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ATTACHED PRIMES OF THE TOP GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, let I be an ideal of R and let M and N be finitely generated R-modules. Assume that $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. First, we give the formula for the attached primes of the top generalized local cohomology module $H_I^{d+n}(M, N)$; later, we prove that if $\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Att}(H_J^{d+n}(M, N))$, then $H_I^{d+n}(M, N) = H_J^{d+n}(M, N)$.

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1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, let I be a proper ideal of R and let M and N be finitely generated R-modules. The generalized local cohomology module

$$H_I^i(M, N) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(M/I^n M, N)$$

was introduced by Herzog [10] and studied further by Yassemi, Suzuki and so on. There are several well-known properties concerning the generalized local cohomology modules. Assume that $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. It is well known that $H_I^i(M, N) = 0$, for all $i > d + n$.

Recall that for an R-module K, a prime ideal p of R is said to be an attached prime of K, if $p = \mathrm{Ann}(K/L)$ for some submodule L of K (see [11]). The set of attached primes of K is denoted by $\mathrm{Att}(K)$. If K is an Artinian R-module, then K admits a reduced secondary representation $K = K_1 + \cdots + K_r$ such that K_i is p_i-secondary, $i = 1, \ldots, r$, then $\mathrm{Att}(K) = \{p_1, \ldots, p_r\}$ is a finite set. Note that $\mathrm{Att}(K) = \emptyset$ if and only if $K = 0$. If $\dim M = n < \infty$, it is well known that $H_I^n(M)$ is an Artinian module. In [11], Macdonald and Sharp studied $H_{\mathfrak{m}}^n(M)$ and proved that

$$\mathrm{Att}(H_{\mathfrak{m}}^n(M)) = \{p \in \mathrm{Ass}(M) \mid \dim R/p = n\}$$

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(the right-hand set is denoted by $\mathrm{Assh}(M)$). Dibaei and Yassemi [6, Theorem A] generalized this result to $\mathrm{Att}(H_I^n(M)) = \{p \in \mathrm{Ass}(M) \mid \mathrm{cd}(I, R/p) = n\}$, where for any R-module K, $\mathrm{cd}(I, K) = \sup\{i \in \mathbb{Z} \mid H_I^i(K) \neq 0\}$.

The object of this paper is the attached primes of the top generalized local cohomology modules $H_I^{d+n}(M, N)$, and we give a formula for $\mathrm{Att}(H_I^{d+n}(M, N))$, which generalizes [6, Theorem A].

Let $E = E_R(R/\mathfrak{m})$, the injective hull of $R/\mathfrak{m}$. As in [14], we define a prime p to be a coassociated prime of L if p is an associated prime of $D(L)$, where $D(\cdot)$ is Matlis' dual functor $\mathrm{Hom}(\cdot, E)$.

It is well known that for any integer i, there is an exact sequence

$$H_{\mathfrak{m}}^i(M) \to H_I^i(M) \to \varinjlim_{t \in \mathbb{N}} \mathrm{Ext}_R^i(\mathfrak{m}^t/I^t, M). \tag{1}$$

By Hartshorne's result (see [8]), we know that the right-hand side module in (1) is zero, so there exists an exact sequence $H_{\mathfrak{m}}^n(M) \to H_I^n(M) \to 0$. So one can deduce that for two ideals $I \subseteq J$, there exists an exact sequence $H_J^n(M) \to H_I^n(M) \to 0$.

For any pair ideals I and J, if $\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Att}(H_J^{d+n}(M, N))$, then we prove that $H_I^{d+n}(M, N) = H_J^{d+n}(M, N)$, from which we can obtain the result of [7].

2. The formula for top generalized local cohomology modules

In this section, we give the formula for the attached primes of the top generalized local cohomology module $H_I^{d+n}(M, N)$, and when $(R, \mathfrak{m})$ is a complete ring with respect to $\mathfrak{m}$-adic topology, we give the formula for $\mathrm{Coass}(H_I^{d+n}(M, N))$.

The following lemma generalizes [13, Lemma 3.4].

LEMMA 2.1 [12, Lemma 2.8]. *Let* $\mathrm{pd}(M) = d < \infty$, *$L$ be an R-module and assume that $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ is an I-filter regular sequence on L. Then*

$$H_I^{i+n}(M, L) \cong H_I^i(M, H_{(x_1,\ldots,x_n)}^n(L)), \quad \forall i \geq d.$$

The next result is important for the main results of this paper.

PROPOSITION 2.2. *Assume that* $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. *Then:*

(i) $H_I^{d+n}(M, N) \cong \mathrm{Ext}_R^d(M, H_I^n(N))$; *in particular,* $H_I^{d+n}(M, N)$ *is Artinian and I-cofinite;*

(ii) $\mathrm{Att}(H_I^{d+n}(M, N)) \subseteq \mathrm{Att}(H_I^n(N))$.

PROOF. (i) Let $x_1, \ldots, x_n$ be an I-filter regular sequence on N. Then

$$H_I^{d+n}(M, N) \cong H_I^d(M, H_{(x_1,\ldots,x_n)}^n(N))$$

by Lemma 2.1. By [4, Exercise 7.1.7], $H_{(x_1,\ldots,x_n)}^n(N)$ is Artinian. So by [13, Lemma 3.4] $H_{(x_1,\ldots,x_n)}^n(N) \cong H_I^0(H_{(x_1,\ldots,x_n)}^n(N)) \cong H_I^n(N)$. Therefore,

$$H_I^{d+n}(M, N) \cong H_I^d(M, H_{(x_1,\ldots,x_n)}^n(N)) \cong H_I^d(M, H_I^n(N)) \cong \mathrm{Ext}_R^d(M, H_I^n(N)).$$

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(ii) Suppose that $p \in \operatorname{Att}(H_I^{d+n}(M, N))$, then

$$H_I^{d+n}(M, N)/pH_I^{d+n}(M, N) \neq 0.$$

By (i), we have that

$$H_I^{d+n}(M, N)/pH_I^{d+n}(M, N) \cong \operatorname{Ext}_R^d(M, H_I^n(N)) \otimes (R/p).$$

Since $\operatorname{Ext}_R^d(M, -)$ is a right exact additive functor,

$$\begin{aligned}\operatorname{Ext}_R^d(M, H_I^n(N)) \otimes (R/p) &\cong \operatorname{Ext}_R^d(M, R) \otimes H_I^n(N) \otimes (R/p) \\ &\cong \operatorname{Ext}_R^d(M, H_I^n(N)/pH_I^n(N)),\end{aligned}$$

thus $H_I^n(N)/pH_I^n(N) \neq 0$, hence $p \in \operatorname{Att}(H_I^n(N))$.

THEOREM 2.3. *Assume that* $\operatorname{pd}(M) = d < \infty$, $\dim N = n < \infty$. *Then*

$$\operatorname{Att}(H_I^{d+n}(M, N)) = \{p \in \operatorname{Ass}(N) \mid \operatorname{cd}(I, M, R/p) = d + n\},$$

where, for any R*-module* K*,*

$$\operatorname{cd}(I, M, K) = \sup\{i \in \mathbb{Z} \mid H_I^i(M, K) \neq 0\}.$$

PROOF. We use induction on n. If $n = 0$, then $\lambda(N) < \infty$. So

$$\operatorname{Att}(H_I^0(N)) = \operatorname{Att}(N) = \{\mathfrak{m}\} = \operatorname{Ass}(N) = \operatorname{Supp}(N),$$

thus, $\operatorname{Att}(H_I^d(M, N)) \subseteq \operatorname{Att}(H_I^0(N)) = \{\mathfrak{m}\} = \operatorname{Ass}(N)$ (where the containment follows from Proposition 2.2(ii)).

(1) If $H_I^d(M, N) = 0$, then $\operatorname{Att}(H_I^d(M, N)) = \emptyset$, $\operatorname{cd}(I, M, N) < d$, thus $\operatorname{cd}(I, M, R/\mathfrak{m}) < d$ by [1, Proposition 2].

(2) If $H_I^d(M, N) \neq 0$, then $\operatorname{Att}(H_I^d(M, N)) = \{\mathfrak{m}\} = \operatorname{Ass}(N)$ and $\operatorname{cd}(I, M, N) = d$, thus $\operatorname{cd}(I, M, R/\mathfrak{m}) = d$ by [1, Proposition 2]. So the result has been proved in this case.

Now let $n > 1$ and the case $n - 1$ is settled. If $H_I^{d+n}(M, N) = 0$, then $\operatorname{cd}(I, M, N) < d + n$, so $\{p \in \operatorname{Ass}(N) \mid \operatorname{cd}(I, M, R/p) = d + n\} = \emptyset$ by [1, Theorem B]. Now let $H_I^{d+n}(M, N) \neq 0$, let L be the largest submodule of N with $\operatorname{cd}(I, M, L) < d + n$. By the short exact sequence $0 \to L \to N \to N/L \to 0$ and [1, Theorem A], we have $\operatorname{cd}(I, M, N) = \operatorname{cd}(I, M, N/L)$. It is easy to prove that N/L has no nonzero submodule K with $\operatorname{cd}(I, M, K) < \operatorname{cd}(I, M, N)$, so

$$\operatorname{Ass}(N/L) \subseteq \{p \in \operatorname{Supp}(N/L) \mid \operatorname{cd}(I, M, R/p) = d + n\}.$$

In addition, if $p \in \operatorname{Supp}(N/L)$ and $\operatorname{cd}(I, M, R/p) = d + n$, then

$$\begin{aligned}d + n = \operatorname{cd}(I, M, R/p) &\leq d + \dim R/p \\ &\leq d + \dim N/L \leq d + \dim N = d + n.\end{aligned}$$

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Therefore, $p \in \min(\mathrm{Supp}(N/L)) \subseteq \mathrm{Ass}(N/L)$, and $p \in \min(\mathrm{Supp}(N)) \subseteq \mathrm{Ass}(N)$, therefore

$$\begin{aligned}\mathrm{Ass}(N/L) &= \{p \in \mathrm{Supp}(N/L) \mid \mathrm{cd}(I, M, R/p) = d + n\}\\ &\subseteq \{p \in \mathrm{Ass}(N) \mid \mathrm{cd}(I, M, R/p) = d + n\}.\end{aligned}$$

If $p \in \mathrm{Ass}(N)$, and $\mathrm{cd}(I, M, R/p) = d + n$, then $p \notin \mathrm{Supp}(L)$, otherwise, $\mathrm{cd}(I, M, R/p) \leq \mathrm{cd}(I, M, L) < d + n$ by [1, Theorem B]. So $p \in \mathrm{Supp}(N/L)$, hence

$$\begin{aligned}&\{p \in \mathrm{Supp}(N/L) \mid \mathrm{cd}(I, M, R/p) = d + n\}\\ &\quad = \{p \in \mathrm{Ass}(N) \mid \mathrm{cd}(I, M, R/p) = d + n\}.\end{aligned}$$

In the following exact sequence

$$H_I^{d+n}(M, L) \to H_I^{d+n}(M, N) \to H_I^{d+n}(M, N/L) \to H_I^{d+n+1}(M, L),$$

since $H_I^{d+n}(M, L) = H_I^{d+n+1}(M, L) = 0$, we have

$$H_I^{d+n}(M, N) \cong H_I^{d+n}(M, N/L).$$

Since $n = \dim N/L$,

$$\begin{aligned}&\{p \in \mathrm{Ass}(N/L) \mid \mathrm{cd}(I, M, R/p) = d + n\}\\ &\quad = \{p \in \mathrm{Ass}(N) \mid \mathrm{cd}(I, M, R/p) = d + n\},\end{aligned}$$

we can assume that $L = 0$, that is, N has no nonzero submodule L such that $\mathrm{cd}(I, M, L) < d + n$. Next we prove that $\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Ass}(N)$.

By Proposition 2.2(ii), we have that $\mathrm{Att}(H_I^{d+n}(M, N)) \subseteq \mathrm{Att}(H_I^{n}(N)) \subseteq \mathrm{Ass}(N)$.

On the other hand, if $p \in \mathrm{Ass}(N)$, then there is a p-primary submodule T of N such that $\mathrm{Ass}(N/T) = \{p\}$. We have $\mathrm{cd}(I, M, N/T) \geq \mathrm{cd}(I, M, R/p) = d + n$ by [1, Theorem B], then $\mathrm{cd}(I, M, N/T) = d + n$, $H_I^{d+n}(M, N/T) \neq 0$. By Proposition 2.2(ii) we obtain

$$\mathrm{Att}(H_I^{d+n}(M, N/T)) \subseteq \mathrm{Att}(H_I^{n}(N/T)) \subseteq \mathrm{Ass}(N/T) = \{p\},$$

so $\{p\} = \mathrm{Att}(H_I^{d+n}(M, N/T))$. Considering the exact sequence $H_I^{d+n}(M, N) \to H_I^{d+n}(M, N/T) \to 0$, then we have $\{p\} \subseteq \mathrm{Att}(H_I^{d+n}(M, N))$, hence $\mathrm{Ass}(N) \subseteq \mathrm{Att}(H_I^{d+n}(M, N))$. Now the proof is complete.

Remark 2.4. Assuming $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. If

$$p \in \mathrm{Att}(H_I^{d+n}(M, N)),$$

then

$$d + n = \mathrm{cd}(I, M, R/p) \leq d + \dim R/p \leq d + \dim N = d + n.$$

So $\dim R/p = n$, and hence $\mathrm{Att}(H_I^{d+n}(M, N)) \subseteq \mathrm{Assh}(N)$.

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In [6, Theorem B], Dibaei and Yassemi proved the following result. Let L be a nonzero module (not necessarily finite) such that $\dim R = \dim L = n < \infty$. Then $\mathrm{Att}(H_I^n(L)) \subseteq \{p \in \mathrm{Ass}(L) \mid \mathrm{cd}(I, R/p) = n\}$.

Assuming that L is finitely generated, we can obtain [6, Theorem A] by Theorem 2.3.

COROLLARY 2.5 [6, Theorem A]. *Assume that L is finitely generated,* $\dim L = n$. *Then* $\mathrm{Att}(H_I^n(L)) = \{p \in \mathrm{Ass}(L) \mid \mathrm{cd}(I, R/p) = n\}$.

We know that for a ring R with $\dim R > 0$, if $H_I^{\dim R}(R) \neq 0$, then it is not finitely generated (see [4, Exercise 8.2.6]). As an application of Theorem 2.3, we have the following proposition.

PROPOSITION 2.6. *Assume that* $\mathrm{pd}(M) = d < \infty$, $0 < \dim N = n < \infty$. *If*

$$H_{\mathfrak{m}}^{d+n}(M, N) \neq 0,$$

then it is not finitely generated.

PROOF. As $H_{\mathfrak{m}}^{d+n}(M, N) \neq 0$, so $\mathrm{Att}(H_{\mathfrak{m}}^{d+n}(M, N)) \neq \emptyset$. We have

$$\mathrm{Att}(H_{\mathfrak{m}}^{d+n}(M, N)) \subseteq \mathrm{Att}(H_{\mathfrak{m}}^{n}(N)) \subseteq \{p \in \mathrm{Ass}(N) \mid \dim R/p = n\}$$

by Proposition 2.2(ii). Since $n > 0$, then $\mathrm{Att}(H_{\mathfrak{m}}^{d+n}(M, N)) \nsubseteq \{\mathfrak{m}\}$. Since $H_{\mathfrak{m}}^{d+n}(M, N)$ is Artinian, it follows that $H_{\mathfrak{m}}^{d+n}(M, N)$ is not finitely generated by [4, Corollary 7.2.12].

In [5, Lemma 3], Delfino and Marley showed that, if $(R, \mathfrak{m})$ is a complete Noetherian local ring, I an ideal of R, M a finitely generated R-module of dimension d, then $\mathrm{Coass}(H_I^d(M)) = \{p \in V(\mathrm{Ann}(M)) \mid \dim R/p = d, \sqrt{I + p} = \mathfrak{m}\}$.

PROPOSITION 2.7. *Let $(R, \mathfrak{m})$ be a complete ring with respect to the $\mathfrak{m}$-adic topology, assume that* $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. *Then*

$$\mathrm{Coass}(H_I^{d+n}(M, N)) \subseteq \{p \in V(\mathrm{Ann}(N)) \mid \dim R/p = n, \sqrt{I + p} = \mathfrak{m}\}.$$

PROOF. Since $H_I^{d+n}(M, N)$ is Artinian, then

$$\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Ass}(D(H_I^{d+n}(M, N)) = \mathrm{Coass}(H_I^{d+n}(M, N)).$$

In addition, when $(R, \mathfrak{m})$ is a complete ring with respect to the $\mathfrak{m}$-adic topology, we can prove that if $p \in \mathrm{Ass}(N)$ with $\mathrm{cd}(I, M, R/p) = d + n$, then $p \in V(\mathrm{Ann}(N))$ with $\dim R/p = n$, and $\sqrt{I + p} = \mathfrak{m}$ by [4, Theorem 8.2.1]. So

$$\mathrm{Coass}(H_I^{d+n}(M, N)) \subseteq \{p \in V(\mathrm{Ann}(N)) \mid \dim R/p = n, \sqrt{I + p} = \mathfrak{m}\}$$

by Theorem 2.3.

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PROPOSITION 2.8. *Assume that* $\mathrm{pd}(M) = d < \infty$, $1 < \dim N = n < \infty$. *Then* $H_I^{d+n-1}(M, N)/I^i H_I^{d+n-1}(M, N)$ *of finite length for any* $i \in \mathbb{N}$.

PROOF. We have that $H_I^{d+n}(M, N) \cong H_{I^i}^{d+n}(M, N)$ for all $i \in \mathbb{N}$, so it is enough to prove the result for $i = 1$. By Proposition 2.2(i), $H_I^{d+n}(M, N)$ is Artinian and I-cofinite. In addition, we know that $H_I^i(M, N) = 0$ for all $i > d + n$. By the same proof as in [2, Theorem 3.3], we get that $H_I^{d+n-1}(M, N)/I H_I^{d+n-1}(M, N)$ is Artinian and I-cofinite, so

$$H_I^{d+n-1}(M, N)/I H_I^{d+n-1}(M, N) \cong \mathrm{Hom}(R/I, H_I^{d+n-1}(M, N)/I H_I^{d+n-1}(M, N))$$

is finitely generated, thus $H_I^{d+n-1}(M, N)/I H_I^{d+n-1}(M, N)$ has finite length.

EXAMPLE 2.9 [2, Example 3.6]. In Proposition 2.8, if $t < \mathrm{pd}(M) + \dim N - 1$, then it can be seen that $H_I^t(M, N)/I H_I^t(M, N)$ is not necessarily of finite length. To see this, let $R = k[[X_1, \ldots, X_4]]$, $I_1 = (X_1, X_2)$, $I_2 = (X_3, X_4)$ and $I = I_1 \cap I_2$, $M = N = R$, where k is a field. Then $H_I^i(M, N) = H_I^i(R)$ for all $i \geq 0$. By the Mayer–Vietoris exact sequence we obtain that $H_I^2(R) = H_{I_1}^2(R) \oplus H_{I_2}^2(R)$. Now consider the following isomorphisms

$$\begin{aligned} H_I^2(R)/I H_I^2(R) &\cong (H_{I_1}^2(R)/I H_{I_1}^2(R)) \oplus (H_{I_2}^2(R)/I H_{I_2}^2(R)) \\ &= H_{I_1}^2(R/I) \oplus H_{I_2}^2(R/I). \end{aligned}$$

By the Hartshorne–Lichtenbaum vanishing theorem, $H_{I_1}^2(R/I) \neq 0$. Therefore, $\mathrm{cd}(I_1, R/I) = 2$, and so by [9, Remark 2.5], $H_{I_1}^2(R/I)$ is not finitely generated. Consequently, $H_I^2(R)/I H_I^2(R)$ is not finitely generated.

3. Top generalized local cohomology modules

In [7, Theorem 1.6], Dibaei and Yssemi show that for any pair of ideals I and J, $\dim N = n$, if $\mathrm{Att}(H_I^n(N)) = \mathrm{Att}(H_J^n(N))$, then $H_I^n(N) = H_J^n(N)$. In this section, we show that, if $\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Att}(H_J^{d+n}(M, N))$, then $H_I^{d+n}(M, N) = H_J^{d+n}(M, N)$.

LEMMA 3.1. *Assume that* $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$, $H_I^{d+n}(M, N) \neq 0$. *Then there exists a homomorphic image* G *of* N *such that:*

(1) $\dim G = n$;
(2) *G has no nonzero submodule of dimension less than n;*
(3) $\mathrm{Ass}(G) = \{p \in \mathrm{Ass}(N) \mid \mathrm{cd}(I, M, R/p) = d + n\}$;
(4) $H_I^{d+n}(M, G) \cong H_I^{d+n}(M, N)$;
(5) $\mathrm{Ass}(G) = \mathrm{Att}(H_I^{d+n}(M, G))$.

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PROOF. By Remark 2.4, $\operatorname{Att}(H_I^{d+n}(M, N)) \subseteq \operatorname{Assh}(N)$. Therefore, there is a submodule L of N such that $\operatorname{Ass}(L) = \operatorname{Ass}(N) \setminus \operatorname{Att}(H_I^{d+n}(M, N))$ and $\operatorname{Ass}(N/L) = \operatorname{Att}(H_I^{d+n}(M, N))$ by [3, p. 263, Proposition 4]. Considering the exact sequence $H_I^{d+n}(M, L) \to H_I^{d+n}(M, N) \to H_I^{d+n}(M, N/L) \to 0$, we claim that $H_I^{d+n}(M, L) = 0$. Otherwise, there is $p \in \operatorname{Att}(H_I^{d+n}(M, L))$ such that $\operatorname{cd}(I, M, R/p) = d + n$. Since $p \in \operatorname{Ass}(L)$ by Theorem 2.3, then $p \in \operatorname{Ass}(N)$, and hence $p \in \operatorname{Att}(H_I^{d+n}(M, N))$ by Theorem 2.3, which is a contradiction. Thus, $H_I^{d+n}(M, N) = H_I^{d+n}(M, N/L)$. Set $G = N/L$. Then (1), (3), (4), (5) are clear. If G has a nonzero submodule K with $\dim K < n$, then $\dim R/p < n$ for some $p \in \operatorname{Ass}(N/L)$, which is contradiction by Remark 2.4. □

PROPOSITION 3.2. *Assume that* $(R, \mathfrak{m})$ *is a complete Noetherian local ring,* $\operatorname{pd}(M) = d < \infty$, $\dim N = n < \infty$. *Then* $\operatorname{Att}(H_I^{d+n}(M, N)) \subseteq \operatorname{Att}(H_J^{d+n}(M, N))$ *if and only if* $H_J^{d+n}(M, N) \to H_I^{d+n}(M, N) \to 0$ *is an exact sequence.*

PROOF. The sufficient part is clear. For the necessary part, there exists a submodule L of N with $\operatorname{Ass}(L) = \operatorname{Ass}(N) \setminus \operatorname{Att}(H_J^{d+n}(M, N))$ and $\operatorname{Ass}(N/L) = \operatorname{Att}(H_J^{d+n}(M, N))$ by [3, p. 263, Proposition 4]. We see that $H_J^{d+n}(M, L) = 0$ by the proof of Lemma 3.1. Hence, we have $H_J^{d+n}(M, N) = H_J^{d+n}(M, N/L)$. Note that for any $p \in \operatorname{Ass}(N/L)$ with $\operatorname{cd}(J, M, R/p) = d + n$, then $p \in \operatorname{Ass}(N)$, $H_J^n(R/p) \neq 0$, so $J + p$ is $\mathfrak{m}$-primary by [4, Theorem 8.2.1]. This induces $J + \operatorname{Ann}(N/L)$ is $\mathfrak{m}$-primary and, hence,

$$H_J^{d+n}(M, N/L) \cong H_{J+\operatorname{Ann}(N/L)}^{d+n}(M, N/L) = H_{\mathfrak{m}}^{d+n}(M, N/L).$$

On the other hand, considering the exact sequence

$$H_I^{d+n}(M, L) \to H_I^{d+n}(M, N) \to H_I^{d+n}(M, N/L) \to 0,$$

if $H_I^{d+n}(M, L) \neq 0$, then there exists $p \in \operatorname{Att}(H_I^{d+n}(M, L))$. Then we have $p \in \operatorname{Ass}(L)$ and $\operatorname{cd}(I, M, R/p) = d + n$ by Theorem 2.3. As $p \in \operatorname{Ass}(N)$, then $p \in \operatorname{Att}(H_I^{d+n}(M, N))$, thus $p \in \operatorname{Att}(H_J^{d+n}(M, N))$, which contradicts with $p \in \operatorname{Ass}(L)$. Therefore, $H_I^{d+n}(M, L) = 0$ and, hence, $H_I^{d+n}(M, N) \cong H_I^{d+n}(M, N/L)$. Since there exists an exact sequence

$$H_{\mathfrak{m}}^n(N/L) \to H_I^n(N/L) \to 0, \quad H_I^{d+n}(M, N/L) \cong \operatorname{Ext}_R^d(M, H_I^n(N/L))$$

and

$$H_{\mathfrak{m}}^{d+n}(M, N/L) \cong \operatorname{Ext}_R^d(M, H_{\mathfrak{m}}^n(N/L))$$

by Proposition 2.2(i), then $H_J^{d+n}(M, N) \to H_I^{d+n}(M, N) \to 0$ is an exact sequence. □

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THEOREM 3.3. *Assume that* $\mathrm{pd}(M) = d < \infty$*,* $\dim N = n < \infty$*. If*

$$\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Att}(H_J^{d+n}(M, N)),$$

then $H_I^{d+n}(M, N) = H_J^{d+n}(M, N)$.

PROOF. Since $H_I^{d+n}(M, N)$ and $H_J^{d+n}(M, N)$ are Artinian, then

$$H_{I\widehat{R}}^{d+n}(\widehat{M}, \widehat{N}) \cong H_I^{d+n}(M, N), \quad H_{J\widehat{R}}^{d+n}(\widehat{M}, \widehat{N}) \cong H_J^{d+n}(M, N),$$

so we can assume that R is complete. We take L to be a submodule of N such that $\mathrm{Ass}(N/L) = \mathrm{Att}(H_I^{d+n}(M, N))$, $\mathrm{Ass}(L) = \mathrm{Ass}(N) \setminus \mathrm{Att}(H_I^{d+n}(M, N))$. By the following two exact sequences

$$H_I^{d+n}(M, L) \to H_I^{d+n}(M, N) \to H_I^{d+n}(M, N/L) \to 0,$$
$$H_J^{d+n}(M, L) \to H_J^{d+n}(M, N) \to H_J^{d+n}(M, N/L) \to 0.$$

As in the proof of Proposition 3.2, we obtain

$$H_I^{d+n}(M, N) = H_{\mathfrak{m}}^{d+n}(M, N/L) = H_J^{d+n}(M, N).$$

In general, there exists an epimorphism $H_{\mathfrak{m}}^n(M) \to H_I^n(M)$, where $\dim M = n$. Next, in a particular case, we obtain that $H_{\mathfrak{m}}^{d+n}(M, N) \cong H_I^{d+n}(M, N)$, where $\mathrm{pd}(M) = d < \infty$, $\dim N = n < \infty$. □

PROPOSITION 3.4. *Assume that* $\mathrm{pd}(M) = d < \infty$*,* $\dim N = n < \infty$ *such that* $\mathrm{Ass}(N) = \mathrm{Att}(H_I^{d+n}(M, N))$*. Then* $H_I^{d+n}(M, N) = H_{\mathfrak{m}}^{d+n}(M, N)$.

PROOF. From

$$\mathrm{Ass}(N) = \mathrm{Att}(H_I^{d+n}(M, N)) \subseteq \mathrm{Assh}(N) \subseteq \mathrm{Ass}(N),$$

we have

$$\mathrm{Att}(H_I^{d+n}(M, N)) = \mathrm{Assh}(N) = \mathrm{Att}(H_{\mathfrak{m}}^n(N)) \supseteq \mathrm{Att}(H_{\mathfrak{m}}^{d+n}(M, N))$$

by Theorem 2.3. On the other hand, since there exists an exact sequence $H_{\mathfrak{m}}^n(N) \to H_I^n(N) \to 0$, then $\mathrm{Ext}_R^d(M, H_{\mathfrak{m}}^n(N)) \to \mathrm{Ext}_R^d(M, H_I^n(N)) \to 0$ is an exact sequence, hence $H_{\mathfrak{m}}^{d+n}(M, N) \to H_I^{d+n}(M, N) \to 0$ is an exact sequence by Proposition 2.2(i). Therefore, $\mathrm{Att}(H_{\mathfrak{m}}^{d+n}(M, N)) \supseteq \mathrm{Att}(H_I^{d+n}(M, N))$, so $H_I^{d+n}(M, N) = H_{\mathfrak{m}}^{d+n}(M, N)$ by Theorem 3.3. □

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