cdots & i & \cdots & j & \cdots & m \\
\hline
1 & & & & & & & & \\
\vdots & * & & & & * & & & \\
n & & & & & & & & \\
\hline
n+1 & & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \ddots & & & & & \vdots \\
i & & 0 & & 0 & & \beta & & 0 \\
\vdots & * & \vdots & & & \ddots & & & \vdots \\
j & & 0 & & -\beta & & 0 & & 0 \\
\vdots & & \vdots & & & & & \ddots & \vdots \\
m & & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{array}.
$$

Figure 1. An example of the form of enhanced solution matrices.

Together with the final remark from Section 3.2, we can thus conclude that the above constructed matrices form a basis for the space of solutions Λ of the homogeneous equation (3).

In addition, we can apply this result to gradings of cluster algebras as introduced in [**16**], since equation (3) is used by Grabowski to define gradings on cluster algebras, see [**16**, Definition 3.1]. In the above setting, any column of any enhanced solution matrix yields a grading of the corresponding cluster algebra.

5. Conclusion. When does a quantisation for a given cluster algebra $\mathcal{A}(\tilde{B})$ exists and how unique is it?

The answer we have seen above: it depends on the rank of $\tilde{B}$, the number of connected components r of the mutable part and the number of frozen indices. If the rank of $\tilde{B}$ is small, then no quantisation exists, see Section 3.1. On the other hand, if $\tilde{B}$ is of full rank, the solution space of matrices satisfying the compatibility equation (1) to a fixed skew-symmetriser is a vector space over the rational numbers of dimension $\binom{m-n}{2}$. Thus, if there is no or only one frozen index, a quantisation only depends on the choice of a skew-symmetriser. Recall that every skew-symmetriser is an $\mathbb{N}^+$-linear combination of the fundamental skew-symmetrisers of the connected components of B. We summarise our findings in the following corollary.

Corollary 5.1. *Let $\tilde{B}$ be an $m \times n$ exchange matrix of full rank and r the number of connected components of the mutable part . Then the solution space of matrices Λ satisfying the compatibility equation $\tilde{B}^T\Lambda = \begin{bmatrix} D' & 0 \end{bmatrix}$ to a given skew-symmetriser D' is a vector space over the rational numbers of dimension $\binom{m-n}{2}$.*

In particular, the set of all quantisations lies in a rational vector space of dimension $r + \binom{m-n}{2}$.

By Theorem 4.4, the vector space of matrix solutions in the first statement of the above corollary can be explicitly constructed:

Corollary 5.2.

(*a*) *All solutions of the compatibility equation $\tilde{B}^T\Lambda = \begin{bmatrix} D' & 0 \end{bmatrix}$ to a fixed skew-symmetriser D' can be constructed as the sum of a solution Λ_0 and a linear combination of all $\mathfrak{M}_{E(i,j)}$ for $i, j \in [m]$.*

(*b*) *In the special case where the principal part of $\tilde{B}$ is already invertible, quantisations of full subquivers with all mutable and two frozen vertices yield a basis of the homogeneous solution space.*

To construct quantum seeds, it is necessary to have integer solutions Λ for the compatibility equation (1). Both Λ from Theorem 3.3 and the enhanced solution matrices $\mathfrak{M}_{E(i,j)}$ have integer entries. General integer solutions Λ to (1) form a semi-group with respect to addition: if Λ_1 and Λ_2 are skew-symmetric $m \times m$ integer matrices satisfying

$$\tilde{B}^T \Lambda_1 = [D_1' \;\; 0] \quad \text{and} \quad \tilde{B}^T \Lambda_2 = [D_2' \;\; 0],$$

where D_1' and D_2' are integer matrices with positive diagonal entries, then $D_1' + D_2'$ has also positive diagonal entries and $\tilde{B}^T(\Lambda_1 + \Lambda_2) = [(D_1' + D_2') \;\; 0]$ holds. However, they do not generate the semi-group of all integer quantisations in general.

What came as a surprise to us is the simple structure of the matrices $\mathfrak{M}_{E(i,j)}$. Their computation only depends on so-called minor blocks which are matrices of size $(n+2) \times (n+2)$ depending on the entries of $\tilde{B}$, which can be realised with little effort. The authors used SAGE in their investigations of the problem and the first author makes a complementary website available at [**14**]. There, one can follow the construction of the matrices above in detail, compute a general solution to (1) and a generating set of matrices to (3).

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