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A 3-DIMENSIONAL NON-ABELIAN COHOMOLOGY OF GROUPS WITH APPLICATIONS TO HOMOTOPY CLASSIFICATION OF CONTINUOUS MAPS

MANUEL BULLEJOS AND ANTONIO M. CEGARRA

Introduction. The general problem of what should be a non-abelian cohomology, what is it supposed to do, and what should be the coefficients, form a set of interesting questions which has been around for a long time. In the particular setting of groups, a comprehensible and well motivated cohomology theory has been so far stated in dimensions ≤ 2, the coefficients for $\mathbb{H}^2$ being crossed modules. The main effort to define an appropriate $\mathbb{H}^3$ for groups has been done by Dedecker [16] and Van Deuren [40]; they studied the obstruction to lifting non-abelian 2-cocycles and concluded with first approach for $\mathbb{H}^3$, which requires "super crossed groups" as coefficients. However, as Dedecker said "some polishing work remains necessary" for his cohomology. In fact, it is a very complicated theory and it is not clear that his cohomology has the properties one hopes a non-abelian cohomology should have; even more, it is not clear how this cohomology solves the obstruction problems (by extending Dedecker's 6-term exact sequence associated to a short exact sequence of crossed modules to a 9-term sequence). At first sight, the coefficients for Dedecker's 3-cohomology are not very much related to other gadgets coming from Geometry, Topology or Algebra; although the system defining a super crossed group is close to that defining a crossed square (in the sense of [23]), neither of these two concepts include the other one.

In this paper we introduce a new 3-dimensional cohomology for groups, in terms of systems much easier and comprehensible. The coefficients we use are reduced *2-crossed modules* in the sense of Conduché [13] (or equivalently crossed modules with a "braiding" structure [6]) and they came into this theory through a problem on homotopy types. The fact that reduced 2-crossed modules are algebraic models for simply-connected 3-types and that our 3-cocycles correspond to pointed continuous maps from aspherical CW-complexes to simply-connected spaces with trivial homotopy groups at dimensions ≥ 4 make possible a classification theorem using our $\mathbb{H}^3$ (Propositions 4.3 and 4.4), which is one of the expected properties a non-abelian cohomology should have. On the other hand, by using our $\mathbb{H}^3$ we will be able to give a natural measure of the deviation from right exactness of the functor $\mathbb{H}^2$, by extending Dedecker's 6-term exact sequence to a 9-term one. So $\mathbb{H}^3$ certainly solves the obstruction problems to lifting 3-cocycles. Two other expected properties that our $\mathbb{H}^3$ has make us think that this is a right choice

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for a 3-dimensional cohomology: firstly that $\mathbb{H}^3$ is a *non-abelian monadic cohomology* (i.e. $\mathbb{H}^3$ can be calculated by using free simplicial resolution—Proposition 1.5—) and secondly that $\mathbb{H}^3$ classifies certain non-abelian 2-fold extensions (Proposition 3.1).

Finally note that any presentation of a non-abelian cohomology at higher dimension is tedious and involves complicated formulas to describe cocycles and cohomologies. There are several motivated ways to present these formulas at dimension 3. Among them, we have chosen that which goes through simplicial group theory and so we conclude with a description of $\mathbb{H}^3(G, \mathcal{E})$, which comes from a *non-homogeneous* parametrization of the set of loop homotopy classes of simplicial group morphisms, from a free simplicial resolution of the group G to the canonical simplicial group associated (according to Conduché [13]) to the reduced 2-crossed module $\mathcal{E}$. To obtain this parametrization is the main objective of Section 1; moreover, this way allows us to give quick arguments (based on homotopy theory) to obtain several results in the following sections. Section 2 is mainly devoted to study long exact sequences involving $\mathbb{H}^i, i \leq 3$. In Section 3 we describe a group theoretic interpretation of the cohomology sets $\mathbb{H}^3$ and in Section 4 we give a topological interpretation of these sets. Finally Section 5 is devoted to compare our 3-cohomology with others already stated.

In this paper we use additive notation for groups.

1. **3-Dimensional non-abelian functors $\mathbb{H}^3$.** In this section we are going to introduce a three dimensional non-abelian cohomology theory for groups by using appropriate notions of *cohomology classes* of *non-abelian 3-cocycles*. These notions will be disentangled from the simplicial group theory by stating a homotopy representability theorem analogous to those established for both Eilenberg-Mac Lane cohomology as well as Dedecker non-abelian cohomology.

In the abelian case, since Eilenberg-Mac Lane cohomology is a cotriple cohomology [1], Duskin's homotopy representability theorem [18] gives, for any group G and any abelian group A, natural isomorphisms

$$H^{n+1}(G,A) \cong [F_., K(A,n)] \tag{1}$$

where $[F_., K(A,n)]$ is the set of loop homotopy classes of simplicial group morphisms from a free resolution $F_.$ of G to the Eilenberg-Mac Lane complex $K(A,n)$. Recall that any two free simplicial resolutions of G are loop homotopically equivalent [29].

There is a similar theorem in the non-abelian case for Dedecker's cohomology of a group G with coefficients in a crossed module $\mathcal{E}$, $\mathbb{H}^2(G, \mathcal{E})$ [16]. This theorem uses the equivalence between the category of crossed modules and the full subcategory of simplicial groups whose objects have trivial Moore complex at dimensions ≥ 2, see [9] or [32], in such a way that if we write $\mathcal{J}(\mathcal{E})$ for the simplicial group associated (by this equivalence) to the crossed module $\mathcal{E}$, there is a natural bijection

$$\mathbb{H}^2(G, \mathcal{E}) \cong [F_., \mathcal{J}(\mathcal{E})], \tag{2}$$

see [19] or [12]. This bijection is a generalization of (1) for the case $n = 1$, since any abelian group A can be considered as a crossed module $\big(A = (A \longrightarrow 0)\big)$ and $\mathcal{J}(A) \cong K(A,1)$.

Let us now note that the simplicial group $K(A,2)$ is strongly related to $\mathcal{J}(A)$, in the sense that there are natural isomorphisms of simplicial groups

$$K(A,2) \cong \bar{W}\mathcal{J}(A) \text{ and } \Omega K(A,2) \cong \mathcal{J}(A),$$

where $\bar{W}$ is the Eilenberg-Mac Lane classifying functor and Ω is the loop complex functor (see for example [14] for a description of these functors). Then the isomorphism (1), for $n = 2$, can be expressed as

$$H^3(G,A) \cong [F_\cdot, \bar{W}\mathcal{J}(A)].$$

Given then a crossed module $\mathcal{E}$, if $\bar{W}\mathcal{J}(\mathcal{E})$ were canonically a simplicial group such that $\Omega\bar{W}\mathcal{J}(\mathcal{E}) = \mathcal{J}(\mathcal{E})$—as simplicial group—it would be natural to think that an adequate notion of $\mathbb{H}^3(G, \mathcal{E})$ will arise from an appropriate parametrization, in a *non-homogeneous* sense, of the set $[F_\cdot, \bar{W}\mathcal{J}(\mathcal{E})]$ in terms of cohomology classes of 3-cocycles.

Unfortunately, in general there is not a canonical group structure on the simplicial set $\bar{W}\mathcal{J}(\mathcal{E})$ which makes $\Omega\bar{W}\mathcal{J}(\mathcal{E})$ isomorphic to $\mathcal{J}(\mathcal{E})$ as simplicial groups; in fact, $\bar{W}\mathcal{J}(\mathcal{E})$ may not admit any (canonical or not) group structure in such conditions (for example take $\mathcal{E}$ a crossed module such that $\Pi_0\big(\mathcal{J}(\mathcal{E})\big)$ is non-abelian). Let us next analyze this question in detail (see Proposition 1.1 for a conclusion).

First recall that a crossed module $\mathcal{E}$ is a group homomorphism $\rho: L \longrightarrow M$ together with a group action of M on L, satisfying:

XM1. $\rho({}^m x) = m + \rho(x) - m$,

XM2. ${}^{\rho(x)}y = x + y - x$,

for all $x, y \in L$ and $m \in M$. A crossed module morphism $\phi: \mathcal{E} \longrightarrow \mathcal{E}'$ is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} L & \xrightarrow{\rho} & M \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ L' & \xrightarrow{\rho'} & M' \end{array}$$

which satisfies

$$\phi_1({}^m x) = {}^{\phi_0(m)}\,\phi_1(x), \text{ for all } x \in L \text{ and } m \in M$$

(i.e. ϕ_1 is a morphism of M-groups, where L' is an M-group via ϕ_0). The corresponding category of crossed modules will be denoted by $\mathcal{XM}$.

A simplicial group $G_\cdot$ will be represented by a diagram

$$G_\cdot = \cdots G_n \;\overset{d_n}{\underset{d_0}{\vdots}}\; G_{n-1} \cdots G_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} G_0,$$

with degeneracies $s_0, \ldots, s_{n-1}: G_{n-1} \to G_n$ and $s_0: G_0 \to G_1$.

in which the face and degeneracy morphisms (d_i and s_i respectively) satisfy the usual simplicial identities. Frequently we will not draw the degeneracy operators in the picture of a simplicial group. The Moore complex, of a simplicial group $G.$, is the group complex

$$\tilde{G}. = \cdots \tilde{G}_n \xrightarrow{\delta_n} \tilde{G}_{n-1} \ldots \tilde{G}_1 \xrightarrow{\delta_1} \tilde{G}_0$$

where $\tilde{G}_0 = G_0$, $\tilde{G}_n = \bigcap_{0\leq i\leq n-1} \mathrm{Ker}(d_i) \subseteq G_n$ and δ_n is the restriction of d_n for all $n \geq 1$.

Simpl Gp will denote the category of simplicial groups and group simplicial morphisms, and Simpl $Gp(G., H.)$ (or just $(G., H.)$) will be the set of group simplicial morphisms from a simplicial group $G.$ to another $H.$.

Let Simpl_1 Gp be the full subcategory of Simpl Gp whose objects are those simplicial groups $G.$ whose Moore complexes are of length one, i.e., $\tilde{G}_n = 0, n \geq 2$. Then, as we said before, the category Simpl_1 Gp is equivalent to the category of crossed modules, $\mathcal{XM}$. An explicit description of this equivalence is given by the functors

$$\mathrm{Simpl}_1\, Gp \underset{\mathcal{N}}{\overset{\mathcal{J}}{\leftrightarrows}} \mathcal{XM},$$

where $\mathcal{N}(G.)$ is the crossed module given by the group homomorphism $\delta_1 : \tilde{G}_1 \longrightarrow \tilde{G}_0$, together with the action of $\tilde{G}_0$ on $\tilde{G}_1$:

$${}^m x = s_0(m) + x - s_0(m).$$

Then $\mathcal{J}$ takes a crossed module $\mathcal{E} = (L \xrightarrow{\rho} M)$ to the simplicial group

$$\mathcal{J}(\mathcal{E}) = \mathrm{cosk}^2(\Lambda_2 \rightrightarrows L \rtimes M \rightrightarrows M)$$

(see [18] for a general definition of the functor cosk in categories of simplicial objects over a category with finite limits), where $L \rtimes M$ is the semidirect product group, Λ_2 is the subgroup of the direct product $(L \rtimes M)^2$ whose elements are of the form $\chi = \big((x,m),(x',m)\big)$, and the face and degeneracy operators are:

$$d_0(x,m) = m, \quad d_1(x,m) = \rho(x) + m, \quad s_0(m) = (0,m),$$

for all $x \in$L and $m \in M$, and

$$d_0(\chi) = (x,m), \quad d_1(\chi) = (x',m), \quad d_2(\chi) = (x' - x, \rho(x) + m)$$
$$s_0(x,m) = \big((x,m),(x,m)\big), \quad s_1(x,m) = \big((0,m),(x,m)\big),$$

for all $\chi = \big((x,m),(x',m)\big) \in \Lambda_2$ and $(x,m) \in L \rtimes M$.

In [13], Conduché shows that the full subcategory of Simpl Gp whose objects are those simplicial groups with trivial Moore complexes at dimensions ≥ 3 is equivalent to the category of 2-crossed modules, where a 2-crossed module is a truncated group complex $L \longrightarrow M \longrightarrow N$ together with certain operators. He also proves that this equivalence gives, by restriction, an equivalence between the full subcategory of Simpl Gp whose object have trivial Moore complexes at dimensions other than one and two, denoted here by $\mathrm{Simpl}_2\, Gp$, and the category of 2-crossed modules with trivial righthand side groups (i.e. $N = 0$). Such 2-crossed modules are called here reduced. Conduché observes that a reduced 2-crossed module is just a crossed module $\mathcal{E} = (L \xrightarrow{\rho} M)$ together with a map $\{-,-\}: M \times M \longrightarrow L$ satisfying the identities:

R2XM1) $\rho\{m, m'\} = m + m' - m - m'$,
R2XM2) $\{\rho(x), m\} = x - {}^{m}x$,
R2XM3) $\{m, \rho(x)\} = {}^{m}x - x$,
R2XM4) $\{m, m' + m''\} = \{m, m'\} + {}^{m'}\{m, m''\}$,
R2XM5) $\{m + m', m''\} = {}^{m}\{m', m''\} + \{m, m''\}$.

Elementary examples of reduced 2-crossed modules are:

-) Any group epimorphism $\rho: L \twoheadrightarrow M$, with central kernel, by taking $\{\rho(\ell), \rho(\ell')\} = [\ell, \ell']$.
-) The cannonical morphism $\rho: G \otimes G \longrightarrow G$, $\rho(g \otimes g') = [g, g']$, from the non-abelian tensor product of a group G with itself, with

$$\{g, g'\} = g \otimes g'.$$

-) The zero morphism $0: L \longrightarrow M$ between any two abelian groups, together with any biadditive map $M \times M \longrightarrow L$.

A morphism of reduced 2-crossed modules is a morphism of crossed modules which is compatible with the corresponding maps $\{-,-\}$. The corresponding category of reduced 2-crossed modules is denoted here by $2 - \mathcal{XM}_{\mathrm{red}}$.

An equivalence between $\mathrm{Simpl}_2\, Gp$ and the category of reduced 2-crossed modules is given by the functor which takes a simplicial group $G. \in \mathrm{Simpl}_2\, Gp$ to the reduced 2-crossed module

$$\mathcal{N}_2(G.) = \tilde{G}_2 \xrightarrow{\delta_2} \tilde{G}_1$$

with action of $\tilde{G}_1$ on $\tilde{G}_2$

$${}^{m}\ell = s_1(m) + \ell - s_1(m),$$

and map $\{-,-\}: \tilde{G}_1 \times \tilde{G}_1 \longrightarrow \tilde{G}_2$ given by

$$\{m, m'\} = s_1(m + m' - m) + s_0(m) - s_1(m') - s_0(m).$$

A quasi-inverse for $\mathcal{N}_2$ takes a reduced 2-crossed module $\mathcal{E} = (L \xrightarrow{\rho} M)$ to the simplicial group

$$\mathcal{J}_2(\mathcal{E}) = \mathrm{cosk}^3(\Lambda_3 \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} (L \rtimes M) \rtimes M \overset{\rightarrow}{\rightrightarrows} M \rightrightarrows 0,);$$

where $L \rtimes M$ is the semidirect product group, according to the given action of M on L, $(L \rtimes M) \rtimes M$ is the semidirect product group with respect to the action of M on $L \rtimes M$:

$${}^{m}(\ell, m') = (\ell - \{m, m'\}, m + m' - m),$$

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and Λ_3 is the subgroup of the direct product $\big((L \rtimes M) \rtimes M\big)$ whose elements are of the form

$$\chi = \big((\ell, m, m'), (\ell', m'', m'), (\ell'', m'' - m, m + m')\big).$$

The face and degeneracy operators are:

$$\begin{aligned}
d_0(\ell, m, m') &= m',\ d_1(\ell, m, m') = m + m',\ d_2(\ell, m, m') = \rho(\ell) + m,\\
s_0(m) &= (0, 0, m),\ s_1(m) = (0, m, 0),\\
d_i &\colon \Lambda_3 \longrightarrow (L \rtimes M) \rtimes M \text{ are the projections for } i = 0, 1, 2,\\
d_3(\chi) &= \big(\ell'' + {}^{m''-m}\ell - \ell', \rho(\ell') + m'' - m - \rho(\ell), \rho(\ell) + m\big),\\
s_0(\ell, m, m') &= \big((\ell, m, m'), (\ell, m, m'), (0, 0, m + m')\big),\\
s_1(\ell, m, m') &= \big((0, 0, m'), (\ell, m, m'), (\ell, m, m')\big),\\
s_2(\ell, m, m') &= \big((0, m', 0), (0, m + m', 0), (\ell, m, m')\big).
\end{aligned}$$

The following square, in which U is the functor which forgets the bracket operation $\{-,-\}$, is clearly commutative

$$\begin{array}{ccc}
\text{Simpl}_1\, Gp & \underset{\mathcal{N}}{\overset{\mathcal{J}}{\rightleftarrows}} & \mathcal{XM}\\
\Omega\uparrow & & \uparrow U\\
\text{Simpl}_2\, Gp & \underset{\mathcal{N}_2}{\overset{\mathcal{J}_2}{\rightleftarrows}} & 2-\mathcal{XM}_{\text{red}}.
\end{array}$$

PROPOSITION 1.1. *Let $\mathcal{E} = (L \xrightarrow{\rho} M)$ be a crossed module. To give a group structure on the simplicial set $\bar{\mathcal{W}}\mathcal{J}(\mathcal{E})$, such that $\Omega\bar{\mathcal{W}}\mathcal{J}(\mathcal{E})$ is isomorphic to $\mathcal{J}(\mathcal{E})$ as a simplicial group, is equivalent to give a reduced 2-crossed module structure in $\mathcal{E}$; i.e., a map $\{-,-\}\colon M \times M \longrightarrow L$ satisfying the identities* R2XM1),..., R2XM5).

PROOF. First observe that if $G.$ is a reduced simplicial group (in the sense that $G_0 = 0$), then $G.$ is isomorphic as simplicial set to $\bar{W}\Omega(G.)$. An isomorphism is given by the maps

$$\begin{gathered}
\bar{W}\Omega(G.)_n \longrightarrow G_n\\
(g_{n-1}, g_{n-2}, \ldots, g_0) \longmapsto g_{n-1} + s_0 g_{n-2} + s_0^2 g_{n-3} + \cdots + s_0^{n-1} g_0,
\end{gathered}$$

$n \geq 0$. Then to give a group structure in $\bar{W}\mathcal{J}(\mathcal{E})$, with $\Omega\bar{W}\mathcal{J}(\mathcal{E}) \cong \mathcal{J}(\mathcal{E})$, is equivalent to find a reduced simplicial group $G.$ with $\Omega(G.) \cong \mathcal{J}(\mathcal{E})$; since such condition forces $G.$ to be in $\text{Simpl}_2\, Gp$, by the commutativity of the above square. This is equivalent to finding a reduced 2-crossed module whose underlying crossed module is $\mathcal{E}$.

Consequently we will consider reduced 2-crossed modules, $\mathcal{E} = (L \xrightarrow{\rho} M, \{-,-\})$, as appropriate coefficients for a non-abelian cohomology theory of groups, $\mathbb{H}^i(G, \mathcal{E})$, $i \leq 3$, where for $i \leq 2$, $\mathbb{H}^i(G, \mathcal{E})$ is just Dedecker's non-abelian cohomology of the group G with coefficients in the underlying crossed module of $\mathcal{E}$.

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As we suggested before, in order to define the sets $\mathbb{H}^3(G,\mathcal{E})$ we need a *non-homogeneous* parametrization of the sets $[F.,\mathcal{J}_2(\mathcal{E})]$ of loop homotopy classes of simplicial group morphisms (see [28]) from a free simplicial resolution $F.$ of G to $\mathcal{J}_2(\mathcal{E})$ (note that $\mathcal{J}_2(\mathcal{E})$ is just $\bar{W}\mathcal{J}(\mathcal{E})$ with the canonical group structure induced by the reduced 2-crossed module structure of $\mathcal{E}$).

Let us recall that a simplicial group $F.$ is called *free* if F_n is a free group with a given basis and the bases are stable under all degeneracy operators (i.e. for every pair of integers (i,n) with $0 \leq i \leq n$ and every generator $x \in F_n$ the element $s_i(x)$ is a generator of F_{n+1}). For any group G, a *free simplicial resolution* of G is a free simplicial group $F.$ such that $\Pi_0(F.) = G$ and $\Pi_i(F.) = 0$, for all $i \geq 1$. A free resolution of G is for example $F. = G\big(K(G,1)\big)$, Kan's construction of the loop free simplicial group associated to the Eilenberg-Mac Lane complex $K(G,1)$ [28], and also the standard resolution of G, $\mathbb{G}.(G)$, corresponding to the cotriple associated to the adjunction $Gp \leftrightarrows \mathrm{Set}$ [1]. Recall also that any truncated free simplicial resolution of G can be extended to a free simplicial resolution and that, by the comparison theorem [29], any two free resolution of G are loop homotopically equivalent.

In order to give a parametrization of the sets $[F.,\mathcal{J}_2(\mathcal{E})]$, let us start by studying the elements in $\big(F.,\mathcal{J}_2(\mathcal{E})\big)$, the set of simplicial group morphisms from $F.$ to $\mathcal{J}(\mathcal{E})$. Recall that

$$\mathcal{J}_2(\mathcal{E}) = \mathrm{cosk}^3\big(\Lambda_3 \equiv (L \rtimes M) \rtimes M \equiv M \rightrightarrows 0\big).$$

Therefore a group simplicial morphism $f.$ from $F.$ to $\mathcal{J}_2(\mathcal{E})$ is determined by its truncation at level three $(f_3, f_2, f_1, f_0 = 0)$, but note that also f_3 is determined by f_2,

$$f_3(x) = \big(f_2 d_0(x), f_2 d_1(x), f_2 d_2(x)\big) \in \Lambda_3.$$

So to give a simplicial morphism $f.$ in $\big(F.,\mathcal{J}_2(\mathcal{E})\big)$ is equivalent to give a truncated simplicial morphism

$$\begin{array}{ccccc} F_2 & \equiv & F_1 & \rightrightarrows & F_0 \\ {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0=0} \\ (L \rtimes M) \rtimes M & \equiv & M & \rightrightarrows & 0 \end{array}$$

such that:

$$d_3\big(f_2 d_0(y), f_2 d_1(y), f_2 d_2(y)\big) = f_2 d_3(y)$$

for all $y \in F_3$. Now, if we denote

$$D_1 = \mathrm{Ker}(\delta_i: \tilde{F}_i \longrightarrow \tilde{F}_{i-1}) = \mathrm{Im}(\delta_{i+1}: \tilde{F}_{i+1} \longrightarrow \tilde{F}_i), i \geq 1,$$

this last condition is equivalent to $f_2\delta_3(\tilde{F}_3) = 0$ and also to $f_2(D_2) = 0$. On the other hand, since the sequence

$$0 \longrightarrow D_2 \longrightarrow F_2 \overset{(d_0,d_1,d_2)}{\longrightarrow} \Delta_2 \longrightarrow 0$$

is exact, where Δ_2 denotes the simplicial kernel of $F_1 \rightrightarrows F_0$, we deduce a natural bijection between $\big(F_., \mathcal{J}(\mathcal{E})\big)$ and the set of truncated simplicial group morphisms $(f_2', f_1, f_0 = 0)$ of the form

$$\begin{array}{ccccc} \Delta_2 & \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & F_1 & \rightrightarrows & F_0 \\ {\scriptstyle f_2'}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0=0} \\ (L \rtimes M) \rtimes M & \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & M & \rightrightarrows & 0 \end{array}$$

We use now that the elements of F_1 and D_2 can be expressed, in a unique way, as sums $a + s_0(w)$ and $z + s_1(a) + s_0(a') + s_1 s_0(w)$, respectively, with $a, a' \in \tilde{F}_1$, $z \in D_1$ and $w \in F_0$. So if we denote by $\mathfrak{f}_2$ and $\mathfrak{f}_1$ the morphisms induced, between the Moore complexes, by f_2' and f_1 respectively, i.e. the compositions

$$\mathfrak{f}_2: D_1 \hookrightarrow \Delta_2 \xrightarrow{f_2'} (L \rtimes M) \rtimes M \xrightarrow{\text{pr.}} L \text{ and } \mathfrak{f}_1: \tilde{F}_1 \hookrightarrow F_1 \xrightarrow{f_1} M,$$

respectively, the simplicial identities allow us to recover f_2' and f_1 from f_2 and f_1 by the formulas:

$$f_2'\big(z + s_1(a) + s_0(a') + s_1 s_0(w)\big) = \big(\mathfrak{f}_2(z), \mathfrak{f}_1(a), \mathfrak{f}_1(a')\big) \text{ and}$$
$$f_1\big(a + s_0(w)\big) = \mathfrak{f}_1(a).$$

Moreover a pair of group morphisms $(\mathfrak{f}_2, \mathfrak{f}_1)$ comes from a simplicial morphism as above, if and only if it satisfies the following conditions:

a) $\mathfrak{f}_1\big(s_0(w) + a - s_0(w)\big) = \mathfrak{f}_1(a)$,

b) $\mathfrak{f}_2(a + z - a) = {}^{\mathfrak{f}_1(a)}\mathfrak{f}_2(z)$,

c) $\mathfrak{f}_2\Big(a + a' - a - \big(s_0\delta_1(a) + a' - s_0\delta_1(a)\big)\Big) = \{\mathfrak{f}_1(a), \mathfrak{f}_1(a)\}$ and

d) $\rho\mathfrak{f}_2(z) = f_1(z)$,

for all $z \in D_1$, $a, a' \in \tilde{F}_1$ and $w \in F_0$. We use that in Δ_2

$$\big(z + s_1(a) + s_0(a')\big) + \big(z' + s_1(b) + s_0(b')\big) = z'' + s_1(c) + s_0(c')$$

where $z'' = z + a + \big(s_0\delta_1(a') + z' - s_0\delta_1(a')\big) + \Big(\big(s_0\delta_1(a') + b - s_0\delta_1(a')\big) + a' - b - a'\Big)$, $c = a + a' + b - a'$ and $c' = a' + b'$.

Let us specialise the above results to the free resolution $F.$ of G, obtained by extending the truncated resolution $F_1 \rightrightarrows F_0 \overset{p}{\twoheadrightarrow} G$, where F_0 is the free group with base $\{\eta_x : x \in G\}$, $\eta_0 = 0$, $F_1 = F_1' * F_0$ the free product of F_0 and the free group F_1', with base $\{\mu_{x,y} : x, y \in G\}$, and relations $\mu_{x,0} = 0 = \mu_{0,x}$, for each $x \in G$; the morphism $p: F_0 \twoheadrightarrow G$ is given by $p(\eta_x) = x$ and the face and degeneracy operators by:

$$d_0(\mu_{x,y}) = 0, \quad d_0(\eta_x) = \eta_x, \quad d_1(\mu_{x,y}) = \eta_x + \eta_y - \eta_{x+y},$$
$$d_1(\eta_x) = \eta_x \text{ and } s_0(\eta_x) = \eta_x.$$

In this case, the group $\tilde{F}_1 = \operatorname{Ker}(d_0: F_1 \longrightarrow F_0)$ is just the free F_0-group with base the set $G^* \times G^*$, where $G^* = G - \{0\}$, and the action is given by

$${}^{w}a = s_0(w) + a - s_0(w), \text{ for all } w \in F_0 \text{ and } a \in \tilde{F}_1.$$

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If we write $K = \mathrm{Ker}(p\colon F_0 \twoheadrightarrow G)$, K is the free subgroup of F_0, with generators the elements $\eta_x + \eta_y - \eta_{x+y} = \delta_1(\mu_{x,y}) \in F_0$, $x, y \in G$, and then D_1 is free with base

$$\left\{ s(k) + {}^{w}\mu_{x,y} - s(w + \eta_x + \eta_y - \eta_{x+y}) - s(k) \in \tilde{F}_1 : k \in K^*,\ w \in \tilde{F}_0^*,\ x, y \in G^* \right\},$$

where $s\colon K \longrightarrow \tilde{F}_1$ is the group homomorphism defined by

$$s(\eta_x + \eta_y - \eta_{x+y}) = \mu_{x,y},$$

which is a section of the morphism $\tilde{F}_1 \twoheadrightarrow K$, induced by δ_1.

Then a morphism $\mathfrak{f}_1\colon \tilde{F}_1 \longrightarrow M$ satisfying the above condition a) is equivalent to a normalized map $\varphi\colon G{\times}G \longrightarrow M\,\big(\varphi(x, y) = \mathfrak{f}_1(\mu_{x,y})\big)$. In the same way, to give a morphism $\mathfrak{f}_2\colon D_1 \longrightarrow L$ satisfying condition b) is equivalent to give the image of the elements of the form

$${}^{w}\mu_{x,y} - s(w + \eta_x + \eta_y - \eta_{x+y} - w).$$

But, for all $w, w' \in F_0$ and $k, k' \in K$, we have:

$$\begin{aligned}
&\mathfrak{f}_2\big({}^{w+w'}s(k) - s(w + w' + k - w' - w)\big)\\
&\qquad = \mathfrak{f}_2\big({}^{w'}s(k) - s(w' + k - w')\big) + \mathfrak{f}_2\big({}^{w}s(w' + k - w') - s(w + w' + k - w' - w)\big)
\end{aligned}$$

and also

$$\begin{aligned}
&\mathfrak{f}_2\big({}^{w}s(k + k') - s(w + k + k' - w)\big)\\
&\qquad = \mathfrak{f}_2\big({}^{w}s(k) - s(w + k - w)\big) + {}^{\mathfrak{f}_1 s(w+k-w)}\mathfrak{f}_2\big({}^{w}s(k') - s(w + k' - w)\big).
\end{aligned}$$

Then $\mathfrak{f}_2$ is determined by the image of the elements of the form

$$\tau_{x,y,z} = {}^{\eta_x}\mu_{y,z} - s(\eta_x + \eta_y + \eta_z - \eta_{y+z} - \eta_x),\ x, y, z \in G,$$

and so to give $\mathfrak{f}_2$ satisfying b) is equivalent to give a normalized map $f\colon G \times G \times G \longrightarrow L\big(f(x, y, z) = \mathfrak{f}_2(\tau_{x,y,z})\big)$.

Conditions c) and d) of $\mathfrak{f}_2$ and $\mathfrak{f}_1$ for basic elements give:

$$\begin{aligned}
\rho f(x, y, z) = \rho\,\mathfrak{f}_2(\tau_{x,y,z}) &= \rho\,\mathfrak{f}_2\big({}^{\eta_x}\mu_{y,z} - s(\eta_x + \eta_y + \eta_z - \eta_{y+z} - \eta_x)\big)\\
&= \mathfrak{f}_1\big({}^{\eta_x}\mu_{y,z} - s(\eta_x + \eta_y + \eta_z - \eta_{y+z} - \eta_x)\big)\\
&= \mathfrak{f}_1({}^{\eta_x}\mu_{y,z}) - \mathfrak{f}_1(\mu_{x,y} + \mu_{x+y,z} - \mu_{x,y+z})\\
&= \mathfrak{f}_1(\mu_{y,z}) + \mathfrak{f}_1(\mu_{x,y+z}) - \mathfrak{f}_1(\mu_{x+y,z}) - \mathfrak{f}_1(\mu_{x,y}),
\end{aligned}$$

that is

$$(CC1) \qquad \rho f(x, y, z) = \varphi(y, z) + \varphi(x, y + z) - \varphi(x + y, z) - \varphi(x, y),\ x, y, z \in G,$$

and, taking $a = \mu_{x,y}$ and $a' = {}^{\eta_{x+y}}\mu_{z,t}$ in c), we have

$$\mathfrak{f}_2(\mu_{x,y} + {}^{\eta_{x+y}}\mu_{z,t} - \mu_{x,y} + {}^{\eta_x+\eta_y}\mu_{z,t}) = \left\{ \varphi(x, y), \varphi(z, t) \right\},$$

if we now express $\mu_{x,y} + {}^{\eta_{x+y}}\mu_{z,t} - \mu_{x,y} + {}^{\eta_x+\eta_y}\mu_{z,t}$ in terms of the basic elements in D_1, we have

$$\begin{aligned}\mu_{x,y} + {}^{\eta_{x+y}}\mu_{z,t} - \mu_{x,y} + {}^{\eta_x+\eta_y}\mu_{z,t} &= (\mu_{x,y} + \tau_{x+y,z,t} - \mu_{x,y}) - \tau_{x,y,z}\\ &+ ({}^{\eta_x}\mu_{y,z} - \tau_{x,y+z,t} - {}^{\eta_x}\mu_{y,z})\\ &+ \big({}^{\eta_x}(\mu_{y,z} + \mu_{y+z,t} - \mu_{y,z+t}) + \tau_{x,y,z+t} - {}^{\eta_x}(\mu_{y,z} + \mu_{y+z,t} - \mu_{y,z+t})\big)\\ &+ \tau_{y,z,t},\end{aligned}$$

or equivalently

$$\begin{aligned}\{\varphi(x,y), \varphi(z,t)\} &= {}^{\varphi(x,y)}f(x+y,z,t) - f(x,y,z) - {}^{\varphi(y,z)}f(x,y+z,t)\\ &+ {}^{\varphi(y,z)+\varphi(y+z,t)-\varphi(y,z+t)}f(x,y,z+t) - f(y,z,t),\end{aligned}$$

what is also equivalent to

$$(CC2)\qquad \begin{aligned}\{\varphi(x,y), \varphi(z,t)\} &= {}^{\varphi(x,y)}f(x+y,z,t) - f(x,y,z) - {}^{\varphi(y,z)}f(x,y+z,t) +\\ &- f(y,z,t) + {}^{\varphi(z,t)}f(x,y,z+t),\end{aligned}$$

for all $x, y, z, t \in G^*$. So a pair of group morphisms $(\mathfrak{f}_2, \mathfrak{f}_1)$ satisfying a), b), c) and d) is equivalent to a pair of normalized maps (f, φ) satisfying CC1) and CC2).

We then give the following definition:

DEFINITION. Given a group G and a reduced 2-crossed module $\mathcal{E} = (L \longrightarrow M)$, a (*normalized*) 3-*cocycle* of G with coefficients in $\mathcal{E}$ is a pair of (normalized) maps $(f: G \times G \times G \longrightarrow L, \varphi: G \times G \longrightarrow M)$ satisfying the above cocycle conditions

$$(CC1)\qquad \rho f(x,y,z) = \varphi(y,z) + \varphi(x,y+z) - \varphi(x+y,z) - \varphi(x,y),$$

$$(CC2)\quad \begin{aligned}\{\varphi(x,y), \varphi(z,t)\} &= {}^{\varphi(x,y)}f(x+y,z,t) - f(x,y,z)\\ &- {}^{\varphi(y,z)}f(x,y+z,t) + -f(y,z,t) + {}^{\varphi(z,t)}f(x,y,z+t),\end{aligned}$$

for all $x, y, z, t \in G^*$.

We will write $\mathbb{Z}^3_N(G, \mathcal{E})$ for the set of normalized 3-cocycles of G with coefficients in $\mathcal{E}$. Then there is a natural bijection

$$\big(F., \mathcal{J}_2(\mathcal{E})\big) = \mathbb{Z}^3_N(G, \mathcal{E}).$$

The following proposition gives the necessary and sufficient conditions that two 3-cocycles in $\mathbb{Z}^3_N(G, \mathcal{E})$ have to satisfy to correspond, by the above bijection, to loop homotopic simplicial morphisms (note that, for any free simplicial group $F.$, to be loop homotopic in Simpl $Gp\big(F., \mathcal{J}(\mathcal{E})\big)$ is an equivalence relation).

PROPOSITION 1.2. *Let G be a group and $\mathcal{E} = (L \xrightarrow{\rho} M)$ a reduced 2-crossed module. Then two 3-cocycles (f, φ) and (g, ψ) in $\mathbb{Z}^3_N(G, \mathcal{E})$ determine loop homotopic simplicial morphisms in $\big(F., \mathcal{J}(\mathcal{E})\big)$ if and only if there exist (normalized) maps $\lambda: G \longrightarrow M$ and*

$\Gamma\colon G \times G \longrightarrow L$, *satisfying:*

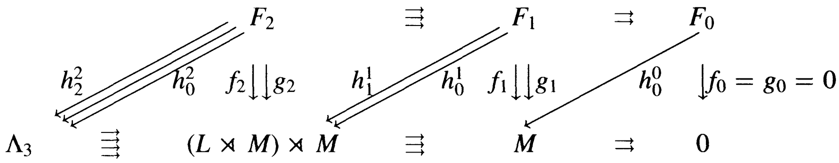
$$(CH1)\quad \psi(x,y) = \rho\Gamma(x,y) + \lambda(x) + \lambda(y) - \lambda(x+y) + \varphi(x,y),$$

$$\begin{aligned}(CH2)\quad g(x,y,z) = {}& {}^{\psi(y,z)}\Gamma(x,y+z) - \{\lambda(x), \psi(y,z)\} + {}^{\lambda(x)}\Gamma(y,z)\\ & - \{\varphi(y,z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(y+z)\}\\ & - {}^{\varphi(y,z)}\{\varphi(x,y+z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(x+y+z)\} + f(x,y,z)\\ & + {}^{\varphi(x,y)}\{\varphi(x+y,z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(x+y+z)\}\\ & + \{\varphi(x,y), \lambda(x) + \lambda(y) - \lambda(x+y)\} - \Gamma(x,y) - {}^{\psi(x,y)}\Gamma(x+y,z),\end{aligned}$$

for all $x, y, z \in G^*$.

PROOF. Let us suppose given simplicial morphisms $f.$ and $g.$ from $F.$ to $\mathcal{J}_2(\mathcal{E})$, which correspond to 3-cocycles (f,φ) and (g,ψ),respectively, by the above bijection $\mathbb{Z}^3_N(G,\mathcal{E}) = (F., \mathcal{J}(\mathcal{E}))$. Let us also suppose that $f.$ and $g.$ are group simplicial morphisms determined, as above, by the pairs of morphisms $(\mathfrak{f}_2, \mathfrak{f}_1)$ and $(\mathfrak{g}_2, \mathfrak{g}_1)$ respectively (then $\mathfrak{f}_2(\tau_{x,y,z}) = f(x,y,z)$, $\mathfrak{g}_2(\tau_{x,y,z}) = g(x,y,z)$, $\mathfrak{f}_1(\mu_{x,y}) = \varphi(x,y)$ and $\mathfrak{g}_1(\mu_{x,y}) = \psi(x,y)$, for all $x,y,z \in G$).

Since $\mathcal{J}_2(\mathcal{E})$ is a 3-coskeleton, a loop homotopy $h.\colon f. \longrightarrow g.$ is equivalent to its truncation at level two



To simplify, let us write h instead of h^0_0 and let χ_i, $i = 0, 1$, be the composition maps

$$\chi_i\colon \tilde{F}_1 \hookrightarrow F_1 \xrightarrow{h^1_i} (L \rtimes M) \rtimes M \xrightarrow{\text{pr.}} L.$$

The triple (χ_0, χ_1, h) determines completely the loop homotopy $h.$. In fact, since any element of F_1 can be expressed, in a unique way, as $a + s_0(w)$, with $a \in \tilde{F}_1$ and $w \in \tilde{F}_0$, we have:

$$\begin{aligned}h^1_0\big(a + s_0(w)\big) &= h^1_0(a) + s_1 h(w)\\ &= \big(\chi_0(a), -\rho\chi_0(a) + h\delta_1(a), \mathfrak{f}_1(a)\big) + \big(0, h(w), 0\big)\\ &= \big(\chi_0(a) - {}^{-\rho\chi_0(a)+h\delta_1(a)}\{\mathfrak{f}_1(a), h(w)\}, -\rho\chi_0(a) + h\delta_1(a) + \mathfrak{f}_1(a)\\ &\quad + h(w) - \mathfrak{f}_1(a), \mathfrak{f}_1(a)\big) \in (L \rtimes M) \rtimes M\end{aligned}$$

and

$$\begin{aligned}h^1_1\big(a + s_0(w)\big) &= h^1_1(a) + s_0 h(w)\\ &= \big(\chi_1(a), -\rho\chi_1(a) + \mathfrak{g}_1(a), 0\big) + \big(0, 0, h(w)\big)\\ &= \big(\chi_1(a), -\rho\chi_1(a) + \mathfrak{g}_1(a), h(w)\big) \in (L \rtimes M) \rtimes M,\end{aligned}$$

and analogously for h_0^2, h_1^2 and h_2^2. On the other hand, the conditions for a triple (χ_0, χ_1, h) to come from a loop homotopy are the following:

i) h is a group homomorphism,
ii) $\chi_0(0) = 0 = \chi_1(0)$,
iii) $\chi(a + a') = {}^{h\delta_1(a)} - \{\mathfrak{f}_1(a), h\delta_1(a')\} + {}^{h\delta_1(a)+\mathfrak{f}_1(a)}\chi_0(a') + \chi_0(a)$,
iv) $\chi_1(a + a') = {}^{\mathfrak{g}_1(a)}\chi_1(a') + \chi_1(a)$,
v) $\chi_0({}^w a) = {}^{h(w+\delta_1(a)-w)}\{\mathfrak{f}_1(a), h(w)\} + {}^{h(w)}\chi_0(a)$,
vi) $\chi_1({}^w a) = -\{h(w), \mathfrak{g}_1(a)\} + {}^{h(w)}\chi_1(a)$,
vii) $\mathfrak{g}_1(a) = \rho\big(\chi_1(a) - \chi_0(a)\big) + h\delta_1(a) + \mathfrak{f}_1(a)$,
viii) $\mathfrak{g}_2(z) = \chi_1(z) - \chi_0(z) + \mathfrak{f}_2(z)$,

for all $w \in F_0$, $a, a' \in \tilde{F}_1$ and $\in D_1$.

Since F_0 is free with base $\{\eta_x : x \in G^*\}$, to give the morphism h is equivalent to give a *map* $\lambda: G \longrightarrow M; \lambda(x) = h(\eta_x)$. On the other hand, $\tilde{F}_1$ is a free F_0-group with base the set $\{\mu_{x,y} : x, y \in G^*\}$, then by the identities ii),...,vi), to know χ_0 and χ_1 is equivalent to know $\chi_0(\mu_{x,y})$ and $\chi_1(\mu_{x,y})$ for all $x, y \in G^*$. Moreover, if (χ_0, χ_1, h) satisfies the identities i),..., viii) so does the triple (χ_0', χ_1', h), where

$$\chi_0'(\mu_{x,y}) = 0 \text{ and } \chi_1'(\mu_{x,y}) = \chi_1(\mu_{x,y}) - \chi_0(\mu_{x,y})$$

for all $x, y \in G^*$. Therefore (χ_0', χ_1', h) determines another loop homotopy h'. from f. to g..

Consequently, f. and g. are loop homotopic if and only if there exist normalized maps $\lambda: G \longrightarrow M$ and $\Gamma: G \times G \longrightarrow L$ such that the triple (χ_0, χ_1, h), defined, using i),...,vi), by: $h(\eta_x) = \lambda(x)$, $\chi_0(\mu_{x,y}) = 0$ and $\chi_1(\mu_{x,y}) = \Gamma(x, y)$, satisfies also vii) and viii) (note that χ_0 is not necessarily the constant map to zero).

Finally, (χ_0, χ_1, h) satisfies the identities vii) and viii) if and only if these identities are satisfied for the elements of the form $a = \mu_{x,y}$ and $z = \tau_{x,y,z}$. But for such elements vii) and viii) reduce to CH1 and CH2 respectively.

This last proposition suggests that we make the following :

DEFINITION. Let G be a group and let $\mathcal{E} = (L \xrightarrow{\rho} M, \{-,-\})$ be a reduced 2-crossed module. Two 3-cocycles (f, φ) and (g, ψ) in $\mathbb{Z}_N^3(G, E)$ are called *cohomologous* if there exist normalized maps $\lambda: G \longrightarrow M$ and $\Gamma: G \times G \longrightarrow L$ satisfying conditions CH1 and CH2 in Proposition 1.2. We will say that the pair (Γ, λ) is a *cohomology* from (f, φ) to (g, ψ) and we will write $(\Gamma, \lambda): (f, \varphi) \longrightarrow (g, \psi)$.

We define the 3*rd non-abelian cohomology (pointed) set* of G with coefficients in $\mathcal{E}$, $\mathbb{H}^3(G, \mathcal{E})$, as the quotient set of $\mathbb{Z}_N^3(G, \mathcal{E})$ by the relation of being cohomologous, pointed by the class of the zero cocycle $(0, 0)$.

A fundamental property of this cohomology is

PROPOSITION 1.3. *(Homotopy representability theorem for $\mathbb{H}^3$). For any group G and any reduced 2-crossed module $\mathcal{E}$ there is a natural bijection*

$$\mathbb{H}^3(G, \mathcal{E}) \cong [F., \mathcal{J}_2(\mathcal{E})],$$

where F. is an arbitrary free simplicial resolution of the group G.

Note that $\mathbb{H}^3(-,-)$ is a bifunctor, with the natural definition on morphisms. In the following section we will show some nice properties of this cohomology functor $\mathbb{H}^3(-,-)$. We end this section by showing how this cohomology is related to the classical Eilenberg-Mac Lane cohomology.

Let us observe that the category of abelian groups can be seen as a full and coreflexive subcategory of the category of reduced 2-crossed modules, by the functors which take an abelian group A to the reduced 2-crossed module $A = (A \longrightarrow 0)$ and a reduced 2-crossed module $\mathcal{E} = (L \xrightarrow{\rho} M, \{-,-\})$ to the abelian group $H_1(\mathcal{E}) = \mathrm{Ker}(\rho)$ respectively. Then the elements in $\mathbb{Z}^3_N(G,A)$ are just (normalized) maps $f\colon G \times G \times G \longrightarrow A$, satisfying:

$$f(x+y,z,t) - f(x,y,z) - f(x,y+z,t) - f(y,z,t) + f(x,y,z+t) = 0,$$

i.e. Eilenberg-Mac Lane 3-cocycles of G with coefficients in A. Moreover two elements f and g in $\mathbb{Z}^3_N(G,A)$ are cohomologous if and only if there exists a map $\Gamma\colon G \times G \longrightarrow A$ satisfying:

$$g(x,y,z) = f(x,y,z) - \Gamma(x,y) + \Gamma(x,y+z) - \Gamma(x+y,z) + \Gamma(y,z).$$

Consequently $\mathbb{H}^3(G,A) = H^3(G,A)$, the usual Eilenberg-Mac Lane cohomology group. We have then:

PROPOSITION 1.4. *The restriction of the functor $\mathbb{H}^3(G,-)$ to the category of abelian groups is just the usual functor $H^3(G,-)$ of Eilenberg-Mac Lane cohomology.*

2. **The nine term exact sequence for non-abelian cohomology of groups.**

Dedecker shows in [16] how a surjective morphism of crossed modules

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{B} \\ \vdots & & \vdots \\ L & \xrightarrow{\Phi_1} & L'' \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho''} \\ M & \xrightarrow{\phi_0} & M'' \end{array}$$

(i.e. a morphism Φ with ϕ_0 and ϕ_1 epimorphisms) gives rise, for any group homomorphism $\theta : G \longrightarrow M$, to a cohomology exact sequence

(3)

$$* \longrightarrow Z^1_\theta(G,\mathcal{K}) \longrightarrow Z^1_\theta(G,\mathcal{E}) \longrightarrow Z^1_{\phi_0\theta}(G,\mathcal{B}) \longrightarrow \mathbb{H}^2(G,\mathcal{K}) \longrightarrow \mathbb{H}^2(G,\mathcal{E}) \longrightarrow \mathbb{H}^2(G,\mathcal{B}),$$

where $\mathcal{K} = \big(\mathrm{Ker}(\phi_1) \xrightarrow{\rho} M\big)$ is the crossed module kernel of Φ and exactness means that the sets in this sequence are endowed with structures (which include sets of distinguished elements) which make it possible to answer the following two questions:

When is an element in one set in the image of the preceding arrow?

When do two elements in one set have the same image through the next arrow?

In this section, we will give a solution to the problem of measuring the deviation from right exactness of the functor $\mathbb{H}^2(G,-)$, by extending the above sequence (3) to a nine term exact sequence in which the last three terms will be 3-dimensional non-abelian cohomology sets, as they where defined in Section 1. Since our non-abelian 3-cohomology requires reduced 2-crossed modules as coefficients, we need such additional structure in the crossed modules $\mathcal{E}$ and $\mathcal{B}$ and also that $\Phi\colon \mathcal{E} \longrightarrow \mathcal{B}$ is a reduced 2-crossed module morphism.

To establish this nine term sequence, we are going to use not only the crossed module kernel $\mathcal{K}$ but also what we call the crossed module *fiber* $\mathcal{F}$, of Φ, which is defined as $\mathcal{F} = \big(\mathrm{Ker}(\phi_1) \xrightarrow{\rho} \mathrm{Ker}(\phi_0)\big)$, where $\mathrm{Ker}(\phi_0)$ acts on $\mathrm{Ker}(\phi_1)$ by restriction of the action of M on L. So if we write $L' = \mathrm{Ker}(\phi_1)$ and $M' = \mathrm{Ker}(\phi_0)$, we will consider the diagram

$$
\begin{array}{ccccc}
\mathcal{K} & \hookrightarrow & \mathcal{E} & \stackrel{\Phi}{\twoheadrightarrow} & \mathcal{B} \\
 & \nwarrow \mathcal{F} \nearrow & & & \\
L' & \hookrightarrow & L & \stackrel{\phi_1}{\twoheadrightarrow} & L'' \\
\downarrow & \nwarrow L' \nearrow & \downarrow \rho & & \downarrow \rho'' \\
M & = \downarrow = & M & \stackrel{\phi_0}{\twoheadrightarrow} & M'' \\
 & \nwarrow M' \nearrow & & &
\end{array} \tag{4}
$$

associated to a surjective morphism Φ of reduced 2-crossed modules. Note that the fiber crossed module $\mathcal{F}$ inherits the 2-crossed module structure of $\mathcal{E}$, since the structure map $\{-,-\}: M \times M \longrightarrow L$ induces by restriction a map $\{-,-\}: M' \times M' \longrightarrow L'$, so $\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{B}$ is a sequence of reduced 2-crossed modules (nevertheless the map $\{-,-\}: M \times M \longrightarrow L$ does not induce a reduced 2-crossed module structure in the kernel $\mathcal{K}$).

REMARK. The category of crossed modules is equivalent to that of internal groupoids in Groups [9]. By this equivalence, a surjective morphism of crossed modules Φ corresponds to a fibration of groupoids which is surjective on objects, the kernel $\mathcal{K}$ and the fiber $\mathcal{F}$ correspond to the groupoids the kernel and fiber at zero of the fibration respectively (cf. [5] and [24]). This is a reason for calling $\mathcal{F}$ the fiber crossed module of Φ.

We then have

PROPOSITION 2.1. *Let Φ be a surjective morphism of reduced 2-crossed modules, with kernel $\mathcal{K}$ and fiber $\mathcal{F}$ as above*

$$
\begin{array}{ccccc}
\mathcal{K} & \hookrightarrow & \mathcal{E} & \stackrel{\Phi}{\twoheadrightarrow} & \mathcal{B} \\
 & \nwarrow \mathcal{F} \nearrow & & &
\end{array}
$$

Then for any group homomorphism $\theta: G \longrightarrow M$ there exists a cohomology sequence

$$
\begin{aligned}
* \longrightarrow Z^1_\theta(G,\mathcal{K}) \longrightarrow Z^1_\theta(G,\mathcal{E}) \longrightarrow Z^1_{\Phi_0\theta}(G,\mathcal{B}) \longrightarrow \mathbb{H}^2(G,\mathcal{K}) \\
\longrightarrow \mathbb{H}^2(G,\mathcal{E}) \xrightarrow{\Phi_*} \mathbb{H}^2(G,\mathcal{B}) \xrightarrow{\delta_*} \mathbb{H}^3(G,\mathcal{F}) \xrightarrow{i_*} \mathbb{H}^3(G,\mathcal{E}) \xrightarrow{\Phi_*} \mathbb{H}^3(G,\mathcal{B}),
\end{aligned}
$$

which extends Dedecker's six term exact sequence and moreover it has the following properties:

i) $\mathbb{H}^2(G, \mathcal{E})$ and $\mathbb{H}^2(G, \mathcal{B})$ have canonical group structures and $\Phi_*: \mathbb{H}^2(G, \mathcal{E}) \longrightarrow \mathbb{H}^2(G, \mathcal{B})$ is a group homomorphism

ii) $\mathrm{Im}(\phi_*) = \delta_*^{-1}\{0\}$ and

$$\delta_*(\alpha) = \delta_*(\beta) \Leftrightarrow \beta = \Phi_*(\mu) + \alpha,$$

for some $\mu \in \mathbb{H}^2(G, \mathcal{E})$.

iii) The group $\mathbb{H}^2(G, \mathcal{B})$ acts canonically on the set $\mathbb{H}^3(G, \mathcal{F})$.

iv) $\mathrm{Im}(\delta_*) = i_*^{-1}\{0\}$ and

$$i_*(\gamma) = i_*(\nu) \Leftrightarrow \nu = {}^{\alpha}\gamma,$$

for some $\alpha \in \mathbb{H}^2(G, \mathcal{B})$.

v) $\mathrm{Im}(i_*) = \phi_*^{-1}\{0\}$

Before attacking the proof of this Proposition 2.1. let us draw attention to particular cases:

A) Any epimorphism of abelian groups $\Phi: A \longrightarrow A''$ can be seen as a surjective morphism of reduced 2-crossed modules (by the identification $A = (A \longrightarrow 0, \{-,-\} = 0)$). In this case the crossed modules fiber and kernel are both identified with the abelian group $A' = \mathrm{Ker}(\Phi)$ and then, using Proposition 1.4, the sequence in Proposition 2.1 gives just the first nine terms of the usual sequence in group cohomology

$$0 \longrightarrow \mathrm{Hom}(G,A') \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,A'') \longrightarrow H^2(G,A') \longrightarrow H^2(G,A)$$
$$\longrightarrow H^2(G,A'') \longrightarrow H^3(G,A') \longrightarrow H^3(G,A) \longrightarrow H^3(G,A'').$$

B) When in Proposition 2.1 $\Phi: \mathcal{E} \longrightarrow \mathcal{B}$ corresponds (as morphism of crossed modules) to a *quotient map of groupoids* in the sense of Higgins [24] (through the equivalence between the categories of crossed modules and internal groupoids in Groups), or equivalently, when Φ satisfies the condition $M' = \rho(L')$ (cf. [17] or [40]), then the canonical inclusion of the center of the kernel crossed module $\mathcal{K}$, $A' = \mathrm{Ker}(L' \longrightarrow M) = \mathrm{Ker}(\rho) \cap L'$, into the fiber crossed module $\mathcal{F}$ is a *weak equivalence*, in the sense that the induced simplicial group morphism

$$K(A',2) = \mathcal{J}_2(A') \longrightarrow \mathcal{J}_2(\mathcal{F})$$

is a weak equivalence, and therefore there are natural bijections

$$H^3(G,A) \cong \mathbb{H}^3(G,A) \cong [F_{\cdot}, \mathcal{J}_2(A)] \overset{(1)}{\cong} [F_{\cdot}, \mathcal{J}_2(\mathcal{F})] \cong \mathbb{H}^3(G, \mathcal{F}),$$

for any free simplicial resolution $F_{\cdot}$ of G (to prove (1) take for example $F_{\cdot} = G\big(K(G,1)\big)$, the group loop complex associated to $K(G,1)$, and then using the adjunction $G(-) \vdash \bar{W}(-)$ we have

$$[F_{\cdot}, \mathcal{J}_2(A')] \cong [K(G,1), \bar{W}\mathcal{J}_2(A')] \text{ and } [F_{\cdot}, \mathcal{J}_2(\mathcal{F})] \cong [K(G,1), \bar{W}\mathcal{J}_2(\mathcal{F})],$$

where square bracket in the right hand side terms denote homotopy classes of simplicial maps; but $\bar{W}\mathcal{J}_2(A') \hookrightarrow \bar{W}\mathcal{J}_2(\mathcal{F})$ is a homotopy equivalence—since it is a weak homotopy equivalence and both complexes satisfy the Kan extension condition. Therefore the cohomology sequence in Proposition 2.1 gives in this case the sequence

$$\begin{aligned} * \longrightarrow Z^1_\theta(G, \mathcal{K}) \longrightarrow Z^1_\theta(G, \mathcal{E}) \longrightarrow Z^1_{\Phi_0\theta}(G, \mathcal{B}) \longrightarrow \mathbb{H}^2(G, \mathcal{K}) \longrightarrow \mathbb{H}^2(G, \mathcal{E}) \\ \longrightarrow \mathbb{H}^2(G, \mathcal{B}) \longrightarrow H^3(G, A') \longrightarrow \mathbb{H}^3(G, \mathcal{E}) \longrightarrow \mathbb{H}^3(G, \mathcal{B}). \end{aligned}$$

So the abelian term $H^3(G,A')$ can be used to give a measure to the obstruction of the image of $\mathbb{H}^2(G, \mathcal{E}) \longrightarrow \mathbb{H}^2(G, \mathcal{B})$. This particular result was already found by F. Kamber in an unpublished paper (see [15]). Nevertheless note that, as was observed by Dedecker, the composition

$$\mathbb{H}^2(G, \mathcal{B}) \longrightarrow H^3(G, A') \longrightarrow H^3(G, A)$$

is not zero and so $H^3(G,A)$ can not be used to extend the sequence.

The rest of this section is essentially devoted to prove the statements in Proposition 2.1. We first give a proof using arguments of algebraic homotopy theory, which really explain the nature of the announced cohomology sequence. Finally we give a constructive and conceptual proof, in terms of cocycles, of this fundamental result in the paper.

Let us recall that Simplicial sets, SimplSet, and Simplicial groups are examples of simplicial homotopy categories in the sense of Quillen [37]. Thus for simplicial sets $L.$ and $K.$ we have the *function complex* simplicial set $L.^K$ whose component set $\Pi_0(L.^K)$ is just the set of homotopy classes of simplicial maps from $K.$ to $L.$; analogously for simplicial groups $G.$ and $H.$ we have the *linear function complex* simplicial set $G.^H$ whose component set $\Pi_0(G.^H)$ is the set $[H., G.]$ of loop homotopy classes of simplicial group morphisms from $H.$ to $G.$. The following lemma will be very useful.

LEMMA 2.2. *For any simplicial group $G.$ and any reduced simplicial set $K.$ there is a natural isomorphism of simplicial sets*

$$G.^{G(K.)} \cong \bar{W}(G.)^K.$$

PROOF. Essentially this isomorphism is a consequence of Kan's loop group functor $G(-)$ being left adjoint to Eilenberg-Mac Lane's classifying functor $\bar{W}(-)$. For any simplicial set $T.$ the set of simplicial maps from $T.$ to $G.^{G(K.)}$ is canonically bijective with the set of simplicial maps $f.: G(K.) \times T. \longrightarrow G.$ which are linear in the sense that $f(x+y, v) = f(x, v) + f(y, v)$. Then considering the natural group structure in $G.^K$ there are natural bijections

$$\mathrm{SimplSet}(T., G.^{G(K.)}) \cong \mathrm{Simpl}\,Gp\big(G(K.), G.^T\big) \cong \mathrm{SimplSet}\big(K., \bar{W}(G.^T)\big) \cong$$

(using that $\bar{W}$ preserves limits and then function spaces [37, II,1.11])

$$\cong \mathrm{SimplSet}\big(K., \bar{W}(G.)^T\big) \cong \mathrm{SimplSet}\big(T., \bar{W}(G.)^K\big).$$

PROPOSITION 2.3. *Let $N. \overset{i}{\hookrightarrow} G. \overset{p}{\twoheadrightarrow} Q.$ be a short exact sequence of simplicial groups and let $F.$ be a free simplicial group. Then there is a sequence of sets of loop homotopy classes*

$$\cdots \longrightarrow [F., \Omega(N.)] \longrightarrow [F., \Omega(G.)] \longrightarrow [F., \Omega(Q.)] \xrightarrow{\delta_*} [F., N.] \xrightarrow{i_*} [F., G.] \xrightarrow{p_*} [F., Q.]$$

which is an exact sequence of groups and pointed sets in the usual sense, and moreover:

i) The group $[F., \Omega(Q.)]$ acts on $[F., N.]$ and

$$i_*(\alpha) = i_*(\alpha') \Leftrightarrow \alpha = {}^{\lambda}\alpha', \textit{ for some } \lambda \in [F., \Omega(Q.)].$$

ii) $\delta_*(\lambda) = \delta_*(\lambda') \Leftrightarrow p_*(\omega) + \lambda = \lambda'$, for some $\omega \in [F., \Omega(G.)]$.

PROOF. Let us first observe that the map $G.^F \longrightarrow Q.^F$, induced by $p.$ between the linear function spaces, is a Kan fibration. In fact, since $F.$ is free, if $K. = \bar{W}(F.)$ the canonical map $G(K.) \longrightarrow F.$ is a loop homotopy equivalence and therefore it is enough to observe that $G.^{G(K.)} \longrightarrow Q.^{G(K.)}$ is a Kan fibration; now by Lemma 2.2 this is equivalent to see that $\bar{W}(G.)^K \longrightarrow \bar{W}(Q.)^K$ is a Kan fibration, but since $p.: G. \longrightarrow Q.$ is surjective the simplicial map $\bar{W}(G.)^K \longrightarrow \bar{W}(Q.)^K$ is a Kan fibration and therefore, by [36, 7.8], $\bar{W}(G.)^K \longrightarrow \bar{W}(Q.)^K$ is also a Kan fibration.

Consequently we have a fiber sequence of simplicial sets

$$N.^F \longrightarrow G.^F \longrightarrow Q.^F,$$

with based point the zero simplicial morphism, this induces a long exact sequence of groups and pointed sets

$$\cdots \longrightarrow \Pi_1(N.^F) \longrightarrow \Pi_1(G.^F) \longrightarrow \Pi_1(Q.^F) \longrightarrow \Pi_0(N.^F) \longrightarrow \Pi_0(G.^F) \longrightarrow \Pi_0(Q.^F)$$

which moreover satisfies the conditions analogous to i) and ii) (see [5] for example).

Finally, note that for arbitrary simplicial groups $H.$ and $P.$ and any integer $i \geq 0$

$$\begin{aligned}\Pi_i(H.^P) &\cong \Pi_0\big(\Omega^i(H.^P)\big) = (\text{ since } \Omega \text{ preserves limits}) \\ &\cong \Pi_0\big(\Omega^i(H.)^P\big) \cong [P., \Omega^i(H.)].\end{aligned}$$

As a direct consequence we have

FIRST PROOF OF PROPOSITION 2.1. It is immediate to observe that the sequence of simplicial groups

$$\mathcal{J}_2(\mathcal{F}) \hookrightarrow \mathcal{J}_2(\mathcal{E}) \twoheadrightarrow \mathcal{J}_2(\mathcal{B})$$

is short exact; therefore, for any simplicial free resolution $F.$ of the group G, by the above Proposition 2.3, we have an exact sequence of pointed sets

$$[F., \Omega\mathcal{J}_2(\mathcal{E})] \xrightarrow{\Phi_*} [F., \Omega\mathcal{J}_2(\mathcal{B})] \xrightarrow{\delta_*} [F., \mathcal{J}(\mathcal{F})] \xrightarrow{i_*} [F., \mathcal{J}_2(\mathcal{E})] \xrightarrow{\Phi_*} [F., \mathcal{J}_2(\mathcal{B})]$$

whose terms can be identified, using the isomorphism $\Omega\mathcal{J}_2(\mathcal{E}) \cong \mathcal{J}(\mathcal{E})$ (Proposition 1.1) and the homotopy representability theorems of Section 1, with cohomology sets as follows:

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$$\begin{array}{ccccccccc}
[F.,\Omega\mathcal{J}_2(\mathcal{E})] & \xrightarrow{\Phi_*} & [F.,\Omega\mathcal{J}_2(\mathcal{B})] & \xrightarrow{\delta_*} & [F.,\mathcal{J}_2(\mathcal{F})] & \xrightarrow{i_*} & [F.,\mathcal{J}_2(\mathcal{E})] & \xrightarrow{\Phi_*} & [F.,\mathcal{J}_2(\mathcal{B})] \\
\wr\| & & \wr\| & & \| & & \| & & \| \\
[F.,\mathcal{J}(\mathcal{E})] & \xrightarrow{\Phi_*} & [F.,\mathcal{J}(\mathcal{B})] & \xrightarrow{\delta_*} & [F.,\mathcal{J}_2(\mathcal{F})] & \xrightarrow{i_*} & [F.,\mathcal{J}_2(\mathcal{E})] & \xrightarrow{\Phi_*} & [F.,\mathcal{J}_2(\mathcal{B})] \\
\wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| \\
\mathbb{H}^2(G,\mathcal{E}) & \xrightarrow{\Phi_*} & \mathbb{H}^2(G,\mathcal{B}) & \xrightarrow{\delta_*} & \mathbb{H}^3(G,\mathcal{F}) & \xrightarrow{i_*} & \mathbb{H}^3(G,\mathcal{E}) & \xrightarrow{\Phi_*} & \mathbb{H}^3(G,\mathcal{B}).
\end{array}$$

Thus by comparing this sequence with Dedecker's one, we get a sequence whose exactness properties are just those stated in Proposition 2.1.

Now we are going to translate into the cohomology situation the above homotopical arguments.

If $\mathcal{E} = (L \xrightarrow{\rho} M)$ is a crossed module enriched with an additional structure of reduced 2-crossed module, given by a map $\{-,-\}: M \times M \longrightarrow L$, we have seen how Dedecker cohomology $\mathbb{H}^2(G,\mathcal{E})$ has a group structure via the natural bijection $\mathbb{H}^2(G,\mathcal{E}) \cong [F.,\Omega\mathcal{J}_2(\mathcal{E})]$. The first question is: how is this group structure described in terms of cocycles?

In order to answer this question let us take, as free simplicial resolution of G, the loop group complex $F. = G\big(K(G,1)\big)$ of the Eilenberg-Mac Lane complex $K(G,1)$. Then

$$\begin{aligned}
\mathbb{H}^2(G,\mathcal{E}) &\cong [G\big(K(G,1)\big),\mathcal{J}(\mathcal{E})] \cong \Pi_0\big(\mathcal{J}_2(\mathcal{E})^{G(K(G,1))}\big) \\
&\cong \Pi_0(\bar{W}\mathcal{J}(\mathcal{E})^{K(G,1)}\big) \cong \Pi_0\big(\mathcal{J}_2(\mathcal{E})^{K(G,1)}\big).
\end{aligned}$$

So we only have to translate the canonical group structure of $\big(\mathcal{J}_2(\mathcal{E})^{K(G,1)}\big)$ to $\mathbb{H}^2(G,\mathcal{E})$. We start by recalling how the set $\mathbb{Z}^2(G,\mathcal{E})$ of Dedecker's 2-cocycles is bijective to

$$\big(\mathcal{J}(\mathcal{E})^{K(G,1)}\big)_0 = \mathrm{SimplSet}\big(K(G,1),\mathcal{J}_2(\mathcal{E})\big).$$

Recall that a Dedecker 2-cocycle of G with coefficients in $\mathcal{E}$ is a pair of maps $(f: G\times G \longrightarrow L, \varphi: G \longrightarrow M)$ satisfying:

CC1. $\varphi(x) + \varphi(y) = \rho f(x,y) + \varphi(x+y)$,

CC2. ${}^{\varphi(x)}f(y,z) + f(x,y+z) = f(x,y) + f(x+y,z)$,

for all $x,y,z \in G$. Then a 2-cocycle (f,φ) corresponds to the simplicial map $f.: K(G,1) \longrightarrow \mathcal{J}(\mathcal{E})$ which is uniquely determined by the truncated simplicial map

$$\begin{array}{ccccc}
G & \substack{\to\\\to\\\to} & G & \rightrightarrows & 1 \\
\big\downarrow f_2 & & \big\downarrow f_1 & & \big\downarrow 0 \\
(L \rtimes M) \rtimes M & \substack{\to\\\to\\\to} & M & \rightrightarrows & 1
\end{array}$$

where $f_1 = \varphi$ and $f_2(x,y) = \big(f(x,y), \varphi(x+y) - \varphi(y), \varphi(y)\big)$. Therefore we have, according to the group structure of $\mathcal{J}_2(\mathcal{E})$ (see pg. 6):

PROPOSITION 2.4. *Let $\mathcal{E} = (L \longrightarrow M, \{-,-\})$ be a reduced 2-crossed module and let G be a group. Then $\mathbb{H}^2(G,\mathcal{E})$ has a canonical group structure which is given by*

$$[(f,\varphi)] + [(g,\psi)] = [(f * g, \varphi * \psi],$$

where square bracket means cohomology class and

$$(\varphi * \psi)(x) = \varphi(x) + \psi(x) \text{ and}$$
$$(f * g)(x,y) = -^{\varphi(x)}\{\varphi(y), \psi(x)\} + f(x,y) +^{\varphi(x+y)} g(x,y),$$

for all $x, y \in G$.

The next step consists in describing, in cohomology terms, the action of the group $\mathbb{H}^2(G, \mathcal{E})$ on the set $\mathbb{H}^3(G, \mathcal{F})$, when $\mathcal{F} \hookrightarrow \mathcal{E} \xrightarrow{\Phi} \mathcal{B}$ is a sequence as in Proposition 2.1. In particular, this action will describe the connecting map $\mathbb{H}^2(G, \mathcal{E}) \longrightarrow \mathbb{H}^3(G, \mathcal{B})$. In the first proof of Proposition 2.1 this action is induced from the canonical action of $\Pi_1\big(\mathcal{J}_2(\mathcal{B})\big)$ on $\Pi_0(\mathcal{J}_2(\mathcal{F})^F)$ which is induced by the fiber sequence of simplicial maps $\mathcal{J}_2(\mathcal{F})^F \longrightarrow \mathcal{J}_2(\mathcal{E})^F \xrightarrow{\Phi_*} \mathcal{J}_2(\mathcal{B})$ and it is given by

$$^{[h]}[f] = [d_1(\tilde{h})],$$

for any $f \in \big(\mathcal{J}_2(\mathcal{F}^F)\big)_0$ and $h \in \big(\mathcal{J}_2(\mathcal{B})^F\big)_1$ with $d_0(h) = 0 = d_1(h)$, where $\tilde{h} \in \big(\mathcal{J}_2(\mathcal{E})\big)_1$ is chosen (using that Φ_* is a Kan fibration) such that $d_0(\tilde{h}) = f$ and $\Phi\tilde{h} = h$.

Now let $F.$ be the free simplicial group resolution of G used in page 272 of this paper. Then we have a canonical equivalence of functors

$$\big(\mathcal{J}_2(-)^F\big)_0 = \big(F., \mathcal{J}_2(-)\big) \cong \mathbb{Z}^2_N(G, -),$$

through which elements in $\big(\mathcal{J}_2(-)^F\big)_1$, i.e. homotopies between simplicial morphisms, correspond to cohomologies between 3-cocycles (according to the results in Section 1). In particular an element $h \in \big(\mathcal{J}_2(\mathcal{B})^F\big)_1$ corresponds to a cohomology between zero and itself, i.e. (see Proposition 1.2) a pair of maps $\lambda: G \longrightarrow M''$ and $\Gamma: G \times G \longrightarrow L$ satisfying:

CH1) $0 = \rho\Gamma(x,y) + \lambda(x) + \lambda(y) - \lambda(x+y)$,

CH2) $0 = \Gamma(x, y+z) +^{\lambda(x)} \Gamma(y,z) + \Gamma(x,y) - \Gamma(x+y, z)$.

Then it is plain to see that the isomorphism $\mathbb{H}^2(G, \mathcal{B}) \cong \Pi_1\big(\mathcal{J}_2(\mathcal{B})^F\big)$ (note that $\Pi_1\big(\mathcal{J}_2(\mathcal{B})^F\big) \cong \Pi_0\big(\Omega\mathcal{J}_2(\mathcal{B})^F\big) \cong \Pi_0\big(\mathcal{J}(\mathcal{B})^F\big) \cong [\tilde{F}., \mathcal{J}(\mathcal{B})] \cong \mathbb{H}^2(G, \mathcal{B})\big)$ is given by $[(f, \varphi)] \longmapsto [(-f, \varphi)]$, where $(f, \varphi) \in \mathbb{Z}^2_N(G, \mathcal{B})$ denotes here a normalized Dedecker 2-cocycle.

Using then the above observation, we have:

PROPOSITION 2.5. *In the hypothesis of Proposition 2.1, the action of* $\mathbb{H}^2(G, \mathcal{B})$ *on* $\mathbb{H}^3(G, \mathcal{F})$ *is as follows:*

Let $(f'', \varphi'') \in \mathbb{Z}^2(G, \mathcal{B})$ be a 2-cocycle and $(g, \psi) \in \mathbb{Z}^3(G, \mathcal{F})$ a 3-cocycle. Then, there are maps $\lambda: G \longrightarrow M$ and $\Gamma: G \times G \longrightarrow L$ satisfying $\phi_0\lambda = \varphi''$ and $\phi_1\Gamma = -f''$ and

$$^{[(f'', \varphi'')]}[(g, \psi)] = [(g', \psi')],$$

where:

$$\psi'(x,y) = \rho\Gamma(x,y) + \lambda(x) + \lambda(y) - \lambda(x+y) + \psi(x,y) \text{ and}$$
$$g'(x,y,z) =^{\psi'(y,z)} \Gamma(x, y+z) - \{\lambda(x), \psi'(y,z)\} +^{\lambda(x)} \Gamma(y,z)$$
$$+ g(x,y,z) - \Gamma(x,y) -^{\psi'(x,y)} \Gamma(x+y, z).$$

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In particular the connecting map $\delta_*\colon \mathbb{H}^2(G,\mathcal{B}) \longrightarrow \mathbb{H}^3(G,\mathcal{F})$ is

$$\delta_*[(f'',\varphi'')] = {}^{[(f'',\varphi'')]}[(0,0)].$$

Now we give a brief proof in terms of cocycles of the exactness of the cohomology sequence

SECOND PROOF OF PROPOSITION 2.1. We already have given the group structures of $\mathbb{H}^2(G,\mathcal{E})$ and $\mathbb{H}^2(G,\mathcal{B})$, the action of $\mathbb{H}^2(G,\mathcal{B})$ on $\mathbb{H}^3(G,\mathcal{F})$ and the connecting map δ_*. Let us then see the exactness properties:

EXACTNESS AT $\mathbb{H}^2(G,\mathcal{B})$. Let $(f,\varphi) \in \mathbb{Z}^2(G,\mathcal{E})$ be a 2-cocycle of G with coefficients in $\mathcal{E}$. Then

$$\delta_*\Phi_*[(f,\varphi)] = {}^{[\phi_1 f,\phi_0\varphi)]}[(0,0)] = [(g',\psi')],$$

where

$$\psi'(x,y) = -\rho f(x,y) + \varphi(x) + \varphi(y) - \varphi(x+y) = 0$$
$$\text{(since } (f,\varphi) \text{ satisfies CC1) and}$$
$$g'(x,y,z) = -f(x,y+z) - {}^{\varphi(x)}f(y,z) + f(x,y) + f(x+y,z) = 0,$$

since (f,φ) satisfies CC2. Therefore $\delta_*\Phi_* = 0$.

Conversely, let $(f'',\varphi'') \in \mathbb{Z}^2(G,\mathcal{B})$ be a 2-cocycle and let $\lambda\colon G \longrightarrow M$ and $\Gamma\colon G\times G \longrightarrow L$ be two maps satisfying $\phi_0\lambda = \varphi''$ and $\phi_1\Gamma = -f''$. Then $\delta_*[(f'',\varphi'')] = [(g',\psi')]$, where

$$\psi'(x,y) = \rho\Gamma(x,y) + \lambda(x) + \lambda(y) - \lambda(x+y) \text{ and}$$
$$g'(x,y,z) = {}^{\psi'(y,z)}\Gamma(x,y+z) - \{\lambda(x),\psi'(y,z)\} +^{\lambda(x)} \Gamma(y,z)$$
$$- \Gamma(x,y) -^{\psi'(x,y)} \Gamma(x+y,z).$$

If $\delta_*[(f'',\varphi'')] = 0$, there must exist maps $\lambda'\colon G \longrightarrow M'$ and $\Gamma'\colon G\times G \longrightarrow L'$ which give a cohomology $(\lambda',\Gamma')\colon (g',\psi') \longrightarrow (0,0)$. On the other hand we can consider (g',ψ') as a cocycle with coefficients on $\mathcal{E}$ (via the inclusion $\mathcal{F} \hookrightarrow \mathcal{E}$) and the pair (Γ,λ) as a cohomology from $(0,0)$ to (g',ψ'). Then by *composing*

$$(0,0) \overset{(\Gamma,\lambda)}{\longrightarrow} (g',\psi') \overset{(\Gamma',\lambda')}{\longrightarrow} (0,0)$$

we have a cohomology $(\Gamma',\lambda')\circ(\Gamma,\lambda) = (\bar{\Gamma},\bar{\lambda})\colon (0,0) \longrightarrow (0,0)$ by the formulas

$$\bar{\lambda}(x) = \lambda'(x) + \lambda(x) \text{ and}$$
$$\bar{\Gamma}(x,y) = \Gamma'(x,y) +^{\lambda'(x)+\lambda'(y)-\lambda'(x+y)} \Gamma(x,y) -^{\lambda'(x)} \{\lambda(x),\lambda'(y)\}$$
$$-^{\lambda'(x)+\lambda(x)+\lambda'(y)-\lambda'(x)} (\{\lambda(x)+\lambda(y), -\lambda(x+y) - \lambda'(x+y) + \lambda(x+y)\}$$
$$+ \{-\lambda(x+y),\lambda(x+y)\}).$$

This cohomology gives a cocycle $(-\bar{\Gamma},\bar{\lambda}) \in \mathbb{Z}^2(G,\mathcal{B})$ which clearly satisfies $\Phi_*[(-\bar{\Gamma},\bar{\lambda})] = [(f'',\varphi'')]$. Therefore we have $\mathrm{Im}(\Phi_*) = \delta_*^{-1}(0)$.

Moreover given $\alpha,\beta \in \mathbb{H}^2(G,\mathcal{B})$, $\delta_*(\alpha) = \delta_*(\beta)$ if and only if ${}^{\alpha}0 =^{\beta} 0$ (or equivalently ${}^{(\alpha-\beta)}0 = 0$), and so there exists $\mu \in \mathbb{H}^2(G,\mathcal{E})$ such that $-\alpha+\beta = \Phi_*(\mu)$.

EXACTNESS AT $\mathbb{H}^3(G,\mathcal{F})$: It is clear that for any $\alpha \in \mathbb{H}^2(G,\mathcal{B})$ and $\gamma \in \mathbb{H}^3(G,\mathcal{F})$,

$$i_*({}^{\alpha}\gamma) = i_*(\gamma),$$

in particular $i_*\delta_* = 0$. On the other hand, given $\gamma = [(g,\psi)]$ and $\nu = [(g',\psi')]$ two elements in $\mathbb{H}^3(G,\mathcal{F})$, if $i_*(\gamma) = i_*(\nu)$ there exists a cohomology $(\Gamma,\lambda)\colon (g,\psi) \longrightarrow (g',\psi')$. Then $(\phi_1\Gamma, \phi_0\lambda)$ is a cohomology from the zero 3-cocycle on $\mathcal{B}$ to itself and taking $\alpha = [(-\phi\Gamma, \phi_0\lambda)] \in \mathbb{H}^2(G,\mathcal{B})$ we have ${}^{\alpha}\gamma = \nu$.

EXACTNESS AT $\mathbb{H}^3(G,\mathcal{E})$. The composition $\Phi_* i_*$ is clearly the constant zero map. Conversely, suppose $\gamma = [(g,\psi)] \in \mathbb{H}^3(G,\mathcal{E})$ an element which is taking by Φ_* to zero, then there exists a cohomology $(\Gamma'',\lambda'')\colon (g,\psi) \longrightarrow (0,0)$. Consider $\lambda\colon \Gamma \longrightarrow M$ and $\Gamma\colon G\times G \longrightarrow L$ two maps satisfying $\phi_0\lambda = \lambda''$ and $\phi_1\Gamma = \Gamma''$; then, by using the identities CH 1 and CH 2, there is a unique 3-cocycle (g',ψ') in $\mathbb{Z}^3(G,\mathcal{E})$ such that $(\Gamma,\lambda)\colon (g,\psi) \longrightarrow (g',\psi')$ is a cohomology. Moreover the cocycle (g',ψ') also represents γ and it factors through the fiber $\mathcal{F}$ (since $\phi_1 g' = 0$ and $\phi_0\psi' = 0$).

Let us now specialise the general sequence in Proposition 2.1.

Given a pair of groups (A,L), with $A \subseteq L$ a central subgroup, the quotient group L/A acts by conjugation on L and the map

$$\{-,-\}\colon (L/A)\times(L/A) \longrightarrow L;\ \{\bar{\ell},\bar{\ell}'\} = [\ell,\ell']$$

give a canonical reduced 2-crossed module structure in $(L \twoheadrightarrow L/A)$, let us denote the above reduced 2-crossed module by the pair (A,L). On the other hand, if (A',L') is another pair of group in the same conditions than (A,L), any epimorphism of groups $p\colon L \twoheadrightarrow L''$ such that $p(A) \subseteq A''$ induces a surjective morphism of reduced 2-crossed modules

$$\begin{array}{ccc} (A,L) & \twoheadrightarrow & (A'',L'') \\ \vdots & & \vdots \\ L & \overset{p}{\twoheadrightarrow} & L'' \\ \downarrow & & \downarrow \\ L/A & \overset{\bar{p}}{\twoheadrightarrow} & L''/A'' \end{array}$$

whose kernel and fibre crossed modules are

$$\mathcal{K} = (L' \longrightarrow L/A) \text{ and } \mathcal{F} = (L' \longrightarrow p^{-1}(A'')/A)$$

respectively, where $L' = \mathrm{Ker}(p)$. Then, specialising the sequence in Proposition 2.1 we obtain an exact sequence

$$\begin{aligned} * &\longrightarrow \mathrm{Hom}_{Gp}(G,L') \longrightarrow \mathrm{Hom}_{Gp}(G,L) \longrightarrow \mathrm{Hom}_{Gp}(G,L/L') \\ &\longrightarrow \mathbb{H}^2(G,\mathcal{K}) \longrightarrow \mathbb{H}^2\big(G,(A,L)\big) \\ &\longrightarrow \mathbb{H}^2\big(G,(A'',L'')\big) \longrightarrow \mathbb{H}^3(G,\mathcal{F}) \\ &\longrightarrow \mathbb{H}^3\big(G,(A,L)\big) \longrightarrow \mathbb{H}^3\big(G,(A'',L'')\big), \end{aligned}$$

where $\mathrm{Hom}_{Gp}(-,-)$ denotes the hom-group functor.

But the inclusions $(A,A) \hookrightarrow (A,L)$ and $(A'',A'') \hookrightarrow (A'',L'')$ induce weak equivalences of simplicial groups when the functors $\mathcal{J}$ and $\mathcal{J}_2$ are applied. Therefore they induce natural isomorphisms

$$\mathbb{H}^i\big(G,(A,L)\big) \cong \mathbb{H}^i(G,A) \cong H^i(G,A) \text{ and}$$
$$\mathbb{H}^i\big(G,(A'',L'')\big) \cong \mathbb{H}^i(G,A'') \cong H^i(G,A''),$$

for $i = 2,3$, and then the above sequence can be written

$$* \longrightarrow \mathrm{Hom}_{Gp}(G,L') \longrightarrow \mathrm{Hom}_{Gp}(G,L) \longrightarrow \mathrm{Hom}_{Gp}(G,L/L') \longrightarrow \mathbb{H}^2(G,\mathcal{K})$$
$$\longrightarrow H^2(G,A) \longrightarrow H^2(G,A'') \longrightarrow \mathbb{H}^3(G,\mathcal{F}) \longrightarrow H^3(G,A) \longrightarrow H(G^3,A'').$$

An interesting special case of this last sequence appears when $A = Z(L)$, $A'' = Z(L'')$ (the centers of L and L'' respectively) and p is any epimorphism of groups. In this case

$$(A,L) = \big(L \twoheadrightarrow \mathrm{Int}(L)\big),$$
$$(A'',L'') = \big(L'' \twoheadrightarrow \mathrm{Int}(L'')\big),$$
$$\mathcal{K} = \big(L' \rightarrow \mathrm{Int}(L)\big) \text{ and}$$
$$\mathcal{F} = \big(L' \twoheadrightarrow p^{-1}\big(Z(L'')\big)/Z(L)\big).$$

Consequently, associated to an epimorphism of groups $p\colon L \twoheadrightarrow L''$ with kernel L' and the zero morphism from G to $\mathrm{Int}(L)$, we have a 9-term exact sequence of pointed sets

$$* \longrightarrow \mathrm{Hom}_{Gp}(G,L') \longrightarrow \mathrm{Hom}_{Gp}(G,L) \longrightarrow \mathrm{Hom}_{Gp}(G,L/L')$$
$$\longrightarrow \mathbb{H}^2(G,\mathcal{K}) \longrightarrow H^2\big(G,Z(L)\big) \longrightarrow H^2\big(G,Z(L'')\big)$$
$$\longrightarrow \mathbb{H}^3(G,\mathcal{F}) \longrightarrow H^3\big(G,Z(L)\big) \longrightarrow H^3\big(G,Z(L'')\big).$$

Note that $\mathbb{H}^3(G,\mathcal{F})$ is also an abelian group; the reason is that $\mathbb{H}^3(G,\mathcal{F}) \cong [F.,\mathcal{J}_2(\mathcal{F})]$ and $\mathcal{J}_2(\mathcal{F})$ is an infinite-loop simplicial group, since $\mathcal{F}$ is a stable crossed module in the sense of Conduché [13]. It is not difficult to see that the above sequence is an exact sequence of abelian groups at the last five terms.

3. **$\mathbb{H}^3$ and 2-dimensional non-abelian torsors and extensions.** The simplicial group $\mathcal{J}_2(\mathcal{E})$, associated to a reduced 2-crossed module $\mathcal{E} = (L \to M, \{-,-\})$, is a 2-hypergroupoid in Glenn's sense [22]. Now, since for any group G there is a natural isomorphism

$$\mathbb{H}^3(G,\mathcal{E}) \cong [\mathbb{G}.(G), \mathcal{J}_2(\mathcal{E})],$$

where $\mathbb{G}.(G)$ is the standard cotriple resolution of G, we have that $\mathbb{H}^3(-,-) \cong H^2_{\mathbb{G}}\big(-,\mathcal{J}_2(-)\big)$ is a particular case of the cohomology with coefficients in a 2-hypergroupoid, which was studied in [11]. In that paper it was shown how the cohomology set $H^2_{\mathbb{G}}\big(G,\mathcal{J}_2(\mathcal{E})\big)$ classifies 2-torsors (in Glenn's sense) under G over the 2-hypergroupoid $\mathcal{J}^2(\mathcal{E})$. Such torsors correspond, in this case, to diagrams of truncated simplicial groups

$$\begin{array}{ccccccc} \Delta_2 & \substack{\to\\ \to\\ \to} & E_1 & \rightrightarrows & E_0 & \twoheadrightarrow & G \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow & & \\ (L \rtimes M) \rtimes M & \substack{\to\\ \to\\ \to} & M & \rightrightarrows & 0 & & \end{array} \tag{4}$$

where Δ_2 and G are the simplicial kernel and the coequalizer, respectively, of the pair of morphisms $E_1 \rightrightarrows E_0$, and α_2 restricts to an isomorphism between the corresponding groups in the Moore complexes. We can then say that the Moore complex of the above diagram is of the form

$$(5)\qquad \begin{array}{ccccccc} \boldsymbol{E}: L & \overset{i}{\hookrightarrow} & E & \overset{\delta}{\longrightarrow} & F & \overset{p}{\twoheadrightarrow} & G \\ \| & & \downarrow{\scriptstyle \alpha} & & & & \\ L & \overset{\rho}{\longrightarrow} & M & & & & \end{array}$$

with $F = E_0$, $E = \mathrm{Ker}(d_0\colon E_1 \longrightarrow E_0)$, α the restriction of α_1 and the top row an exact sequence of groups. In this last diagram (5), E is an F-group, with action

$$^{x}y = s_0(x) + y - s_0(x),$$

δ is an F-group morphism, where F is considered as an F-group by conjugation, and moreover α satisfies the following conditions:

E1) $\alpha(^{x}y) = a(y)$,

E2) $\{\alpha(y), \alpha(z)\} = y + z - y -^{\delta(y)} z$,

for all $x \in F$ and $y, z \in E$ (we have identified L with its image in E). Conversely, a diagram as in (5), with E an F-group, δ an F-group morphism and satisfying E1) and E2) determines, up to isomorphism, a 2-torsor as in (4), with $E_0 = F$, $E_1 = E \rtimes F$, d_0 and $d_1\colon E_1 \longrightarrow E_0$ given by:

$$d_0(y, x) = x,$$
$$d_1(y, x) = \delta_1(y) + x,$$

$s_0\colon E_0 \longrightarrow E_1$ by $s_0(x) = (0, x)$, $\alpha_1\colon E_1 \longrightarrow M$ by $\alpha_1(y, x) = \alpha_1(y)$ and $\alpha_2\colon \Delta_2 \longrightarrow (L \rtimes M) \rtimes M$ by $\alpha_2(x_0, x_1, x_2) = (x_2 - s_0 d_0 x_0 + x_0 - x_1, \alpha_1 x_1 - \alpha_1 x_0, \alpha_1 x_0)$.

So we define a 2-*extension* of the reduced 2-crossed module $\mathcal{E}$ by the group G as a commutative diagram as in (5) with the top row exact, E an F-group, δ an F-group morphism and α a group morphism satisfying E1 and E2. We have then:

PROPOSITION 3.1. *For any group G and any reduced 2-crossed module $\mathcal{E}$ the cohomology set $\mathbb{H}^3(G, \mathcal{E})$ classifies 2-extensions of $\mathcal{E}$ by G.*

PROOF. The proof of this proposition is an immediate consequence of [12, Theorem 2.18]. Nevertheless we give here a brief description of the correspondence between 2-extensions and cocycles.

Suppose $\boldsymbol{E}$ is a 2-extension of $\mathcal{E}$ by G and consider, for any $x \in G$, an element $u(x) \in F$ such that $p\big(u(x)\big) = x\big(u(0) = 0\big)$. The element $u(x)+u(y)-u(x+y)$ is in $\mathrm{Ker}(p) = \mathrm{Im}(\delta)$, for all $x, y \in G$, and therefore there exists $v(x, y) \in E$ such that

$$\delta\left(v(x, y)\right) = u(x) + u(y) - u(x + y)$$

with $v(0, x) = 0 = v(x, 0)$. Then, for any $x, y, z \in G$, the element

$$f(x, y, z) = {}^{u(x)}v(y, z) + v(x, y + z) - v(x + y, z) - v(x, y)$$

is in $\mathrm{Ker}(\delta) = L$, and so we have a map $f: G \times G \times G \longrightarrow L$ which, together with $\varphi: G \times G \longrightarrow M$ defined by $\varphi(x, y) = \alpha\big(v(x, y)\big)$, form a cocycle in $\mathbb{Z}^3_N(G, \mathcal{E})$. Note that this cocycle (f, φ) corresponds by the bijection

$$\mathbb{Z}^3_N(G, \mathcal{E}) \cong \mathrm{Simpl}\, Gp\big(F.\,, \mathcal{J}_2(\mathcal{E})\big)$$

to the simplicial morphism obtained from the composition

$$\begin{array}{ccccccc}
F_3 & \substack{\to\\ \to\\ \to} & F_1 & \rightrightarrows & F_0 & \twoheadrightarrow & G \\
\big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_0} & & \| \\
\Delta_2 & \substack{\to\\ \to\\ \to} & E \rtimes F & \rightrightarrows & F & \twoheadrightarrow & G \\
\big\downarrow{\scriptstyle \alpha_2} & & \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow & & \\
(L \rtimes M) \rtimes M & \substack{\to\\ \to\\ \to} & M & \rightrightarrows & 0 & &
\end{array}$$

where $f_0(\eta_x) = u(x)$ and $f_1(\mu_{x,y}) = \big(v(x, y), 0\big)$. Since two liftings of the 1_G, to $F.$, are homotopic, the class of (f, φ) in $\mathbb{H}^3(G, \mathcal{E})$ does not depend of the choice of the maps u and v.

Conversely, given a cocycle $(f, \varphi) \in \mathbb{Z}^3_N(G, \mathcal{E})$, its class in $\mathbb{H}^3(G, \mathcal{E})$ can be realized in the above sense by the extension

$$\begin{array}{ccccccc}
L & \overset{i}{\hookrightarrow} & L \rtimes K & \overset{\delta}{\longrightarrow} & F_0 & \overset{p}{\twoheadrightarrow} & G \\
\| & & \big\downarrow{\scriptstyle \alpha} & & & & \\
L & \overset{\rho}{\longrightarrow} & M & & & &
\end{array}$$

where F_0 is the free group based on $\{\eta_x : x \in G\}$, $\eta_0 = 0$, the morphism p is determined by $p(\eta_x) = x$, K is the kernel of p (therefore K is free with base the elements $\eta_x + \eta_y - \eta_{x+y}$ of F_0), $L \rtimes K$ is the semidirect product group, where K acts on L via the group morphism

$$\theta: K \longrightarrow M : \theta(\eta_x + \eta_y - \eta_{x+y}) = \varphi(x, y),$$

and the morphism $\alpha: L \rtimes K \longrightarrow M$ is given by $\alpha(\ell, k) = \rho(\ell) + \theta(k)$. Finally, to describe the action of F_0 on $L \rtimes K$, note that there is a unique map $\beta: F_0 \times K \longrightarrow L$ satisfying:

i) $\beta(\eta_x, \eta_y + \eta_z - \eta_{y+z}) = f(x, y, z)$,

ii) $\beta(0, k) = 0 = \beta(w, 0)$,

iii) $\beta(w + w, k) = \beta(w', k) + \beta(w, w' + k - w')$,

iv) $\beta(w, k + k') = \beta(w, k) +^{w-k+w} \beta(w, k')$,

for all $x, y, z \in G$, $w, w' \in F_0$ and $k, k' \in K$. This map β also satisfies:

$$\rho\beta(w, k) = \theta(k) - \theta(w + k - w).$$

Then, the action of F_0 on $L \rtimes K$ is given by

$${}^{w}(\ell, k) = (\ell + \beta(w, k),\ w + k - w).$$

Note that for any abelian group A, the concept of 2-extension of the reduced 2-crossed module $A \longrightarrow 0$ by the group G is just the classical concept of 2-fold extension of A by G and so the above Proposition 3.1 gives just the well known interpretation of $H^3(G, A)$ in terms of 2-fold extensions [21], [33], [25], [26] or [27].

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4. **Non-abelian cohomology and homotopy classification of continuous maps.** In this section all spaces are pointed and path connected CW-complexes.

Given spaces X and Y, a fundamental problem in homotopy theory is to find algebraic methods for classifying the set $[X, Y]$ of homotopy classes of continuous maps from X to Y. Eilenberg-Mac Lane cohomology gives an apropriate solution for aspherical spaces X and spaces Y with a unique non-trivial homotopy group, at dimension ≥ 2; in this case if $\Pi = \Pi_1(X)$ is the fundamental group of X and $A = \Pi_n(Y)$ is the unique non-trivial homotopy group of Y, $n \geq 2$, Eilenberg-Mac Lane's classification theorem states the existence of a natural isomorphism

$$[X, Y] \cong H^n(\Pi, A).$$

The main goal of this section is to show how this theorem can be generalized for $n = 2$ and $n = 3$, by using the non-abelian cohomology groups $\mathbb{H}^n$, to spaces Y with $\Pi_1(Y) = 0$ for all $i \neq n, n-1$.

The fact (proved by Mac Lane-Whitehead in [35]) that crossed modules are adequate algebraic models for the homotopy types of spaces Y with $\Pi_1(Y) = 0$, for all $i \neq 2, 1$, makes our classification theorem easy for n=2. Recall that the *classifying space* $B(\mathcal{E})$ of a crossed module $\mathcal{E} = (L \longrightarrow M)$ is defined as the classifying space of the associated simplicial group $\mathcal{J}(\mathcal{E})$, i.e. $B(\mathcal{E}) = \left|\bar{W}\big(\mathcal{J}(\mathcal{E})\big)\right|$ the geometric realization of the simplicial set $\bar{W}\big(\mathcal{J}(\mathcal{E})\big)$ (see [32] or [35]); this space is a pointed and path connected CW-complex whose homotopy invariants are determined by $\mathcal{E}$ (for example $\Pi_1 B(\mathcal{E}) = 0$, for $i \geq 3$, $\Pi_2 B(\mathcal{E}) = \operatorname{Ker}(\rho)$ and $\Pi_1 B(\mathcal{E}) = \operatorname{Coker}(\rho)$). On the other hand, any space Y has associated a *fundamental crossed module*

$$\mathcal{E}(Y) = \Big(\Pi_2\big(Y, \mathrm{Sk}^1(Y)\big) \xrightarrow{\rho} \Pi_1\big(\mathrm{Sk}^1(Y)\big)\Big),$$

where $\mathrm{Sk}^1(Y)$ is the one dimensional skeleton of Y, ρ is the boundary map and the action of $\Pi_1\big(\mathrm{Sk}^1(Y)\big)$ on $\Pi_2\big(Y, \mathrm{Sk}^1(Y)\big)$ is the standard one, in such a way that Y has the homotopy type of $B\mathcal{E}(Y)$ if and only if $\Pi_i(Y) = 0$ for all $i \neq 1, 2$. This fact is essentially given in [35].

A space Y is said of the *homotopy type* of a crossed module $\mathcal{E}$ if it is homotopically equivalent to $B(\mathcal{E})$.

PROPOSITION 4.1 (CLASSIFICATION THEOREM). *Let X be an aspherical space with fundamental group Π and let Y be a space with the homotopy type of a crossed module $\mathcal{E}$. Then there is a natural bijection*

$$[X, Y] \cong \mathbb{H}^2(\Pi, \mathcal{E}).$$

PROOF. The proof is an immediate consequence of the homotopy representation theorem for Dedecker $\mathbb{H}^2$ since

$$\begin{aligned}\mathbb{H}^2(\Pi, \mathcal{E}) &\cong \big[G\big(K(\Pi, 1)\big), \mathcal{J}(\mathcal{E})\big] \cong \mathrm{SimplSet}[K(\Pi, 1), \bar{W}\mathcal{J}(\mathcal{E})] \\ &\cong [|K(\Pi, 1)|, B(\mathcal{E})] \cong [X, Y].\end{aligned}$$

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(An alternative proof can be deduced from results in [7] and [8]).

To extend the above Proposition 4.1 to $n = 3$ we first have to obtain adequate algebraic models for the homotopy types of spaces Y with $\Pi_i(Y) = 0, i \neq 3, 2$. In fact we are going to prove that reduced 2-crossed modules can be used as such models.

If $\mathcal{E} = (L \xrightarrow{\rho} M, \{-,-\})$ is a reduced 2-crossed module, its *classifying space* $B_2(\mathcal{E})$ is defined as the classifying space of the simplicial group $\mathcal{J}_2(\mathcal{E})$, i.e. $B_2(\mathcal{E}) = \left|\bar{W}\big(\mathcal{J}_2(\mathcal{E})\big)\right|$; this space is a pointed and path connected CW-complex with trivial homotopy groups at dimensions other than 2 and 3 (moreover $\Pi_2\big(B_2(\mathcal{E})\big) = \mathrm{Coker}(\rho)$ and $\Pi_3\big(B_2(\mathcal{E})\big) = \mathrm{Ker}(\rho)\big)$.

A space Y is said to be of the *homotopy type of a reduced 2-crossed module* $\mathcal{E}$ if it is homotopically equivalent to $B_2(\mathcal{E})$.

PROPOSITION 4.2. *A space Y has the homotopy type of a reduced 2-crossed module if and only if* $\Pi_i(Y) = 0$, $i \neq 3, 2$.

PROOF. Let Y be a space with $\Pi_i(Y) = 0$, $i \neq 3, 2$. To find a reduced 2-crossed module, $\mathcal{E}_2(Y)$, whose classifying space is homotopically equivalent to Y, let us consider the simplicial group $G. = GS_*(Y)$ (i.e. Kan's simplicial group of the reduced singular complex of Y). This simplicial group has $\Pi_i(G.) = \Pi_{i+1}(Y)$ for all $i \geq 0$, and the canonical morphism $Y \longrightarrow |\bar{W}(G.)|$ is a homotopy equivalence. Let $K.$ be the largest simplicial subgroup of $G.$ whose 2-simplices are in $\delta_3(\tilde{G}_3)$. Then $K.$ has $K_0 = 0 = K_1$, $K_2 = \delta_3(\tilde{G}_3) \subseteq G_2$, and inductively $K_{n+1} = \{x \in G : d_i(x) \in K_n,\ 0 \leq i \leq n+1\}$, for $n \geq 2$. Since $\delta_3(\tilde{G}_3)$ is a normal subgroup of G_2, we have that $K.$ is a normal simplicial subgroup of $G.$. Moreover the Moore complex of $K.$ is

$$\cdots \longrightarrow \tilde{G}_n \longrightarrow \tilde{G}_{n-1} \longrightarrow \cdots \tilde{G}_3 \longrightarrow \delta_3(\tilde{G}_3) \longrightarrow 0 \longrightarrow 0.$$

Then the quotient simplicial group G./K. has Moore complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots 0 \longrightarrow \tilde{G}_2/\delta_3(\tilde{G}_3) \xrightarrow{\delta_2} \tilde{G}_1 \xrightarrow{\delta_1} G_0,$$

and therefore the projection $G. \twoheadrightarrow G./K.$ is a weak equivalence. Consequently, Y is homotopy equivalent to $|\bar{W}(G./K.)|$.

Consider now $P.(Y)$ the simplicial subgroup of $G./K.$ whose simplices have all their 0-faces equal to zero, i.e. $P_0(Y) = 0$ and $P_{n+1}(Y) = \{x \in G_{n+1}/K_{n+1} : d_i(x) \in P_n(Y)\}$, for $n \geq 0$. Its Moore complex is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots 0 \longrightarrow \tilde{G}_2/\delta_3(\tilde{G}_3) \longrightarrow \mathrm{Ker}(\delta_1) \longrightarrow 0,$$

so $P.(Y)$ is an object in $\mathrm{Simpl}_2\, Gp$ and

$$\mathcal{E}_2(Y) = \mathcal{N}_2\big(P.(Y)\big) = \big(\tilde{G}_2/\delta_3(\tilde{G}_3) \longrightarrow \mathrm{Ker}(\delta_1)\big)$$

is a reduced 2-crossed module (with $\mathcal{J}_2\mathcal{E}_2(Y) = P.(Y)$). Moreover the inclusion $P.(Y) \hookrightarrow G./K.$ is a weak equivalence and therefore Y is homotopy equivalent to $|\bar{W}P.(Y)| = B_2\mathcal{E}_2(Y)$.

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Note that the construction of $\mathcal{E}(Y)$ (the fundamental reduced 2-crossed module of Y) is functorial. Thus $\mathcal{E}_2(-)$ is a functor from the category of spaces to the category of reduced 2-crossed modules.

The *homology groups* H_i, $i = 0, 1$, of a reduced 2-crossed module $\mathcal{E} = (L \xrightarrow{\rho} M)$ are defined to be the homology groups of its underlying crossed module (i.e. $H_0(\mathcal{E}) = \mathrm{Coker}(\rho)$ and $H_1(\mathcal{E}) = \mathrm{Ker}(\rho)$). A morphism of reduced 2-crossed modules is called a *weak equivalence* if it is a weak equivalence as a morphism of crossed modules (i.e. it induces isomorphisms between the homology groups). Clearly weak equivalences correspond to weak equivalences by the above functors $\mathcal{E}_2$ and B_2 and so the above Proposition 4.2. can be rewritten as follows:

PROPOSITION 4.3. *The functors $\mathcal{E}_2$ and $\mathcal{B}_2$ induce an equivalence between the homotopy category of pointed and connected CW-complexes with trivial homotopy groups at dimensions other than 2 and 3 and the homotopy category of reduced 2-crossed modules (i.e. the corresponding category of fractions where all weak equivalences have being inverted).*

PROPOSITION 4.4 (CLASSIFICATION THEOREM). *Let X be an aspherical space with fundamental group Π and let Y be a space with the homotopy type of a reduced 2-crossed module $\mathcal{E}$. Then there is a natural bijection*

$$[X, Y] \cong \mathbb{H}^3(\Pi, \mathcal{E}).$$

PROOF. Using the homotopy representation theorem for $\mathbb{H}^3$ (Proposition 1.3) we have

$$\begin{aligned}\mathbb{H}^3(\Pi, \mathcal{E}) &\cong \big[G\big(K(\Pi, 1)\big), \mathcal{J}_2(\mathcal{E})\big] \cong \mathrm{SimplSet}[K(\Pi, 1), \bar{W}\mathcal{J}_2(\mathcal{E})] \\ &\cong \big[\big|K(\Pi, 1)\big|, B_2(\mathcal{E})\big] \cong [X, Y].\end{aligned}$$

5. $\mathbb{H}^3$ **and other 3-dimensional cohomologies for groups.** As we said at the end of Section 1, our cohomology $\mathbb{H}^3$ coincides with the classical Eilenberg-Mac Lane cohomology when the coefficients are abelian groups. In this section, we are going to see how this cohomology $\mathbb{H}^3$ is related to other two already established cohomology theories: Dedecker thick cohomology $\mathbb{H}^3$ whose coefficients are super crossed groups, [17], and Ulbrich's cohomology with coefficients in Picard categories, [39].

Let us start with Ulbrich cohomology $\mathcal{H}^3$.

Fröhlich and Wall, see [20], defined a cohomology with coefficients in graded categories. Later on Ulbrich, [39], specialized this cohomology to Picard categories and obtained interesting applications of it. Let us see now that our cohomology $\mathbb{H}^3(G, \mathcal{E})$, under certain restrictions on the coefficients $\mathcal{E}$, coincides with Ulbrich's cohomology at dimension three of G with coefficients in a Picard category with a trivial G-module structure.

The first step is then to find appropriate conditions on the reduced 2-crossed module $\mathcal{E}$ to be able to associate to it a symmetric Picard category. These conditions were pointed out by Conduché [13], and they reduce to the following condition on the map $\{-,-\}$:

$$\{m', m\} = -\{m, m'\}, \text{ for all } m, m' \in M.$$

A reduced 2-crossed module $\mathcal{E}$, in which the map $\{-,-\}$ verifies the above identity is called a *stable crossed module*. Let then $\mathcal{E} = (L \xrightarrow{\rho} M)$ be a stable crossed module. By the equivalence between the category of crossed modules and the category of internal categories in Groups, [9], the crossed module $\mathcal{E}$ has associated an internal category in Groups $\mathcal{A}(\mathcal{E})$, whose groups of objects and arrows are M and $L \rtimes M$ respectively, the source and target of an arrow (ℓ, m) are m and $\rho(\ell)+m$ respectively, and the composition is given by

$$(\ell, m) \circ (\ell', \rho(\ell) + m) = (\ell' + \ell, m);$$

this category $\mathcal{A}(\mathcal{E})$ has then a coherent group structure in Ulbrich's sense, with all coherent isomorphism identities. Moreover the map $\{-,-\}$ induces coherent isomorphisms for commutativity, by taking, $c_{m,m'} = (\{m, m'\}, m' + m)$. So $\mathcal{A}(\mathcal{E})$ is a symmetric Picard category and the correspondence $\mathcal{E} \longmapsto \mathcal{A}(\mathcal{E})$ is functorial. Considering then $\mathcal{A}(\mathcal{E})$ with the trivial G-module structure we will see here that there is a natural bijection $\mathbb{H}^3(G, \mathcal{E}) \cong \mathcal{H}^3\big(G, \mathcal{A}(\mathcal{E})\big)$, where $\mathcal{H}$ denotes Ulbrich's cohomology.

Let us now specialize Ulbrich's definition of $\mathcal{H}^3$ to the symmetric Picard category with trivial G-module structure $\mathcal{A}(\mathcal{E})$:

The abelian group $\mathcal{H}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ is defined as the quotient group $Z^3\big(G, \mathcal{A}(\mathcal{E})\big)/B^3\big(G, \mathcal{A}(\mathcal{E})\big)$, where $Z^3\big(G, \mathcal{A}(\mathcal{E})\big)$ is the group of connected components (isomorphism classes) of the Picard category $\mathcal{Z}^3\big(G, \mathcal{A}(\mathcal{E})\big)$. This category has as objects "3-cocycles", i.e. pairs $(P_{x,y}, \sigma_{x,y,z})$ where $P_{x,y}, x, y \in G$, is a family of objects of $\mathcal{A}(\mathcal{E})$, that is a family of elements of M, and $\sigma_{x,y,z}$ is a family of morphisms in $\mathcal{A}(\mathcal{E})$ of the form $\sigma_{x,y,z}\colon P_{y,z} + P_{x,y+z} \longrightarrow P_{x,y} + P_{x+y,z}$, such that the following diagram

(6)

$$\begin{array}{ccccc}
P_{z,t} + P_{y,z+t} + P_{x,y+z+t} & \xrightarrow{\mathrm{Id}+\sigma_{x,y,z+t}} & P_{z,t} + P_{x,y} + P_{x+y,z+t} & \xrightarrow{C+\mathrm{Id}} & P_{x,y} + P_{z,t} + P_{x+y,z+t} \\
\downarrow{\scriptstyle \sigma_{y,z,t}+\mathrm{Id}} & & & & \downarrow{\scriptstyle \mathrm{Id}+\sigma_{x+y,z,t}} \\
P_{y,z} + P_{y+z,t} + P_{x,y+z+t} & \xrightarrow{\mathrm{Id}+\sigma_{x,y+z,t}} & P_{y,z} + P_{x,y+z} + P_{x+y+z,t} & \xrightarrow{\sigma_{x,y,z}+\mathrm{Id}} & P_{x,y} + P_{x+y,z} + P_{z+y+z,t}
\end{array}$$

is commutative for all $x, y, z, \in G$. An arrow $\nu\colon (P, \sigma) \longrightarrow (Q, \tau)$ in $\mathcal{Z}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ is a family of arrows $\nu_{x,y}\colon P_{x,y} \longrightarrow Q_{x,y}$, making the diagram

$$\begin{array}{ccc}
P_{y,z} + P_{x,y+z} & \xrightarrow{\sigma_{x,y,z}} & P_{x,y} + P_{x+y,z} \\
{\scriptstyle \nu_{y,z}+\nu_{x,y+z}}\downarrow & & \downarrow{\scriptstyle \nu_{x,y}+\nu_{x+y,z}} \\
Q_{y,z} + Q_{x,y+z} & \xrightarrow{\tau_{x,y,z}} & Q_{x,y} + Q_{x+y,z}
\end{array}$$

commutative, for all $x, y, z \in G$.

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Any family (λ_x), $x \in G$, of objects in $\mathcal{A}(\mathcal{E})$, defines a 3-cocycle by taking the pair $(\lambda_x + \lambda_y - \lambda_{x+y}, \chi_{x,y,z})$, $x, y, z \in G$, where $\chi_{x,y,z}$ is the morphism in $\mathcal{A}(\mathcal{E})$ given by $\chi_{x,y,z} = (\{\lambda_y + \lambda_z - \lambda_{y+z}, \lambda_x\}, \lambda_x + \lambda_y + \lambda_z - \lambda_{x+y+z})$. The classes in $Z^3\big(G, \mathcal{A}(\mathcal{E})\big)$ of such cocycles form the subgroup $B^3\big(G, \mathcal{A}(\mathcal{E})\big)$.

Now, given a cocycle $(P_{x,y}, \sigma_{x,y,z})$, the arrows $\sigma_{x,y,z}$ are elements in $L \rtimes M$ necessarily in the form $\big(f(x,y,z), P_{x,y} + P_{x+y,z}\big)$, since its target is $P_{y+z} + P_{x,y+z}$. Moreover the source of $\sigma_{x,y,z}$ is $P_{y+z} + P_{x,y+z}$ and therefore we have

$$\rho f(x,y,z) + P_{x,y} + P_{x+z,z} = P_{y+z} + P_{x,y+z}.$$

On the other hand, the commutativity of the above diagram (6) is equivalent to the identity

$$\begin{aligned} &{}^{P_{z,t}}f(x,y,z+t) + \{P_{z,t}, P_{x,y}\} + {}^{P_{x,y}}f(x+y,z,t) \\ &\quad = f(y,z,t) + {}^{P_{y,z}}f(x,y+z,t) + f(x,y,z) \end{aligned}$$

or equivalently

$$\begin{aligned} \{P_{x,y}, P_{z,t}\} = {}&{}^{P_{x,y}}f(x+y,z,t) - f(x,y,z) - {}^{P_{y,z}}f(x,y+z,t) \\ &- f(y,z,t) + {}^{P_{z,t}}f(x,y,z+t). \end{aligned}$$

Therefore the correspondence $(P_{x,y}, \sigma_{x,y,z}) \longmapsto (f, \varphi)$, where $\varphi(x,y) = P_{x,y}$, gives a natural bijection between the set of objects of $\mathcal{Z}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ and $\mathbb{Z}^3(G, \mathcal{E})$. Also it is easy to see that a morphism $\nu: (P, \sigma) \longrightarrow (Q, \tau)$ is equivalent to a map $\Gamma: G \times G \longrightarrow L$ verifying:

$$g(x,y,z) = {}^{\psi(y,z)}\Gamma_{x,y+z} + \Gamma_{y,z} + f(x,y,z) - \Gamma_{x,y} - {}^{\psi(x,y)}\Gamma_{x+y,z},$$

where (g, ψ) corresponds by the above bijection to (Q, τ). So we have a Picard category (whose group structure is denoted by +) isomorphic to $\mathcal{Z}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ and whose objects are the elements of $\mathbb{Z}^3(G, \mathcal{E})$. Moreover, the elements of $B^3\big(G, \mathcal{A}(\mathcal{E})\big)$ correspond to isomorphism classes of the cocycles $\lambda = (\varphi_\lambda, f_\lambda)$ in $\mathbb{Z}^3(G, \mathcal{E})$ determined by maps $\lambda: G \longrightarrow L$, by

$$\varphi_\lambda(x,y) = \lambda_x + \lambda_y - \lambda_{x+y} \text{ and } f_\lambda(x,y,z) = \{\lambda_y + \lambda_z - \lambda_{y+z}, \lambda_x\}.$$

and the cocycle $\lambda + (f, \varphi)$ is isomorphic to (g, ψ), where

$$\begin{aligned} \psi(x,y) &= \lambda(x) + \lambda(y) - \lambda(x+y) + \varphi(x,y), \\ g(x,y,z) &= -\{\lambda(x), \psi(y,z)\} - \{\varphi(y,z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(y+z)\} \\ &\quad - {}^{\varphi(y,z)}\{\varphi(x,y+z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(x+y+z)\} + f(x,y,z) \\ &\quad + {}^{\varphi(x,y)}\{\varphi(x+y,z), \lambda(x) + \lambda(y) + \lambda(z) - \lambda(x+y+z)\} \\ &\quad + \{\varphi(x,y), \lambda(x) + \lambda(y) - \lambda(x+y)\}. \end{aligned}$$

So two 3-cocycles (f, φ) and (g, ψ) correspond to 3-cocycles in $\mathcal{Z}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ which gives the same classes in $\mathcal{H}^3\big(G, \mathcal{A}(\mathcal{E})\big)$ if and only if there exists a map $\lambda: G \longrightarrow M$ such that $\lambda + (f, \varphi)$ and (g, ψ) are isomorphic, i.e. if and only if there are maps $\lambda: G \longrightarrow M$ and $\Gamma: G \times G \longrightarrow L$ in such a way that Γ gives an arrow from (g, ψ) to $\lambda + (f, \varphi)$ and this is equivalent to the conditions CH1 and CH2. Thus we have:

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PROPOSITION 5.1. *For any group G and any stable crossed module $\mathcal{E}$, there is a natural bijection*

$$\mathcal{H}^3\big(G, \mathcal{A}(\mathcal{E})\big) \cong \mathbb{H}^3(G, \mathcal{E}).$$

Now let us compare with Dedecker cohomology $\mathbb{H}^3$.

The coefficients for $\mathbb{H}^3$ are called *super crossed groups*. A super crossed group $\mathbb{A}$ consists of a commutative diagram of group homomorphisms

$$\begin{array}{ccc} H & \xrightarrow{\rho'} & \Lambda \\ {\scriptstyle k}\downarrow & \searrow_{\rho} & \downarrow \\ E & \xrightarrow[\theta]{} & \Pi \end{array}$$

where $\Lambda \subseteq \Pi$ is a normal subgroup, together with actions of Π on H and E and a map $\Delta: E \longrightarrow S^1(\Lambda, H)$, from E to the set $S^1(\Lambda, H)$ of inverse crossed homomorphisms from Λ to H (i.e. maps $d: \Lambda \longrightarrow H$ satisfying $d(\lambda + \lambda') =^{\lambda} d(\lambda') + d(\lambda)$, note that Π acts on $S^1(\Lambda, H)$ by

$$({}^{x}d)(\lambda) = {}^{x}\big(d(-x + \lambda + x)\big)$$

for any $x \in \Pi$, $\lambda \in \Lambda$ and $d \in S^1(\Lambda, H)$), such that:

1. $(H \xrightarrow{\rho} \Pi)$ and $(E \xrightarrow{\theta} \Pi)$ are crossed modules and the pair of homomorphisms $(k, 1_{\Pi})$ is a crossed module morphism.
2. Δ is an equivariant map and the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{k} & E \\ & {\scriptstyle \partial}\searrow & \downarrow{\scriptstyle \Delta} \\ & & S^1(\Lambda, H) \end{array}$$

   where $\partial(h)(\lambda) = h -^{\lambda} h$, $\lambda \in \Lambda$ and $h \in H$.
3. For any $\lambda \in \Lambda$ and $e \in E$, $\rho\big(\Delta(e)(\lambda)\big) = [\lambda, \theta(e)]$.

These coefficients arose when Dedecker, [17], tried to give a constructive definition of non-abelian cohomology at dimension 3, with the main propose of extending his 6-terms exact sequence and so give a measure of the obstructions to lifting 2-cocycles. This layout led Dedecker to the above systems whose only motivation is that they *probably* work.

Given a reduced 2-crossed module $\mathcal{E} = (L \to M, \{-,-\})$, the diagram

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & \searrow & \| \\ M & = & M \end{array}$$

together with the action of M on L and the map

$$\Delta: M \longrightarrow S^1(M, L);\; x \longmapsto \Delta(x): M \longrightarrow L$$
$$y \longmapsto \{y, x\}$$

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is a super crossed group $\mathbb{A}(\mathcal{E})$. It is not difficult to observe that with an appropriate definition of super crossed group morphism $\mathbb{A}(-)$ defines a functor from the category 2-$\mathcal{XM}_{\text{red}}$ to the corresponding category of super crossed groups (note that not all super crossed group with $\theta = 1_{\Pi}$, $\Lambda = \Pi$ and $\rho = \rho' = k$ is equal to $\mathbb{A}(\mathcal{E})$, for some reduced 2-crossed module $\mathcal{E}$).

We are not going to write here a complete description of a Dedecker's 3-cocycle, just let us say that a Dedecker's 3-cocycle of a group G with coefficients in the super crossed module $\mathbb{A}(\mathcal{E})$ consists of a 5-tuple of maps $(k, \lambda, \phi, K, \eta)$

$$\begin{aligned} k&: G \times G \times G \longrightarrow L, \\ \lambda&: G \times G \longrightarrow M, \\ \phi&: G \longrightarrow M, \\ K&: G \times G \longrightarrow S^1(M, L) \text{ and} \\ \eta&: G \times G \longrightarrow M, \end{aligned}$$

satisfying certain delicate formulas (see [17, 3.20.I to 3.20.VI]). Now given a 3-cocycle (f, φ) of G with coefficients in the reduced 2-crossed module $\mathcal{E}$, it is straightforward to observe that the 5-tuple $(f, -\varphi, 0, K, \varphi)$ gives a Dedecker's 3-cocycle of G with coefficients in $\mathbb{A}(\mathcal{E})$, where $K(x, y)(m) = \{m, \eta(x, y)\}$. This correspondence determines a well defined map

$$\mathbb{H}^3(G, \mathcal{E}) \longrightarrow \mathbb{H}^3\big(G, \mathbb{A}(\mathcal{E})\big)$$

which relates our 3-cohomology with Dedecker's one. It is not clear whether this map is injective or surjective.

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Departamento de Algebra
Facultad de Ciencias, University of Granada
Granada 18075, Spain.