Hence a circle, of radius ρ, surrounding the new origin, becomes a circle of radius $\frac{\rho^2}{2D}$ surrounding the point $\left(0, -\frac{D}{2}\right)$ half-way between the new and old origins. The ϕ of any point in the circle becomes 2ϕ.

Hence the whole surface is opened up like a fan round the new origin, *every* radius through this origin having its inclination to the axis of *y doubled*. Thus the parts of a diameter, on opposite sides of the centre, are brought to coincide; and an infinitely extended line, through the centre, becomes limited at the centre. Thus what was a single sheet becomes duplex, as was said above.

7. It suffices to have indicated, by a partial examination of some of the curious features of a single case, the stores of novelties which are thus easily reached. See especially, for additional materials of the same kind, the investigation in §§ 706-7 of Thomson and Tait's *Natural Philosophy.*

38 George Square, Edinburgh,
15*th January* 1885.

Boole's and other proofs of Fourier's Double-Integral Theorem.

By Peter Alexander, M.A.

In my former paper on Fourier's double-integral I remarked that Poisson's form of the integral gave the same incorrect result as Fourier's form in an example by which I tested it, and seemed subject to the same limitations.

This, I now find, is not the case. The assertion, though quite true of the form

$$f(x) = \mathbf{L}_{\kappa=0} \frac{1}{\pi}\int_0^\infty dq \int_{-\infty}^\infty d\alpha\, \epsilon^{-\kappa q} \cos(qx - q\alpha).f(\alpha)$$

which I then dealt with, and which I understood to be Poisson's formula, having it on the authority of Freeman (*Fourier's Theory of Heat*, p. 351), is not true of the correct form.

Dr Muir having a difficulty in accepting my statement, asked me to reconsider the matter, and kindly referred me to several papers on

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Fourier's theorem. After perusing these I have come to the conclusion that Poisson's formula is

$$f(x) = \mathrm{L}_{\kappa=0} \frac{1}{\pi} \int_{-\infty}^{\infty} da f(a) \int_{0}^{\infty} dq\, \epsilon^{-\kappa q} \cos(qx - qa),$$

and that my remarks do not apply to it. I have in fact tested it by the example I used as a test in my former paper, and find that it stands the test, provided κ be not put equal to 0 immediately after the integration with respect to q, but after the integration with respect to a.

It having been suggested that I might subject to a critical examination the proofs in the papers to which I had been referred, I now proceed to do so.

The first, and in my opinion the most important of these papers, is one by Boole in the *Transactions of the Royal Irish Academy*, vol. xxi., pp. 124-130, entitled "On the Analysis of Discontinuous Functions." The other papers are those by J. W. L. Glaisher and by G(regory ?) referred to in a foot-note to my former paper.

Boole's proof is beautiful and masterly, and every one interested in this subject should read it. But though the result he arrives at is correct, I have found a serious error at a certain point of it. To render intelligible what I have to say, a sketch of the proof must be given. It runs thus :—

"If in the function $\tan^{-1}\frac{a-x}{\kappa}$ we suppose $x < a$ and κ a positive "quantity, then as κ is diminished the limit of the values of the func- " tion will be $\frac{\pi}{2}$. This is evident.

" If $x = a$, the limit is 0, the entire series of values being 0.

" If $x > a$, the limit is $-\frac{\pi}{2}$.

" Let $\Delta f(a) = f(a + \Delta a) - f(a)$. Then

$$\Delta \tan^{-1}\frac{a-x}{\kappa} = \tan^{-1}\frac{a + \Delta a - x}{\kappa} - \tan^{-1}\frac{a-x}{\kappa}, \qquad \text{(I)}$$

and applying what precedes to each term of the second member we find that the limit of their sum is π, or $\frac{\pi}{2}$, or 0, according as x lies be-

" tween a and $a+\Delta a$, or is equal to a or $a+\Delta a$, or lies entirely without " these limits.

" In what follows we shall suppose that κ is thus diminished, so " that by any expression involving κ we shall understand the limit to " which it approaches as κ approaches to 0. Then

$$\frac{1}{\pi}\left(\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa}\right)f(x)=f(x),\text{ or }\frac{f(x)}{2},\text{ or }0,\qquad\text{(II)}$$

" according as x lies between, upon, or without the limits a and $a+\Delta a$."

Thus far the reasoning is unexceptionable, but what now follows is faulty. It runs thus:—

" When Δa becomes infinitesimal it may be replaced by da, and " the symbol Δ by d, whence by (I) and its consequences,

$$d\tan^{-1}\frac{a-x}{\kappa}=\pi,\text{ or }\frac{\pi}{2},\text{ or }0,\qquad\text{(III)}$$

" according as x lies between, upon, or without the limits a and $a+da$.

" Effecting the operation in the first member, we have, under the " same conditions,

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\pi,\text{ or }\frac{\pi}{2},\text{ or }0\,;\qquad\text{(IV)}$$

" and since under the first two conditions the values of a and x are in- " definitely near to each other

$$\frac{1}{\pi}\cdot\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=f(x),\text{ or }\frac{f(x)}{2},\text{ or }0.\qquad\text{(V)}$$

" Extend this by integration from p to q, then observing that " each half value, $\frac{f(x)}{2}$, occurs in two contiguous elements, except the " first of them and the last, we have

$$\frac{1}{\pi}\int_p^q\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=f(x),\text{ or }\frac{f(x)}{2},\text{ or }0,\qquad\text{(VI)}$$

" according as x lies within, upon, or without the limits p and q."

This when read in an ordinary way seems to be faultless, but careful examination shows that it is not so.

Let us first examine (IV) without regard to how it was obtained, remembering that κ is infinitesimally small compared with da.

It is plain that if x and a are not equal

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\frac{0.da}{0^2+(a-x)^2}=0,$$

and that if $x=a$

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\frac{\kappa da}{\kappa^2+0^2}=\frac{da}{\kappa}=\frac{da}{0}=\infty.$$

Consequently (IV) is not true, and therefore the process by which it was obtained must be faulty.

Let us now examine the process.

$$\Delta\tan^{-1}\frac{a-x}{\kappa}=\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa},$$

$$=\tan^{-1}\left(\frac{\frac{a+\Delta a-x}{\kappa}-\frac{a-x}{\kappa}}{1+\frac{a+\Delta a-x}{\kappa}\cdot\frac{a-x}{\kappa}}\right),$$

$$=\tan^{-1}\left(\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\right).$$

Now z and $\tan^{-1}z$ are interchangeable if both are infinitesimally small and of the same sign, but not in any other case. But if

$$z=\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

this is the case only when x lies without the limits a and $a+\Delta a$. If x lie between the limits z is infinitesimally small and negative, in which case putting

$$\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\quad\text{for}\quad\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

is equivalent to putting -0 for π (see the lines immediately following (I) above), and putting

$$\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}\text{ instead of }\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

for $\tan^{-1}\dfrac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$, or $+0$ for π, makes it no better.

Lastly, when $x=a$ or $a+\Delta a$ the putting of

$$\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\text{ for }\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

is equivalent to putting ∞ for $\frac{\pi}{2}$, and the putting of $\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}$ instead of $\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$ for $\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$, or ∞ for $\frac{\pi}{2}$, is no better.

Hence we conclude that we cannot substitute $\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}$ for $\Delta\tan^{-1}\frac{a-x}{\kappa}$ except when x lies without the limits a and $a+\Delta a$. Therefore the process by which (IV) was obtained from (III), namely, the putting of $\frac{\kappa da}{\kappa^2+(a-x)^2}$ for $d\tan^{-1}\frac{a-x}{\kappa}$ is illegitimate.

Now, though (IV) is not true, (VI) is true. To prove it let us return to the beginning.

From (II) we have

$$f(x),\ \frac{f(x)}{2}\text{ or }0=\frac{1}{\pi}f(x)\left\{\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa}\right\},$$
$$=\frac{1}{\pi}f(x)\int_a^{a+\Delta a}\frac{\kappa da}{\kappa^2+(a-x)^2},$$
$$=\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da\,f(a)}{\kappa^2+(a-x)^2},$$
$$=\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da\,f(a)}{\kappa^2+(a-x)^2},$$

provided Δa be infinitesimally small and κ be infinitesimally small compared with Δa; because when x lies between a and $a+\Delta a$, or is equal to either of them, $f(a)$ will differ infinitesimally little from $f(x)$ while a changes to $a+\Delta a$, and when x lies without the limits a and $a+\Delta a$ the substitution of $f(a)$ for $f(x)$ makes no difference because the other factor $\int_a^{a+\Delta a}\frac{\kappa da}{\kappa^2+(a-x)^2}=0$ in this case. Hence

$$\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=f(x),\ \frac{f(x)}{2},\text{ or }0, \tag{VII}$$

according as x lies within, upon, or without the limits a and $a+\Delta a$. From this it follows that

$$\left.\begin{aligned}\frac{1}{\pi}\int_{x}^{x+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}&=\frac{f(x)}{2}\\ \frac{1}{\pi}\int_{x-\Delta a}^{x}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}&=\frac{f(x)}{2}\\ \frac{1}{\pi}\int_{x+m\Delta a}^{x+(m+1)\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}&=0\end{aligned}\right\} \qquad \text{(VIII)}$$

where m denotes any positive integer or any negative integer except -1.

(VII) and (VIII) are true on the supposition that $f(a)$ is continuous between the limits of integration which has been tacitly assumed.

If x lie between the limits p and q, and if $f(a)$ be continuous between these limits, it follows from (VIII) that

$$\frac{1}{\pi}\int_p^q\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{1}{\pi}\left\{\int_p^{p+\Delta a}+\int_{p+\Delta a}^{p+2\Delta a}+\ldots\ldots+\int_{x-\Delta a}^{x}+\int_x^{x+\Delta a}+\right.$$

$$\left.\ldots\ldots+\int_{q-2\Delta a}^{q-\Delta a}+\int_{q-\Delta a}^{q}\right\}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2},$$

$$=0+0+\ldots\ldots+\frac{f(x)}{2}+\frac{f(x)}{2}+\ldots\ldots+0+0,$$

$$=f(x).$$

Similarly it may be shown that if x is equal to p or q, or lies without the limits p and q, the result will be $\frac{f(x)}{2}$ or 0. Hence

$$\frac{1}{\pi}\int_p^q\frac{\kappa da f(a)}{\kappa^2+(a-x)^2}=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(IX)}$$

according as x lies within, upon, or without the limits p and q.

This is the same as (VI), which is thus proved to be true universally, provided $f(a)$ is continuous between the limits p and q.

It could easily be shown that if $f(a)$ is discontinuous at a point x between p and q

$$\frac{1}{\pi}\int_p^q\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{f(x-0)}{2}+\frac{f(x+0)}{2}. \qquad \text{(X)}$$

(IX) might also be written in this form, so that (X) is true universally whether x be a point of continuity or discontinuity, if x lie between p and q.

Now (Todhunter's *Integral Calculus*, § 291)

$$\frac{\kappa}{\kappa^2+(a-x)^2}=\int_0^\infty\epsilon^{-\kappa b}\cos b(a-x)db.$$

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Therefore (IX) becomes

$$\frac{1}{\pi}\int_p^q da f(a)\int_0^\infty \epsilon^{-\kappa b}\cos b(a-x)db = f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XI)}$$

according as x lies within, upon, or without the limits p and q.

There might be a little doubt as to the truth of (XI) for the reason that in arriving at it we have made Δa infinitesimally small, that is, *we have put* $\Delta a = 0$ *before putting* $\kappa = 0$, *which is contrary to our initial supposition*, namely, that $\frac{a-x}{\kappa}$ and $\frac{a+\Delta a-x}{\kappa}$ are both infinitely great when x lies between a and $a+\Delta a$, and consequently that κ is infinitesimally small compared with Δa. Glaisher in his paper signifies this by saying that κ is an infinitesimal of a higher grade than Δa, and appears to think that this must be emphasised so as to remove any doubt as to Boole's result. Now, in whatever way the condition is expressed, it simply amounts to this, *that* κ *should be put equal to* 0 *before* Δa *is put equal to* 0. But in (XI) Δa has *already* been put equal to 0, and κ has *not yet* been put equal to 0, therefore *the initial condition has been violated.* If, therefore, we cannot justify this violation, we can place no dependence on (XI).

This justification might be attempted as follows :—

Going back to (VII), which is the foundation for the whole, it must be observed that it is *only an approximation.*

Now, if we put x_1 and x_2 for the values of a which make $f(a)$ respectively the greatest and least as a increases to $a+\Delta a$, then instead of (VII) we might write

$$\left.\begin{array}{l} \dfrac{1}{\pi}\displaystyle\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} < f(x_1) \text{ and } > f(x_2) \\ \text{if } x \text{ lie between } a \text{ and } a+\Delta a\,; \\ \dfrac{1}{\pi}\displaystyle\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} < \frac{f(x_1)}{2} \text{ and } > \frac{f(x_2)}{2} \\ \text{if } x \text{ equals either } a \text{ or } a+\Delta a\,; \\ \dfrac{1}{\pi}\displaystyle\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} = 0 \end{array}\right\} \qquad \text{(VII*)}$$

if x lies without the limits a and $a+\Delta a$.

Hence instead of (VIII) we might write

$$\left.\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_{x}^{x+\Delta a}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\dfrac{f(x_1)}{2}\text{ and }>\dfrac{f(x_2)}{2}\\ \dfrac{1}{\pi}\displaystyle\int_{x-\Delta a}^{x}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\dfrac{f(x_1')}{2}\text{ and }>\dfrac{f(x_2')}{2}\\ \dfrac{1}{\pi}\displaystyle\int_{x+m\Delta a}^{x+(m+1)\Delta a}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=0\end{array}\right\}\quad\text{(VIII*)}$$

and going through the process by which we passed from (VIII) to (IX), we obtain instead of (IX)

$$\left.\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_{p}^{q}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\dfrac{f(x_1)+f(x_1')}{2}\text{ and }>\dfrac{f(x_2)+f(x_2')}{2}\\ \text{if } x \text{ lies between } p \text{ and } q\,;\\ \dfrac{1}{\pi}\displaystyle\int_{p}^{q}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\dfrac{f(x_1)}{2}\text{ or }\dfrac{f(x_1')}{2}\text{ and }>\dfrac{f(x_2)}{2}\text{ or }\dfrac{f(x_2')}{2}\\ \text{according as } x=p \text{ or } q\,;\\ \dfrac{1}{\pi}\displaystyle\int_{p}^{q}\dfrac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=0\end{array}\right\}\quad\text{(IX*)}$$

if x lies without the limits p and q.

In all this it must be observed that Δa is not equal to 0, and must not be confounded with da, for da did not enter (IX*) as Δa indefinitely diminished, but entered it in the passage from (II) to (VII*), in the latter of which are found both da and Δa as independent quantities. The danger of confounding these might have been avoided by writing (VII) thus

$$\frac{1}{\pi}\int_{a}^{a+\Delta a}\frac{\kappa dv.f(v)}{\kappa^2+(v-x)^2}=f(x),\ \frac{f(x)}{2},\text{ or }0$$

according as, &c.

Now, in (IX*) we may suppose Δa to be as small as we please, provided we do not make it absolutely 0. But the smaller Δa is made the more nearly will $f(x_1)$ and $f(x_2)$ be equal to each other and to $f(x)$ if $x=a$ or $a+\Delta a$ or lie between them. The same is true of $f(x_1')$ and $f(x_2')$. Therefore (IX*) can be made as nearly (IX) as we please by diminishing Δa sufficiently, and therefore I *think* we may conclude that (IX) is true, and therefore that (XI) is true.

Assuming then that (XI) is true, and putting $p=\infty$ and $\ \ = -\infty$ we shall have

$$\frac{1}{\pi}\int_{-\infty}^{x} da f(a)\int_0^{\infty}\epsilon^{-\kappa b}\cos b(a-x)db = f(x) \qquad \text{(XII)}$$

for all real and finite values of x, with the same caution as before as in the case of points where $f(x)$ is discontinuous.

I shall now show that the order of operation in (XI) is reversible. To do this let us return to

$$\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa}=\pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XIII)}$$

according as x lies within, upon, or without the limits a and $a+\Delta a$.

Now (Todhunter's *Integral Calculus*, § 285)

$$\tan^{-1}\left(\frac{r}{\kappa}\right)=\int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin rb}{b}db.$$

Therefore

$$\begin{aligned}\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa} &= \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a+\Delta a-x)}{b}db \\ &\quad -\int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a-x)}{b}db, \\ &= \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a+\Delta a-x)-\sin b(a-x)}{b}db, \\ &= \int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}\cos b(v-x)dv.\end{aligned}$$

Therefore (XIII) becomes

$$\int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}dv\cos b(v-x)=\pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XIV)}$$

according as x lies within, upon, or without the limits a and $a+\Delta a$. If we suppose Δa to be exceedingly small $f(v)$ will vary exceedingly little while v increases from a to $a+\Delta a$, provided $f(v)$ is continuous between these limits. Therefore

$$\frac{1}{\pi}\int_0^{\infty}db.\epsilon^{-\kappa b}\int_a^{a+\Delta a}dvf(v)\cos b(v-x)$$

will differ exceedingly little from

$$\frac{f(a)}{\pi}\int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}dv\cos b(v-x),$$

that is from $\qquad f(a),\ \dfrac{f(a)}{2},\ \text{or } 0,$

according as x lies within, upon, or without the limits a and $a+\Delta a$.

Hence we have

$$\frac{1}{\pi}\int_0^\infty db\epsilon^{-\kappa b}\int_a^{a+\Delta a} dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XV)}$$

according as x lies between, upon, or without the limits a and $a+\Delta a$.

We can now, in the same way as we passed from (VII) to (IX), pass from (XV) to

$$\frac{1}{\pi}\int_0^\infty db\epsilon^{-\kappa b}\int_p^q dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XVI)}$$

according as x lies within, upon, or without the limits p and q; and this is the same as (XI) with the order of integration reversed.

The same doubt might be felt regarding the truth of (XVI) as regarding the truth of (XI), and may be removed in the same way.

If now p and q be put respectively equal to $-\infty$ and $+\infty$ we will have

$$\frac{1}{\pi}\int_0^\infty db.\epsilon^{-\kappa b}\int_{-\infty}^\infty dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XVII)}$$

where it must be observed that p and q are not to be put equal to ∞ until after both integrations, in those cases at least where

$\int_{-\infty}^\infty dvf(v)\cos b(v-x)$ becomes infinite or indeterminate without the arbitrary introduction of such a multiplier as $\epsilon^{-\lambda v}$ under the integral (where λ is put $=0$ after the integration). This I showed in my former paper to be true in the case where

$f(x)=t_1$ for positive values of x

and $\qquad = t_2$ „ negative „ „

in which case the integration with respect to v between the limits $-\infty$ and $+\infty$ is indeterminate, and when rendered determinate by exponential multipliers the result comes out $\pm\frac{t_1-t_2}{2}$ according as x is positive or negative (see end of my former paper). But if we put $p=-l$ and $q=+l$ we get

$$\frac{1}{\pi}\int_0^\infty db\epsilon^{-\kappa b}\int_{-l}^{l} dvf(v)\cos b(v-x)=\frac{1}{\pi}\left[t_1.\tan^{-1}\frac{l-x}{\kappa}+t_2.\tan^{-1}\frac{l+x}{\kappa}\right.$$
$$\left.+(t_1-t_2)\tan^{-1}\frac{x}{\kappa}\right]$$

which, on making l infinite, becomes

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$$\frac{1}{\pi}\left[t_1 . \frac{\pi}{2} + t_2 . \frac{\pi}{2} + (t_1 - t_2)\tan^{-1}\frac{x}{\kappa}\right]$$

or
$$\frac{t_1+t_2}{2} + \frac{t_1-t_2}{\pi}\tan^{-1}\frac{x}{\kappa};$$

and this gives $\frac{t_1+t_2}{2} + \frac{t_1-t_2}{\pi}.\frac{\pi}{2} = t_1$, when x is positive,

and $\frac{t_1+t_2}{2} + \frac{t_1-t_2}{\pi}\left(-\frac{\pi}{2}\right) = t_2$ when x is negative;

which is as it should be.

Solving the same problem by (XI) we get

$$\frac{1}{\pi}\int_{-l}^{l} da f(a) \int_0^{\infty} \epsilon^{-\kappa b} \cos b(a-x) db$$
$$= \frac{1}{\pi}\left\{ t_1 \left(\tan^{-1}\frac{l-x}{\kappa} - \tan^{-1}\frac{-x}{\kappa}\right) + t_2\left(\tan^{-1}\frac{-x}{\kappa} - \tan^{-1}\frac{-l-x}{\kappa}\right)\right\},$$

which when l is put $=\infty$ becomes

$$\frac{1}{\pi}\left\{ t_1 \left(\frac{\pi}{2} - \tan^{-1}\frac{-x}{\kappa}\right) + t_2\left(\tan^{-1}\frac{-x}{\kappa} + \frac{\pi}{2}\right)\right\},$$

or
$$\frac{t_1+t_2}{2} - \frac{t_1-t_2}{\pi}\tan^{-1}\frac{-x}{\kappa};$$

which is also correct.

I had written a draft copy of this before I had the pleasure of seeing Mr Glaisher's paper. Dr Muir, who kindly lent it to me, had previously called my attention to the fact that my proposed demonstration was substantially the same as Glaisher's. I have, however, retained it in the present communication, as the procedure I adopt shows that it has the same foundation as Boole's proof—a point not brought out in Glaisher's paper.

Glaisher starting with

$$\int_0^{\infty} \frac{\sin rb}{b} db = \pm \frac{\pi}{2}, \qquad \text{(XVIII)}$$

according as r is positive or negative, and therefore

$$\int_0^{\infty} \frac{\sin b(a+\Delta a - x) - \sin b(a-x)}{b} db = \pi, \ \frac{\pi}{2}, \text{ or } 0, \qquad \text{(XIX)}$$

according as x lies within, upon, or without the limits a and $a + \Delta a$, shows that this is equivalent to

$$\Delta a\int_0^\infty \cos b(a-x)db=\pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XX)}$$

according as x is within, &c. Therefore

$$f(a).\Delta a\int_0^\infty \cos b(a-x)db=\pi f(a),\ \frac{\pi}{2}f(a), \text{or } 0, \qquad \text{(XXI)}$$

according as x is within, &c. Hence we have

$$f(p).\Delta a\int_0^\infty \cos b(p-x)db+f(p+\Delta a).\Delta a\int_0^\infty \cos b(p+\Delta a-x)db$$

$$+\ldots\ldots\ldots\ldots+f(p+n\Delta a)\Delta a\int_0^\infty \cos b(p+n\Delta a-x)\,db$$

$$=\pi f(x),\ \frac{\pi}{2}f(x), \text{or } 0, \qquad \text{(XXII)}$$

according as x lies within, upon, or without the limits p and $p+n\Delta a$; or putting q for $p+n\Delta a$ and diminishing Δa indefinitely this becomes

$$\int_p^q da f(a)\int_0^\infty \cos b(a-x)db=\pi f(x),\ \frac{\pi}{2}f(x), \text{or } 0, \qquad \text{(XXIII)}$$

according as x lies within, upon, or without the limits p and q.

He now remarks that $\int_0^\infty \cos b(a-x)db$ is indeterminate and proposes to substitute for it the determinate form $\int_0^\infty \epsilon^{-\kappa b}\cos b(a-x)db$ to obtain which he proposes to start with $\int_0^\infty \epsilon^{-\kappa b}\frac{\sin rb}{b}db=\pm\frac{\pi}{2}$ instead of with $\int_0^\infty \frac{\sin rb}{b}db=\pm\frac{\pi}{2}$. But then it must be noted that

$$\int_0^\infty \epsilon^{-\kappa b}\frac{\sin rb}{b}db=\tan^{-1}\frac{r}{k}.$$

So that Glaisher's second departure is the same as Boole's.

If Glaisher's result (XXIII) be written in its equivalent form

$$\int_0^\infty db\int_p^q da f(a)\cos b(a-x)=\pi f(x), \frac{\pi}{2}f(x), \text{or } 0, \qquad \text{(XXIV)}$$

it is no longer indeterminate; and if in this for p and q we substitute $-l$ and $+l$ and increase l indefinitely, we obtain

$$\mathrm{L}_{l=\infty}\int_0^\infty db\int_{-l}^{l} da f(a)\cos b(a-x)=\pi f(x) \qquad \text{(XXV)}$$

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which gives correct results in every example I have tested it by *provided l be not put $=\infty$ till after both integrations.* It has, moreover, the advantage of being more easily handled than any of the other forms.

This same form has been proved by G(regory?) by a symbolical method, but owing to his using vague considerations of grades of infinity he has obtained $\int_0^\infty db \int_0^\infty da f(a)\cos b(a-x) = \frac{\pi}{2}f(x)$ instead of $\pi f(x)$. Besides he makes no restriction as to *when* the superior limit of a is to be put $=\infty$, without which the theorem is unsatisfactory in its application to examples. I have succeeded in modifying G(regory?)'s proof so as to avoid doubtful considerations of grades of infinity and zero, and have obtained the proper value $\pi f(x)$, and the nature of the investigation points out that l should not be put $=\infty$ till after both integrations. As however my paper is already too long I shall not give this investigation.

Equation (XXV) also follows from (XVI) by putting $\kappa=0$ *before the integrations*; and this is lawful, because (XVI) is derived from (XIV), which is true if κ be put equal to 0 before integrating. For

$$\int_0^\infty db \int_a^{a+\Delta a} dv \cos b(v-x) = \int_0^\infty db \left\{ \frac{\sin(a+\Delta a-x)b - \sin(a-x)b}{b} \right\},$$

$$= \int_0^\infty db \frac{\sin b(a+\Delta a-x)}{b} - \int_0^\infty db \frac{\sin b(a-x)}{b},$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right),\ \frac{\pi}{2} - 0 \text{ or } 0 - \left(-\frac{\pi}{2}\right),$$

$$\frac{\pi}{2} - \frac{\pi}{2} \text{ or } -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi,\ \frac{\pi}{2},\ \text{or } 0,$$

according as x lies within, upon, or without the limits a and $a+\Delta a$. We can thus avoid the doubtful proceeding of putting $\Delta a=0$ before putting $\kappa=0$, which is a violation of the initial conditions.

After objecting to the arbitrary introduction of such a factor as $\epsilon^{-\lambda v}$ to make such an integral as $\int_0^\infty \cos b(v-x)dv$ determinate I ought to justify my own *seeming* use of this method in my former paper.

Taking equation (A) of said paper, which I observe is wrongly stated there,

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$$\phi(x) = \frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv + \frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l}\cos w(x-v).\phi(v)dv\Delta w\right\}, \qquad \text{(A)}$$

should the integral in the second term of the right hand member become indeterminate when l is put $=\infty$, I suppose $\epsilon^{\mp\lambda x}\phi(x)$ to be put for $\phi(x)$ according as x is positive or negative, when (A) becomes

$$\epsilon^{\mp\lambda x}\phi(x) = \frac{1}{2l}\int_{-l}^{+l}\epsilon^{\mp\lambda v}\phi(v)\,dv + \frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l}\cos w(x-v).\epsilon^{\mp\lambda v}\phi(v)dv\Delta w\right\}, \qquad \text{(A*)}$$

which will now remain determinate when l becomes infinite, and, after the integrations are performed, we can obtain $\phi(x)$ by putting $\lambda = 0$ in the result.

In justice to De Morgan I ought to notice that he has given Poisson's formula in the same order as (XII) in which there is no danger of putting $l=\infty$ too soon. But he does not say at what stage κ is to be put equal to 0. His words, if they indicate anything as to this point, seem to me to say that this is to be done before any of the integrations are effected. Besides this, he does not seem to see that his form labours under the same fault, if fault it be, with which he charges the verifications he discusses, namely, the order of integration is inverted.

Theorems on three mutually tangent circles.

By Thomas Muir, M.A., LL.D.

This paper will appear later.

Mr William Peddie gave some notes on Reflected Rainbows, in which the bow and its reflection due to the image of the sun were discussed in relation to the ordinary bow and its reflection.

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