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CHARACTERIZATION OF A CLASS OF EQUICONTINUOUS SETS OF FINITELY ADDITIVE MEASURES WITH AN APPLICATION TO VECTOR VALUED BOREL MEASURES

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Let V denote a ring of subsets of an abstract space X, let R^+ denote the nonnegative reals, and let N denote the set of positive integers. We denote by $C(V)$ the space of all subadditive and increasing functions, from the ring V into R^+, which are zero at the empty set. The space $C(V)$ is called the space of contents on the ring V and elements are referred to as contents.

A sequence of sets $A_n \in V$, $n \in N$ is said to be dominated if there exists a set $B \in V$ such that $A_n \subseteq B$, for $n = 1, 2, \ldots$. A content $p \in C(V)$ is said to be Rickart on the ring V if $\lim_n p(A_n) = 0$ for each dominated, disjoint sequence $A_n \in V$, $n \in N$. Note that each finitely additive content is Rickart on the ring V. A set of contents $P \subset C(V)$ is said to be uniformly Rickart on the ring V if the limit above holds uniformly with respect to the contents $p \in P$. This condition is an abstraction of the condition of strong boundedness (often abbreviated s-bounded in the literature) introduced by Rickart [**22**] for a finitely additive vector measure on a σ-algebra. A content $p \in C(V)$ is said to vanish at infinity on the ring V if for each number $\epsilon > 0$, there exists a set $A \in V$ such that $p(B) < \epsilon$ for each set $B \in V$, $B \subseteq X \backslash A$. A set of contents $P \subset C(V)$ is said to vanish uniformly at infinity on the ring V if the above relation holds uniformly with respect to the contents $p \in P$.

The ring V is an abelian group with respect to the symmetric difference operation $\div$ and each content $p \in C(V)$ generates a semimetric on the group $(V, \div)$ by the relation

$$\rho(A, B) = p(A \div B)$$

for sets $A, B \in V$. This semimetric is invariant in the sense that

$$\rho(A, B) = \rho(A \div C, B \div C)$$

for sets $A, B, C \in V$. Therefore, any family of contents $P \subset C(V)$ generates a topology on the group $(V, \div)$. A base of neighborhoods is given by the family of sets

$$N(A_0, p_1, \ldots, p_n, \epsilon) = \{A \in V : p_k(A \div A_0) < \epsilon, \quad \text{for} \quad k = 1, \ldots, n\}$$

where $A_0 \in V$, $p_1, \ldots, p_n \in P$ and $\epsilon > 0$. A pair (V, P), where $P \subset C(V)$

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and $(V, \div)$ is given the topology generated by the family P, will be called a topological ring of sets. A study of topological rings of sets generated by uniformly Rickart families of contents with applications to finitely additive vector measures was initiated by Oberle [**19**], and developed by Bogdanowicz and Oberle [**7**; **8**].

Since contents generate an invariant semimetric on the group $(V, \div)$, a content $q \in C(V)$ is (V, P)-continuous if and only if it is continuous at the origin, the empty set $\emptyset$. Consequently, a content $q \in C(V)$ is (V, P)-continuous if and only if for each number $\epsilon > 0$ there exists a number $\delta > 0$ and a finite set $p_1, \ldots, p_n \in P$ such that $A \in V$ and $p_k(A) < \delta$, for $k = 1, \ldots, n$ yields $q(A) < \epsilon$. Two topological rings (V, P) and (V, Q) are equivalent if each content $p \in P$ is (V, Q)-continuous and conversely. A set of contents $Q \subset C(V)$ is said to be (V, P)-equicontinuous if for each number $\epsilon > 0$ there exists a number $\delta > 0$ and a finite set $p_1, \ldots, p_n \in P$ such that $A \in V$ and $p_k(A) < \delta$, for $k = 1, 2, \ldots, n$ yields $q(A) < \epsilon$ for all contents $q \in Q$.

Let Y be a Banach space and let $a(V, Y)$ and $ca(V, Y)$ denote respectively the spaces of finitely additive and countably additive Y-valued functions on the ring V. Elements of the space $a(V, Y)$ are referred to as vector charges and elements of the space $ca(V, Y)$ are referred to as vector volumes. For each vector charge $\mu \in a(V, Y)$, the semivariation $p(\cdot, \mu) : V \to [0, \infty]$ is defined by the relation

$$p(A, \mu) = \sup (|\mu(B)| : B \in V, B \subseteq A)$$

for $A \in V$. The semivariation is subadditive and increasing on the ring V. A vector charge $\mu \in a(V, Y)$ is said to be Rickart on the ring V if $\lim_n \mu(A_n) = 0$ for each dominated, disjoint sequence $A_n \in V, n \in N$. For each Rickart charge $\mu \in a(V, Y)$, the semivariation $p(\cdot, \mu)$ is a Rickart content on the ring V (see Rickart [**22**]). A vector charge $\mu \in a(V, Y)$ is said to vanish at infinity on the ring V if for each number $\epsilon > 0$ there exists a set $A \in V$ such that $|\mu(B)| < \epsilon$ for all sets $B \in V, B \subseteq X\backslash A$. Let W be an algebra and let $V \subseteq W$ be a subring. A charge $\mu \in a(W, Y)$ is said to be continuous at infinity relative to V if for each number $\epsilon > 0$ there exists a set $A \in V$ such that $|\mu(X) - \mu(B)| < \epsilon$ for all sets $B \in W$ with $A \subseteq B$. The following spaces of vector charges will be referred to in the sequel.

$$\begin{aligned}
R(V, Y) &= \{\mu \in a(V, Y) : \mu\text{-Rickart on the ring } V\} \\
R_\infty(V, Y) &= \{\mu \in R(V, Y) : \mu\text{-vanishes at infinity on } V\} \\
caR(V, Y) &= ca(V, Y) \cap R(V, Y) \\
caR_\infty(V, Y) &= ca(V, Y) \cap R_\infty(V, Y) \\
ab(V, Y) &= \{\mu \in a(V, Y) : p(\cdot, \mu) \in C(V)\} \\
ab_\infty(V, R) &= \{\mu \in ab(V, R) \colon \mu\text{-vanishes at infinity on } V\} \\
cab(V, R) &= ab(V, R) \cap ca(V, R) \\
cab_\infty(V, R) &= ab_\infty(V, R) \cap ca(V, R).
\end{aligned}$$

The symbols $ab^+(V, R)$, $ab_\infty{}^+(V, R)$ and $cab^+(V, R)$ are used to denote the cone of nonnegative elements. The main result of this paper is a characterization of each pointwise bounded set in $ab_\infty(V, R)$ as equicontinuous if and only if it is uniformly Rickart and vanishes uniformly at infinity on the ring V. This characterization is then used to characterize the class of vector valued regular Borel measures on a locally compact space as the set of extensions of Rickart vector volumes which vanish at infinity on the ring generated by the compact G_δ sets.

A vector charge $\mu \in a(V, Y)$ is said to be strongly bounded (see Rickart [**22**]), if for each disjoint sequence $A_n \in V$, $n \in N$, we have $\lim_n \mu(A_n) = 0$. It has been established that strongly bounded vector charges admit a nonnegative, finitely additive control measure (see Brooks [**9**; **10**]). Uhl [**23**] showed that for countably additive, strongly bounded vector measures on an algebra of sets, the existence of a finitely additive control measure is equivalent to the weak relative compactness of the range of the vector measure which in turn is equivalent to the existence of a countably additive extension to the generated σ-algebra. In the development given by Brooks [**9**; **10**; **11**], and Uhl [**24**], either the Stone representation of an algebra of sets as the algebra of open/closed subsets of a totally disconnected, compact Hausdorff space and/or the weak compactness criteria developed by Bartle, Dunford and Schwartz [**1**] is used. The relation between the vector charges considered in this note and those studied by Brooks in [**9**; **10**] and [**11**] is contained in the following proposition.

PROPOSITION 1. *Let V be a ring of subsets of a space X and let Y be a Banach space. The following are equivalent.*

(1) *The charge $\mu \in a(V, Y)$ is Rickart and vanishes at infinity on the ring V.*

(2) *The charge $\mu \in a(V, Y)$ is strongly bounded.*

The basic result of this paper is given in the following theorem.

THEOREM 1. *Let V be a ring of subsets of a space X. A pointwise bounded set $M \subset ab(V, R)$ is uniformly Rickart and vanishes uniformly at infinity on the ring V if and only if there exists a charge $v \in ab_\infty{}^+(V, R)$ such that the set M is v-equicontinuous. In addition, $M \subset cab(V, R)$ if and only if $v \in cab_\infty{}^+(V, R)$.*

Proof. If the family $M \subset ab(V, R)$ is uniformly Rickart and vanishes uniformly at infinity on the ring V, then the family M is weakly relatively compact in the Banach space $fa(V, R)$ of real valued charges with totally bounded variation (see [**11**]). Consequently, there exists a control charge $v \in ab^+(V, R)$ such that the family M is v-equicontinuous and

$$v(A) \leqq \sup (|\mu|(A) : \mu \in M)$$

for sets $A \in V$ where $|\mu|(\cdot)$ denotes the variation of the charge μ. Since the family M vanishes uniformly at infinity, the charge $v \in ab^+(V, R)$ vanishes at infinity.

If each charge $\mu \in M$ is countably additive, then the uniform Rickart condition insures that the family $\{|\mu|(\cdot) : \mu \in M\}$ is uniformly countably additive [**19**]. Consequently, the control charge $v \in ab^+(V, R)$ is countably additive.

Remark 1. The existence of the control charge given by Brooks, [**9**; **10**] is established by transferring the problem to the σ-algebra generated by the Stone representation algebra and then applying the Bartle-Dunford-Schwartz [**1**] weak compactness criteria. A direct construction of the control charge may be found in [**7**; **8**], and [**19**].

Let V be a ring (σ-ring) of subsets of an abstract space X. Then the smallest algebra (σ-algebra) containing the ring V is given by the relation

$$\mathscr{A}(V) = \{A \in P(X) : A \in V \quad \text{or} \quad X \backslash A \in V\}.$$

The referee suggested the following proposition to clarify the structure of the class of charges under study.

PROPOSITION 2. *Let V be a ring of subsets of an abstract space X and let $\mathscr{A}(V)$ denote the smallest algebra containing the ring V. Then for each Banach space Y, there is a one-to-one correspondence between the space of Y-valued charges on the ring V vanishing at infinity and the space of Y-valued charges on the algebra $\mathscr{A}(V)$ continuous at infinity relative to the ring V. The correspondence preserves the Rickart condition and the semivariation. For Rickart charges the correspondence also preserves countable additivity.*

Proof. Let $\mu \in a(V, Y)$ vanish at infinity and let V be ordered by inclusion. Then the net $\langle \mu(A) : A \in V \rangle$ is Cauchy (hence convergent) in the space Y. The extension is defined for each set $A \in \mathscr{A}(V)$ by the relation

$$\bar{\mu}(A) = \begin{cases} \mu(A) & \text{if} \quad A \in V, \\ \lim_{B \in V} \mu(B) - \mu(X \backslash A) & \text{if} \quad X \backslash A \in V. \end{cases}$$

It is clear that the function $\bar{\mu}$ is a charge extending the charge μ and from the definition, the charge $\bar{\mu}$ is continuous at infinity relative to V. The restriction defines the inverse mapping.

Assume that the charge $\mu \in a(V, Y)$ is Rickart on the ring V and that $A_n \in \mathscr{A}(V)$, $n \in N$ is a disjoint sequence. If $X \notin V$, it follows from the definition of the algebra $\mathscr{A}(V)$ that at most one term in the sequence (say A_1) satisfies the relation $X \backslash A_1 \in V$. Using Proposition 1, $\lim_n \bar{\mu}(A_n) = 0$. The fact that the semivariation of the extension is an extension of the semivariation follows from the relation

$$\{A \cap B : B \in \mathscr{A}(V)\} = V(A)$$

for all sets $A \in V$ where $V(A) = \{C \in V : C \subseteq A\}$. The fact that countably additive, Rickart charges vanishing at infinity extend to countably additive charges will not be proven since this observation is not necessary for the later development.

Proposition 1 and the characterization of unconditionally converging series in terms of the weak relative compactness of the unordered finite sums given by McArthur [**18**] ,and Robertson [**23**], yields the "weakly relatively compact range" theorem.

PROPOSITION 3. *Let V be a ring of subsets of a space X, let Y be a Banach space and let $p \in C(V)$ be Rickart and vanish at infinity on the ring V. Then each p-continuous charge $\mu \in a(V, Y)$ is strongly bounded and has weakly relatively compact range. Conversely, any charge $\mu \in a(V, Y)$ with weakly relatively compact range is strongly bounded and its semivariation $p(\cdot, \mu)$ is a Rickart content vanishing at infinity on the ring V.*

Proof. Let $\mu \in a(V, Y)$ be p-continuous and assume that the content $p \in C(V)$ is Rickart and vanishes at infinity on the ring V. It is clear that the charge $\mu \in a(V, Y)$ is strongly bounded. We show that its range is weakly relatively compact. Let $R(\mu)$ denote the range of the charge μ and let $F(N)$ denote the family of finite subsets of the positive integers N. From Proposition 2, the charge μ admits an extension to a strongly bounded vector charge on the algebra $\mathscr{A}(V)$ and consequently has weakly relatively compact range [**24**].

The converse assertion was first observed by Kluvanek [**17**]. The strong boundedness of a vector charge with weakly relatively compact range may be established by applying the McArthur-Robertson characterization. Note first that for each disjoint sequence $A_n \in V$, $n \in N$, the set

$$\left\{ \sum_{k\in\Delta} \mu(A_k) : \Delta \in F(N) \right\} \subset R(\mu)$$

is weakly relatively compact. Consequently, the series $\sum_k \mu(A_k)$ converges unconditionally. Hence, the charge $\mu \in a(V, Y)$ is strongly bounded.

Let V be a ring of subsets of a space X and for a volume $v \in cab_\infty(V, R)$, let (X, V_c, v_c) denote the completion (see Bogdanowicz, [**2**; **3**]). Let V_σ denote the class of countable unions of sets from the family V and let $\bar{V}$ denote the class of v-measurable sets (see Bogdanowicz, [**4**; **5**]). If we denote by λ the measure on the σ-ring $\bar{V}$ extending the volume v, from the condition $V_\sigma \subset V_c$ (which is true for all volumes $v \in cab_\infty{}^+(V, R)$) and the fact that measurable sets have V_σ-support, we conclude that the measure λ is finite valued (and therefore bounded) on the σ-ring $\bar{V}$. For such measures, we denote by $S(\lambda)$ the support of the measure λ (note that the measurable set $S(\lambda)$ is defined uniquely up to sets of measure zero). Since the measure λ is finite on the σ-ring $\bar{V}$, the construction insures that λ coincides with the completion v_c (by [**4**, Theorem 3(6)]), that is, $V_c = \bar{V}$ and $v_c(\cdot) = \lambda(\cdot)$. Using Theorem 1 and the fact that the ring V is dense in the σ-ring V_c (with respect to the semi-metric

$$\rho(A, B) = v_c(A \div B)$$

for sets $A, B \in V_c$) we have the following extension theorem for vector volumes.

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THEOREM 2. *Let V be a ring of subsets of a space X and let Y be a Banach space. For each vector volume $\mu \in caR_\infty(V, Y)$ there exists a σ-ring $\bar{V}$ and a vector measure $\bar{\mu} \in ca(\bar{V}, Y)$ such that $V \subset \bar{V}$ and $\mu \subset \bar{\mu}$.*

Proof. Set

$$M = \{|y' \circ \mu|(\cdot) : y' \in Y', |y'| = 1\}$$

and note that the set $M \subset cab_\infty(V, R)$ is uniformly Rickart on the ring V and vanishes uniformly at infinity on the ring V. Using Theorem 1, there exists a volume $v \in cab_\infty{}^+(V, R)$ such that the set M is v-equicontinuous. Consequently, the vector volume $\mu \in caR_\infty(V, Y)$ is v-continuous. Since the volume $v \in cab_\infty{}^+(V, R)$ has an extension to a bounded, scalar measure $\bar{v} \in cab^+(\bar{V}, R)$ and the vector volume μ is $\bar{v}$-continuous (and hence uniformly continuous) on the ring V, which is dense in the σ-ring $\bar{V}$, there exists an extension $\bar{\mu} \in ca(\bar{V}, Y)$ which is $\bar{v}$-continuous on the σ-ring $\bar{V}$.

Let (X, τ) be a locally compact, Hausdorff space and let $C_\infty(X, R)$ denote the space of bounded, continuous real valued functions on the space X which vanish at infinity. The space $C_\infty(X, R)$ is a Banach space with respect to the uniform norm, defined for functions $f \in C_\infty(X, R)$ by the relation

$$\| \ \|_\infty : f \to \sup (|f(x)| : x \in X).$$

Moreover, each function $f \in C_\infty(X, R)$ has σ-compact support. Denote by $\mathscr{B}$ the Borel σ-ring and denote by V (respectively V_0) the ring generated by the compact (respectively the compact G_δ) sets. The dual of the space $(C_\infty(X, R), \| \ \|_\infty)$ is the space $M(X, R)$ of finite (and hence bounded) Radon measures on X (see [**16**]). Moreover, each such measure has σ-compact support (see [**12**]) so that each vanishes at infinity on the σ-ring $\mathscr{B}$ relative to the class of compact sets. Since each compact set is a subset of a compact G_δ set (see [**13**, Proposition 11, p. 294]), such measures vanish at infinity on the Borel σ-ring with respect to the family of compact G_δ sets.

THEOREM 3. *For each vector volume $\mu \in caR_\infty(V_0, Y)$ there exists a bounded vector measure $\bar{\mu} \in ca(\mathscr{B}, Y)$ extending the vector volume μ. The extension $\bar{\mu}$ is regular in the sense that for each number $\epsilon > 0$, there exists a compact set F and an open Borel set G such that*

$$|\bar{\mu}(B)| < \epsilon$$

for all Borel sets $B \in \mathscr{B}$ such that $B \subset G \backslash F$.

Proof. By Theorems 1 and 2, the vector volume $\mu \in caR_\infty(V_0, Y)$ is continuous with respect to a volume $v \in cab_\infty{}^+(V, R)$. This volume is Baire regular in the sense that for each set $A \in V_0$ and each number $\epsilon > 0$, there exists a compact set K and an open set G such that $k \subset A \subset G$ and $|v(A) - v(B)| < \epsilon$ for each set $B \in V_0$ with $K \subset B \subset G$ (see [**13**]). Also [**13**, p. 351] the volume v generates a unique, regular volume $\bar{v}$ on the delta ring $\mathbf{b}$ generated by the compact sets. For each compact set $Q \subset X$, the extension $\bar{v}$ is given by the formula

$$\bar{v}(Q) = \inf (v(A) : A \in V_0, Q \subseteq A)$$

and for each set $E \in \mathbf{b}$,

$$\bar{v}(E) = \sup (\bar{v}(Q) : Q\text{-compact}, Q \subseteq E).$$

We will show that the extension $\bar{v} : \mathbf{b} \to R^+$ vanishes at infinity on the delta ring $\mathbf{b}$. Consider any number $\epsilon > 0$ and use the fact that the volume v vanishes at infinity on the ring V_0 to choose a set $K \in V_0$ such that $A \in V_0, A \subset X \backslash K$ yields $v(A) < \epsilon$. Since each set in the ring V_0 is contained in a compact G_δ set, we may assume that the set K is a compact G_δ set. Let $Q \subset X \backslash K$ be an arbitrary compact set. Then there exists an open F_σ set G and a compact G_δ set C such that $Q \subset G \subset C \subset X \backslash K$ [**13**, p. 294]. Consequently, $\bar{v}(Q) < \epsilon$ so that for any set $A \in \mathbf{b}$ with $A \subset X \backslash K$, we have

$$\bar{v}(A) = \sup (\bar{v}(Q) : Q \subseteq A) \leqq \epsilon.$$

Since the extension $\bar{v} : \mathbf{b} \to R^+$ is countably additive and vanishes at infinity on the delta ring $\mathbf{b}$, it is strongly bounded. Therefore $\bar{v}$ is bounded on the class $\mathbf{b}_\sigma = \sigma(\mathbf{b}) = \mathscr{B}$. That is, for each disjoint sequence $A_n \in \mathbf{b}$, $n \in N$, the sequence of numbers $\bar{v}(\bigcup_{k=1}^n A_k)$, $n \in N$, is bounded. Moreover, from the regularity of $\bar{v}$ on $\mathbf{b}$ the restriction of the measure $\bar{v}$ to the lattice $\mathscr{C}$ of compact sets is a regular content in the sense of Halmos [**15**, pp. 224-240]. This restriction generates a measure $\tilde{v}$ on the Borel σ-ring $\mathscr{B}$ which is regular in the sense that

$$\tilde{v}(A) = \sup (\tilde{v}(C) : C \in \mathscr{C}, C \subseteq A)$$

and

$$\tilde{v}(A) = \inf (\tilde{v}(G) : A \subset G, G\text{-open}, G \in \mathscr{B})$$

for each set $A \in \mathscr{B}$. Also, the restriction of the measure $\tilde{v}$ to the delta ring $\mathbf{b}$ coincides with the measure $\bar{v}$. Consequently, the measure $\tilde{v}$ is finite on the σ-ring $\mathscr{B} = \mathbf{b}_\sigma$. From regularity and [**13**, Proposition 11, p. 234], the ring V_0 is dense in the σ-ring $\mathscr{B}$. Therefore the vector measure $\mu \in ca(V_0, Y)$ admits an extension to a vector measure $\bar{\mu} \in ca(\mathscr{B}, Y)$. Moreover, the extension $\bar{\mu} \in ca(\mathscr{B}, Y)$, obtained via the $\tilde{v}$-continuity (actually the uniform $\tilde{v}$-continuity) is Borel regular and the proof is complete.

Let $\mathscr{B}_r$ be the largest σ-ring in which the σ-ring $\mathscr{B}$ is an ideal. The family $\mathscr{B}_r$ is characterized by the equality

$$\mathscr{B}_r = \{A \subset X : A \cap B \in \mathscr{B} \text{ for all } B \in \mathscr{B}\}.$$

The σ-ring $\mathscr{B}_r$ is a σ-algebra containing the delta ring $\mathbf{b}$ of relatively compact Borel sets and consequently [**13**, Proposition 5, pp. 290-291] the σ-algebra $\mathscr{B}_r$ contains the open and the closed sets.

PROPOSITION 4. *Let $p \in C(\mathscr{B}_r)$ be a content which is regular on the Borel sets*

$\mathscr{B}$ and vanishes at infinity on the σ-algebra $\mathscr{B}_r$ relative to the family of compact sets (i.e., for each number $\epsilon > 0$, there exists a compact set K such that $p(A) < \epsilon$ for each set $A \in \mathscr{B}_r$, $A \subset X \backslash K$). Then the content p is outer regular on the σ-algebra $\mathscr{B}_r$.

Proof. For any set $A \in \mathscr{B}_r$, it must be shown that the set A can be approximated from above by open sets. Let $\epsilon > 0$ be arbitrary and choose a compact set Q for which $p(B) < \epsilon/2$ for each set $B \in \mathscr{B}_r$, $B \subset X \backslash Q$. From the regularity of the content p on the σ-ring of Borel sets, there exists an open set $G_1 \in \mathscr{B}$ with $A \cap Q \subset G_1$ and $p(G_1) < p(A \cap Q) + \epsilon/2$. Then for any open set G_2 with $A \backslash Q \subset G_2 \subset X \backslash Q$ we have for the set $G = G_1 \cup G_2$, $A \subset G$ and

$$p(G) \leqq p(G_1) + p(G_2) < p(A \cap Q) + \epsilon/2 + \epsilon/2$$

so that

$$p(G) < p(A \cap Q) + \epsilon < p(A) + \epsilon.$$

Since the number $\epsilon > 0$ is arbitrary, the content p is outer regular.

Let $w \in ca^+(\mathscr{B}, R)$ be regular. Then the measure w vanishes at infinity on the σ-ring $\mathscr{B}$ relative to the compact G_δ sets. For each set $A \in \mathscr{B}_r$, we set

$$\bar{w}(A) = \sup \ (w(Q) : Q \subset A, Q\text{-compact}) < \infty.$$

The function $\bar{w} : \mathscr{B}_r \to R^+$ is a countably additive, regular extension of the Borel measure w. Indeed, the countable additivity on the σ-algebra $\mathscr{B}_r$ follows easily from the countable additivity and the regularity of the measure w on the σ-ring $\mathscr{B}$. In addition the regularity of the measure w insures that the function $\bar{w}$ extends the measure w. Consequently, the function $\bar{w}$ is a countably additive content which is regular on the σ-ring $\mathscr{B}$ and inner regular on the σ-algebra $\mathscr{B}_r$. We show that the content $\bar{w}$ vanishes at infinity on the σ-algebra $\mathscr{B}_r$ relative to the family of compact G_δ sets. However, this follows from the definition of the content $\bar{w}$ and the fact that the measure w vanishes at infinity on the σ-ring $\mathscr{B}$ relative to the compact G_δ sets. From Proposition 4 the countably additive content $\bar{w}$ is a finite regular measure extending the measure w. Moreover, since any regular extension w' of the measure w to the σ-algebra $\mathscr{B}_r$ satisfies the relation

$$w'(A) = \sup \ (w'(Q) : Q \subseteq A, Q\text{-compact})$$

for each set $A \in \mathscr{B}_r$, the extension $\bar{w}$ is necessarily unique.

We have established the essentials of the following general theorem.

THEOREM 4. *Let V_0 denote the ring generated by the compact G_δ sets and let $\mathscr{B}_r$ be the largest σ-ring in which the Borel σ-ring $\mathscr{B}$ (the σ-ring generated by the compact sets) is an ideal. Then corresponding to each vector volume $\mu \in caR_\infty(V_0, Y)$ there exists a unique, regular measure $\bar{\mu} \in ca(\mathscr{B}_r, Y)$ which extends the volume μ.*

Proof. Let $v \in cab_\infty{}^+(V_0, R)$ be a control volume for the vector volume μ and let $\bar{v} \in ca^+(\mathscr{B}_r, R)$ be the unique regular extension of the volume v. The volume $\tilde{v}$ generates a semimetric on the σ-algebra $\mathscr{B}_r$ for which the lattice $\mathscr{C}$ of compact sets is dense. By Theorem 3, the volume μ admits an extension to a $\tilde{v}$-uniformly continuous, regular, vector measure $\mu_1 \in ca(\mathscr{B}, Y)$. Since the σ-ring $\mathscr{B}$ contains the compact sets, there exists a vector measure $\bar{\mu} \in ca(\mathscr{B}_r, Y)$ for which $\bar{\mu}$ is $\tilde{v}$-continuous on the σ-algebra $\mathscr{B}_r$ and $\bar{\mu}$ extends μ. Consequently, $\bar{\mu} \in ca(\mathscr{B}_r, Y)$ is a regular vector measure extending the volume $\mu \in caR_\infty(V_0, Y)$. The uniqueness follows from the fact that the compact sets are $\tilde{v}$-dense in the σ-algebra $\mathscr{B}_r$.

Remark 2. In a recent paper, Ohba [**20**] noted that a Borel regular vector measure on the σ-ring $\mathscr{B}$ generates an inner regular measure on the σ-algebra generated by the closed sets via the above process.

References

1. R. G. Bartle, N. Dunford, and J. T. Schwartz, *Weak compactness and vector measures*, Can. J. Math. *7* (1955), 287–305.

2. W. M. Bogdanowicz, *A generalization of the Lebesque-Bochner-Stieltjes integral and a new approach to the theory of integration*, Proc. Nat. Acad. Sci. U.S.A. *53* (1965), 492–498.

3. ——— *Existence and uniqueness of extensions of volumes and the operation of completion of a volume I*, Proc. Japan Acad. *42* (1966), 571–575.

4. ——— *An approach to the theory of Lebesque-Bochner measurable functions and the theory of measure*, Math. Ann. *164* (1966), 251–269.

5. ——— *An approach to the theory of integration and theory of Lebesque-Bochner measurable functions on locally compact spaces*, Math. Ann. *171* (1967), 219–238.

6. ——— *An approach to the theory of integration generated by Daniell functionals and representation of linear continuous functionals*, Math. Ann. *173* (1967), 34–52.

7. W. M. Bogdanowicz and R. A. Oberle, *Topological rings of sets generated by families of Rickart contents* (to appear).

8. ——— *Uniformly Rickart families of contents on rings of sets admitting an upper complete content* (to appear).

9. J. K. Brooks, *On the existence of a control measure for strongly bounded vector measures*, Notices Amer. Math. Soc. *18* (1971), 415.

10. ——— *On the existence of a control measure for strongly bounded vector measures*, Bull. Amer. Math. Soc. *77* (1971), 999–1001.

11. ——— *Weak compactness in the space of vector measures*, Bull. Amer. Math. Soc. *78* (1972), 284–287.

12. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. *5* (1958), 95–104.

13. N. Dinculeanu, *Vector measures* (Pergamon Press, New York, 1967).

14. N. Dunford and J. T. Schwartz, *Linear operators*, part I (Interscience, New York, 1958).

15. P. R. Halmos, *Measure theory* (Van Nostrand, New York, 1950).

16. E. Hewitt and K. A. Ross, *Abstract harmonic analysis I* (Academic Press, Inc., New York, 1963).

17. I. Kluvanek, *The theory of vector valued measures, part II*, Mat. Časopis Sloven. Akad. Vied. *16* (1966), 76–81 (Russian).

18. C. W. McArthur, *On a theorem of Orlicz-Pettis*, Pacific J. Math. *22* (1967), 292–302.

19. R. A. Oberle, *Theory of a class of vector measures on topological rings of sets and generalizations of the Vitali-Hahn-Saks theorem*, Ph.D. Thesis, The Catholic University of America, 1971.

20. S. Ohba, *Decompositions of vector measures*, The Reports of the Faculty of Technology, Kanagawa University, No. *10* (1972).

21. A. L. Peressini, *Ordered topological vector spaces* (Harper and Row Co., New York, 1967).

22. C. E. Rickart, *Decompositions of additive set functions*, Duke Math. J. *10* (1943), 653–665.

23. A. P. Robertson, *On unconditional convergence in topological vector spaces*, Proc. Roy. Soc. Edinburgh Sect. A *68* (1969), 145–147.

24. J. J. Uhl, *Extensions and decompositions of vector measures*, J. London Math. Soc. *3* (1971), 672–676.

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