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ON THE ISOMORPHISMS BETWEEN CERTAIN CONGRUENCE GROUPS, II

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For integral domains of characteristic not 2, we prove here that the symplectic and unitary congruence groups are not isomorphic if the Witt indices are at least 3. This is Theorem 2.1; Theorem 3.3 describes the isomorphisms of unitary congruence groups.

Preliminaries. Let V be an n-dimensional vector space over the commutative field F of characteristic not 2. We shall assume $f(x, y)$ is a non-degenerate skew-hermitian form on V with respect to an involutory automorphism J of F. We allow the possibility that J is the identity on F; in that case the form $f(x, y)$ is skew-symmetric. *We assume throughout $f(x, y)$ has index at least 3.*

The unitary group $U_n(V, f)$ of the skew-hermitian form $f(x, y)$ is defined as all the non-singular linear transformations σ of V onto V such that $f(\sigma x, \sigma y) = f(x, y)$ for all x, y in V. Since we allow the possibility that J is the identity map of F, we see that $U_n(V, f)$ is the symplectic group of V when J is the identity. When J is not the identity we can multiply $f(x, y)$ by a suitable scalar factor λ in F so that the resulting form $\lambda f(x, y)$ is hermitian. The group $U_n(V, f)$ defined above is then the unitary group of the hermitian form $\lambda f(x, y)$.

Let U be a subspace of V. We define $U^* = \{x \in V | f(x, U) = 0\}$. The radical of U, rad U, is defined by $\text{rad}\, U = U \cap U^*$. U is called non-degenerate (or regular) if $\text{rad}\, U = 0$. If U is non-degenerate, 2-dimensional, and contains a non-zero vector x such that $f(x, x) = 0$ we call U a hyperbolic plane. Two subspaces U_1 and U_2 are called orthogonal if $f(U_1, U_2) = 0$. A non-zero vector x is called isotropic if $f(x, x) = 0$, and a subspace of V is called isotropic if it contains isotropic vectors; otherwise it is called anisotropic.

For any $\sigma \in U_n(V, f)$ we let R and P be the residual and fixed spaces of σ; i.e., $P = \ker (\sigma - 1)$ and $R = P^*$. Note $R = (\sigma - 1)V$. Similarly R_i and P_i denote the residual and fixed spaces of any $\sigma_i \in U_n(V, f)$. Note if $\sigma_1, \sigma_2 \in U_n(V, f)$, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if $R_1 \subseteq R_2^*$. This follows from [**4**, 1.4].

For any $\sigma \in GL_n(V)$ we let $\bar{\sigma}$ denote the coset of σ in $PGL_n(V)$; $PGL_n(V)$ is of course the quotient group of the general linear group $GL_n(V)$ by its center $\dot{F}$. If S is a subset of $GL_n(V)$ we define $\bar{S} = \{\bar{\sigma} \in PGL_n(V) | \sigma \in S\}$.

We now define the projective unitary group $PU_n(V, f)$ as the image of $U_n(V, f)$ in $PGL_n(V)$ under the natural map $^-$ of $GL_n(V)$ onto $PGL_n(V)$. We say two elements σ_1 and σ_2 anticommute if $\bar{\sigma}_1$ and $\bar{\sigma}_2$ do commute, but

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σ_1 and σ_2 do not; i.e., $\sigma_1\sigma_2 = \lambda\sigma_2\sigma_1$ for some scalar λ in F unequal to 0 or 1. And if a, b, are any two elements of a group, $[a, b]$ denotes $aba^{-1}b^{-1}$.

Next consider an element σ of $GL_n(V)$ which leaves a hyperplane pointwise invariant. Such a σ is called a shearing, and the corresponding element $\bar{\sigma}$ in $PGL(V)$ is called a projective shearing. A shearing of determinant 1 is called a transvection; and a non-trivial shearing in $U_n(V, f)$ which is not a transvection is called a quasi-symmetry. Projective quasi-symmetries and projective transvections are defined in the obvious way.

It is well-known [**1**, p. 25] that every transvection τ in $U_n(V, f)$ has the form $\tau = \tau_{a,\lambda}$ where $\tau_{a,\lambda}(x) = x + \lambda f(a, x) \cdot x$, for $x \in V$, λ being an element in F with $\lambda = J(\lambda)$ and a being an isotropic vector of V. If $\tau \neq 1$, the isotropic line Fa is called the proper line of the transvection τ.

1. Centralizer results. In all that follows G shall denote a subgroup of $PU_n(V, f)$ which *has enough projective transvections*. This means for each isotropic line L of V there is a nontrivial projective transvection $\bar{\tau}$ in G with proper line L. If λ is any scalar, then τ and $\lambda\tau$ cannot be distinct transvections and so the proper line of a projective transvection is unique.

We put

$$\Delta = \{\sigma \in U_n(V, f) | \bar{\sigma} \in G\}.$$

For any subspace U of V, we define $E(U) = \{\sigma \in \Delta | R \subseteq U\}$, where R is the residual space of σ; i.e. $R = [\ker\ (\sigma - 1)]^*$.

If S is any subset of Δ, $C(S)$ will denote the centralizer of S in Δ. And if X is any subset of G, $C(X)$ will denote the centralizer of X in G.

1.1 *Let* $\sigma \in U_n(V, f)$ *and let* σ *have residual space* R. *Then* $\sigma^2 = 1$ *if and only if* $\sigma | R = -1_R$.

Proof. Apply [**4**, **1.7**].

1.2 *Let* $\sigma \in \Delta$ *be such that* $\dim R = 2$ *and* $\sigma | R$ *is not a scalar. Then* $E(R) \subseteq CDC(\sigma)$.

Proof. Proceed as in [**9**, 3.1], the only difference being that we are here dealing with a skew-hermitian form $f(x, y)$ instead of with an alternating form.

1.3 *Let* $\sigma \in \Delta$ *and suppose* $\dim R \leqq 2$. *Then* $CDC(\bar{\sigma}) \subseteq \overline{E(R)}$.

Proof. We proceed along the lines of [**9**, 3.2]. Consider any element $\bar{\Sigma}$ in $CDC(\bar{\sigma})$. The argument used in the first paragraph of the proof of [**9**, 3.2], when applied here, will show that Σ acts on all isotropic lines of P. (Here P is the fixed space of σ.)

Since Σ acts on all isotropic lines of P, we can conclude that Σ acts on all hyperbolic planes in P since a hyperbolic plane is always spanned by two isotropic lines. Now every anisotropic line in P is the intersection of two hyperbolic planes in P (cf. [**1**, Lemma 3, p. 43]). Thus Σ acts on all anisotropic lines in P, and therefore Σ acts on all lines in P. Therefore, $\Sigma | P$ is a scalar, say

$\Sigma|P = \alpha$. So the fixed space of $\alpha^{-1} \cdot \Sigma$ contains P. Since $R = P^*$, the residual space of $\alpha^{-1} \cdot \Sigma$ is contained in R. Thus $\bar{\Sigma} = \overline{\alpha^{-1} \cdot \Sigma} \in \overline{E(R)}$.

1.4 *Let $\sigma \in \Delta$ be such that R is a hyperbolic plane, and $\sigma|R$ is not a scalar. Then $CDC(\bar{\sigma}) = \overline{E(R)}$.*

Proof. Apply 1.2 and 1.3.

1.5 *Let $\sigma \in \Delta$ be such that* $\dim R \leqq 2$.

(a) *If R is totally isotropic, $CDC(\bar{\sigma})$ is abelian.*

(b) *If R is a non-isotropic line, $CDC(\bar{\sigma})$ is abelian.*

(c) *If R is a hyperbolic plane and $\sigma|R$ is not a scalar, then $CDC(\bar{\sigma})$ is non-abelian.*

Proof. (a) follows from 1.3 and the fact that if σ_1 and σ_2 are in $U_n(V, f)$ and $R_1 \subseteq R_2^*$ then σ_1 and σ_2 commute. (b) follows from 1.3. To prove (c) observe that 1.2 implies $\overline{E(R)} \subseteq CDC(\bar{\sigma})$. Since R is a hyperbolic plane there are two non-orthogonal isotropic lines L_1 and L_2 in R. If we choose two projective transvections in G whose proper lines are L_1 and L_2, then these two projective transvections will be in $\overline{E(R)}$, hence in $CDC(\bar{\sigma})$, but will not commute.

1.6 *Suppose that $\Sigma \in U_n(V)$ is such that $\Sigma(Fa) \neq Fa$ for some isotropic line Fa of V. Let $\tau_{a,\lambda}$ be a non-trivial transvection in $U_n(V)$. Then $\bar{\Sigma}$ and $\bar{\tau}_{a,\lambda}\bar{\Sigma}^{-1}\bar{\tau}_{a,-\lambda}$ do not commute.*

Proof. This follows as in [**9**, 2.1].

Now let V and W be two finite-dimensional vector spaces over fields F_1 and F_2 respectively, each field of characteristic not two. Let V and W each have defined on them non-degenerate skew-hermitian forms f_1 and f_2 respectively, each form having index at least three.

1.7 *Under the assumptions above, if H is a subgroup of $PU(W, f_2)$ and G is a subgroup of $PU(V, f_1)$ such that G and H both have enough projective transvections, then any isomorphism Λ of G onto H maps projective shearings to projective shearings.*

Proof. Let $\bar{\sigma} \in G$ be a projective shearing with proper line L. We can assume $\bar{\sigma} \neq \bar{1}$. Put $\bar{\Sigma} = \Lambda\bar{\sigma}$; since $\bar{\sigma} \neq \bar{1}$ there is an isotropic line L_1 in V such that $\Sigma L_1 \neq L_1$. Let $\bar{\tau}_{a,\lambda}$ be a nontrivial transvection in H with line L_1. Put $T = \tau_{a,\lambda}$; by 1.6 $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ do not commute. Put $\Lambda\bar{\tau} = \bar{T}$, $h = [\Sigma, T]$, and $g = [\sigma, \tau]$. Since $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ cannot commute, σ and $\tau\sigma^{-1}\tau^{-1}$ cannot commute; hence $L \neq \tau L$ and $f_1(L, \tau L) \neq 0$. So $L + \tau L$ is the residual space of g by [**4**, 1.2].

A simple computation shows the composition of two non-commuting shearings cannot be a scalar when restricted to its residual space. Hence $g = \sigma(\tau\sigma^{-1}\tau^{-1})$ satisfies the hypotheses of 1.2 and so $E(L + \tau L) \subseteq CDC(g)$. Thus

$$\overline{E(L + \tau L)} \subseteq \overline{CDC(g)} \subseteq CD\overline{C(g)} = CDC(\bar{g})$$

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since $\overline{C(g)} = C(\bar{g})$. Thus $CDC(\bar{g})$ is non-abelian since both $\bar{\sigma}$ and $\bar{\tau}\bar{\sigma}^{-1}\bar{\tau}^{-1}$ are in $\overline{E(L + \tau L)}$, and they do not commute.

Let us denote the residual space of h by R; h is the product of the two transvections $\Sigma T \Sigma^{-1}$ and T^{-1}. But the proper line of $\Sigma T \Sigma^{-1}$ is ΣL_1 and the proper line of T^{-1} is L_1. Since $\Sigma L_1 \neq L_1$, [**4**, 1.2] implies the residual space of $h = \Sigma T \Sigma^{-1} T^{-1}$ is $\Sigma L_1 + L_1$. Thus $R = \Sigma L_1 + L_1$.

Since L_1 is isotropic, R is either a hyperbolic plane or totally degenerate plane. But $CDC(\bar{g})$ is non-abelian, hence $CDC(\bar{h})$ is non-abelian. So if R were totally degenerate, 1.5 would imply $CDC(\bar{h})$ is abelian, a contradiction. Thus R is a hyperbolic plane.

Now we show $\Lambda\bar{\sigma}$ is a projective shearing. We saw above that

$$\bar{\sigma} \in \overline{E(L + \tau L)} \subseteq \overline{CDC(g)} \subseteq CD\overline{C(g)} = CDC(\bar{g}).$$

So we have $\Lambda\bar{\sigma} \in CDC(\Lambda\bar{g}) = CDC(\bar{h})$. By 1.3,

$$\bar{\Sigma} = \Lambda\bar{\sigma} \in CDC(\bar{h}) \subseteq \overline{E(R)};$$

thus we may assume the residual space of Σ is contained in R. If R is the residual space of Σ and $\Sigma|R$ is a scalar, then since $CDC(\bar{h}) \subseteq \overline{E(R)}$ we see that $\bar{\Sigma}$ centralizes $CDC(\bar{h})$ which contradicts the fact that $\bar{\sigma} \notin CCDC(\bar{g})$; if R is the residual space of Σ and $\Sigma|R$ is not a scalar, then 1.2 shows $E(R) \subseteq CDC(\Sigma)$ contradicting the fact $CDC(\bar{\sigma})$ is abelian. So Σ has residual space a line.

For the rest of this paper let us assume the hypotheses of 1.7 are in force. Thus, in particular, Λ is an isomorphism between the subgroups G and H of $PU(V, f_1)$ and $PU(W, f_2)$ respectively, and further, G and H both have enough projective transvections.

We will show that in fact Λ maps projective transvections to projective transvections. Note that if σ_1 and σ_2 are shearing $\neq 1$ with residual spaces L_1 and L_2, then by [**4**, 1.4 and 1.5], $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if and only if $L_1 = L_2$ or $f_1(L_1, L_2) = 0$. We also see that if $\sigma \in \Delta$ is a nontrivial shearing with residual space the line L, then $CC(\bar{\sigma}) = \overline{E(L)}$. For to prove $CC(\bar{\sigma}) = \overline{E(L)}$, note that $CC(\bar{\sigma}) \subseteq CDC(\bar{\sigma}) \subseteq \overline{E(L)}$ by 1.3. The inclusion $CC(\bar{\sigma}) \supseteq \overline{E(L)}$ is easily checked, and so $CC(\bar{\sigma}) = \overline{E(L)}$.

Definition. For a subspace U of V let $S(U)$ be all projective shearings in G whose residual lines are contained in U. For a subset X of G, let $C'(X)$ be all projective shearings in G which commute with each element of X.

1.8. *Let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be non-trivial commuting shearings in G with distinct residual lines L_1 and L_2. Then $C'C'(\bar{\sigma}_1, \bar{\sigma}_2) \subseteq S(L_1 + L_2)$, and $C'C'(\bar{\sigma}_1, \bar{\sigma}_2) = S(L_1 + L_2)$ if $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are both transvections.*

Proof. Clearly

$$C'(\bar{\sigma}_1, \bar{\sigma}_2) \supseteq S((L_1 + L_2)^*) \cup S(L_1) \cup S(L_2).$$

Thus $C'C'(\bar{\sigma}_1, \bar{\sigma}_2) \subseteq S(L_1 + L_2)$. However if σ_1 and σ_2 are both transvections then $C'(\bar{\sigma}_1, \bar{\sigma}_2) = S((L_1 + L_2)^*)$ and this implies $S(L_1 + L_2) = C'C'(\bar{\sigma}_1, \bar{\sigma}_2)$.

1.9 *Under the hypotheses of* 1.7, *if* $\bar{\sigma}$ *is a projective transvection in* G *then* $\Lambda\bar{\sigma}$ *is also a projective transvection.*

Proof. We may suppose $\bar{\sigma} \neq \bar{1}$. Let $\bar{\sigma}$ have residual line L_1 and choose an isotropic line L_2 in V such that $f_1(L_2, L_1) = 0$ and $L_2 \neq L_1$. Choose a non-trivial projective transvection $\bar{\sigma}_2$ in G with line L_2. Let the projective shearings $\Lambda\bar{\sigma}$ and $\Lambda\bar{\sigma}_2$ have lines L_1' and L_2'.

Next note that the totally degenerate plane $L_1 + L_2$ contains at least three distinct pairwise orthogonal isotropic lines. Thus $S(L_1 + L_2) = C'C'(\bar{\sigma}, \bar{\sigma}_2)$ contains at least three distinct pairwise commuting projective transvections with pairwise distinct double centralizers. So $C'C'(\Lambda\bar{\sigma}, \Lambda\bar{\sigma}_2)$ contains at least three distinct pairwise commuting projective shearings with pairwise distinct double centralizers. We know that $C'C'(\Lambda\bar{\sigma}, \Lambda\bar{\sigma}_2) \subseteq S(L_1' + L_2')$ so the plane $L_1' + L_2'$ contains at least three pairwise distinct lines K_1, K_2, K_3 such that $f_2(K_i, K_j) = 0$ if $i \neq j$. This implies the plane $L_1' + L_2'$ is totally degenerate which implies $\Lambda\bar{\sigma}$ is a projective transvection as desired.

Thus Λ in fact maps transvections in G to transvections in H.

Now for any hyperbolic plane R of V we define a second hyperbolic plane $\Psi(R)$ of W as follows. Choose a transformation $\bar{\sigma}$ in G such that σ has residual space R and such that σ is the product of two non-commuting transvections, $\sigma = \tau_1\tau_2$. By 1.4, $CDC(\bar{\sigma}) = \overline{E(R)}$. But then $\Lambda\bar{\sigma}$ is the product of two non-commuting transvections so $CDC(\Lambda\bar{\sigma}) = \overline{E(\Psi(R))}$ for some hyperbolic plane $\Psi(R)$ in W, again by 1.4. Clearly $\Psi(R)$ depends only on R and is independent of the particular non-commuting transvections τ_1 and τ_2 chosen in $E(R)$. If L is an isotropic line such that $L \subset R$ then $L' \subset \Psi(R)$, where $L \rightarrow L'$ is the bijection of the isotropic lines of V onto the isotropic lines of W obtained from the fact that Λ maps transvections in G to transvections in H. It is also easy to see that the map $R \rightarrow \Psi(R)$ of hyperbolic planes just defined is a bijection of the hyperbolic planes of V onto the hyperbolic planes of W such that $f_1(R_1, R_2) = 0$ if and only if $f_2(\Psi(R_1), \Psi(R_2)) = 0$, for any two hyperbolic planes R_1 and R_2 of V.

1.10 *If* R_1 *and* R_2 *are any two hyperbolic planes of* V, *then:*

$$R_1 \cap R_2 \neq 0 \Leftrightarrow \Psi(R_1) \cap \Psi(R_2) \neq 0$$

where $R \rightarrow \Psi(R)$ *is the bijection of hyperbolic planes previously defined.*

Before proving 1.10, we will first show how 1.10 can be used to prove the non-isomorphism of the symplectic and unitary congruence groups.

2. The symplectic and unitary congruence groups. Now we demonstrate our main theorem on the non-isomorphism of the symplectic and unitary congruence groups; the definitions of these congruence groups will be taken as in [**10**]. Under those definitions each symplectic or unitary congruence group has enough transvections.

THEOREM 2.1. *Let S_1 be a symplectic congruence group whose associated vector space has dimension at least* 6 *and whose associated field is of characteristic not* 2. *Let S_2 be a unitary congruence group whose associated hermitian form f_2 has index at least* 3 *and whose associated field is of characteristic not* 2. *Then S_1 and S_2 are not isomorphic.*

Proof. Let f_1 be the alternating form associated to the symplectic congruence group S_1. Define $G = \bar{S}_1$ and $H = \bar{S}_2$. Then if S_1 and S_2 were isomorphic, G and H would be isomorphic. Now we can assume that $G \subseteq PSp(V, f_1)$ and $H \subseteq PU(W, f_2)$ where f_2 is a skew-hermitian form which is not skew symmetric, and where f_1 is skew-symmetric. Now if Λ were an isomorphism from G onto H, then we could choose hyperbolic planes $\Psi(R_1)$ and $\Psi(R_2)$ in W which intersect in an anisotropic line of W. Then by 1.10, we see that $R_1 \cap R_2$ is a line in V, which is necessarily an isotropic line of V. Let $L = R_1 \cap R_2$. Then $L' \subseteq \Psi(R_1) \cap \Psi(R_2)$ and L' is an isotropic line of W by 1.9, since L is isotropic. But this contradicts the fact $\Psi(R_1) \cap \Psi(R_2)$ is anistropic. Thus no isomorphism Λ exists.

Now to complete the proof of 2.1, it only remains to establish Theorem 1.10. Theorem 1.10 can be proved using the argument of [**2**, p. 87]. For convenience we repeat that argument here, as a sequence of short lemmas. The following proposition is easily proved.

2.2. *Let R_1 and R_2 be hyperbolic planes in V with* $\dim (R_1 + R_2) = 3$. *Then* $(R_1 + R_2)^*$ *is the linear sum of its hyperbolic planes.*

For any subspace U of V let U_h denote the set of all hyperbolic planes contained in U. And if S is any set of hyperbolic planes in V, let S^* denote all hyperbolic planes in V orthogonal to each hyperbolic plane in the set S. We denote $(S^*)^*$ by S^{**}.

2.3 *Let R_1 and R_2 be hyperbolic planes in V with* $\dim (R_1 + R_2) = 3$. *Then* $\{R_1, R_2\}^{**} = (R_1 + R_2)_h$.

Proof. We always have $(R_1 + R_2)_h \subseteq \{R_1, R_2\}^{**}$. Now $\{R_1, R_2\}^*$ is the same as the set of all hyperbolic planes in the orthogonal complement of $R_1 + R_2$. Let $R \in \{R_1, R_2\}^{**}$. Then R is orthogonal to every hyperbolic plane in the orthogonal complement of $R_1 + R_2$, and hence R is orthogonal to the orthogonal complement of $R_1 + R_2$. Hence $R \subseteq R_1 + R_2$ and so $R \in (R_1 + R_2)_h$. Thus

$$\{R_1, R_2\}^{**} \subseteq (R_1 + R_2)_h.$$

COROLLARY 2.3(a) *If* $\dim (R_1 + R_2) = 3$, *then* $\{R_3, R_4\}^{**} = \{R_1, R_2\}^{**}$ *for any two distinct hyperbolic planes R_3, R_4, in $\{R_1, R_2\}^{**}$.*

2.4. *Let R_1 and R_2 be hyperbolic planes in V. If* $\dim (R_1 + R_2) = 4$, *there is a hyperbolic plane R_3 in $\{R_1, R_2\}^{**}$ such that $\{R_1, R_3\}^{**} \neq \{R_1, R_2\}^{**}$.*

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Proof. Choose a hyperbolic plane R_3 lying in $R_1 + R_2$ with dim $(R_1 + R_3) = 3$. Since $R_2 \not\subseteq R_1 + R_3$, 2.2 implies $R_2 \notin \{R_1, R_3\}^{**}$. But it always is true that $R_2 \in (R_1 + R_2)_h \subseteq \{R_1, R_2\}^{**}$. So $\{R_1, R_3\}^{**} \neq \{R_1, R_2\}^{**}$.

Proof of 1.10. Assume R_1 and R_2 are distinct hyperbolic planes in V with $R_1 \cap R_2 \neq 0$. Then dim $(R_1 + R_2) = 3$, so by 2.3(a) $\{R_3, R_4\}^{**} = \{R_1, R_2\}^{**}$ for any two distinct hyperbolic planes R_3 and R_4 in $\{R_1, R_2\}^{**}$. Hence, since Ψ is a bijection of the hyperbolic planes of V onto those of W which maps orthogonal hyperbolic planes to orthogonal hyperbolic planes, we have

$$\{\Psi(R_3), \Psi(R_4)\}^{**} = \{\Psi(R_1), \Psi(R_2)\}^{**}$$

for any two distinct hyperbolic planes $\Psi(R_3)$ and $\Psi(R_4)$ in $\{\Psi(R_1), \Psi(R_2)\}^{**}$. Thus dim $(\Psi(R_1) + \Psi(R_2)) = 3$ by 2.4. So $R_1 \cap R_2 \neq 0 \Rightarrow \Psi(R_1) \cap \Psi(R_2) \neq 0$. That $\Psi(R_1) \cap \Psi(R_2) \neq 0 \Rightarrow R_1 \cap R_2 \neq 0$, follows from consideration of the inverse isomorphism Λ^{-1} and the inverse bijection Ψ^{-1}.

3. Explicit description of the isomorphisms. Next we are going to examine more closely the possible isomorphisms between unitary congruence groups and give an explicit description of those isomorphisms that do exist.

3.1 *Let the hypotheses be as in Theorem* 1.7. *Let R_1 and R_2 be two distinct hyperbolic planes of V such that $R_1 \cap R_2 \neq 0$. Let R_3 be any hyperbolic plane of V. Then*

$$R_3 \subseteq R_1 + R_2 \Leftrightarrow \Psi(R_3) \subseteq \Psi(R_1) + \Psi(R_2).$$

Proof. Assume $R_3 \subseteq R_1 + R_2$. By 1.10, dim $(\Psi(R_1) + \Psi(R_2)) = 3$ since dim $(R_1 + R_2) = 3$. By 2.3, $R_3 \in (R_1 + R_2)_h = \{R_1, R_2\}^{**}$. Hence $\Psi(R_3) \in \{\Psi(R_1), \Psi(R_2)\}^{**} = (\Psi(R_1) + \Psi(R_2))_h$, again by 2.3. Thus $\Psi(R_3) \subseteq \Psi(R_1) + \Psi(R_2)$. The converse implication follows by considering Ψ^{-1}.

3.2 *Let R_1, R_2, R_3 be three distinct hyperbolic planes in V. Then*

$$R_1 \cap R_2 \cap R_3 \neq 0 \Leftrightarrow \Psi(R_1) \cap \Psi(R_2) \cap \Psi(R_3) \neq 0.$$

Proof. We need only prove $\Rightarrow$ since the converse implication follows by considering Ψ^{-1}. So assume $R_1 \cap R_2 \cap R_3 \neq 0$. Note that dim $(R_1 + R_2 + R_3)$ equals 3 or 4.

Case 1. dim $(R_1 + R_2 + R_3) = 4$: Then $R_3 \not\subseteq R_1 + R_2$ and so $\Psi(R_3) \not\subseteq \Psi(R_1) + \Psi(R_2)$ by 3.1. Now by 1.10, we have $\Psi(R_3) \cap \Psi(R_1) = L_1$ and $\Psi(R_3) \cap \Psi(R_2) = L_2$, for certain lines L_1 and L_2. If $L_1 \neq L_2$, then $\Psi(R_3) = L_1 + L_2 \subseteq \Psi(R_1) + \Psi(R_2)$, a contradiction. Hence $L_1 = L_2$. Thus $\Psi(R_1) \cap \Psi(R_2) \cap \Psi(R_3) = L_1$.

Case 2. dim $(R_1 + R_2 + R_3) = 3$: Let $R_1 \cap R_2 \cap R_3 = F_1 \cdot x$. Since x is in the hyperbolic plane R_1, there is an isotropic vector y in R_1 such that $f_1(x, y) \neq 0$. A dimension argument shows $(F_1x)^* \cap (F_1y)^*$ is not contained in $R_1 + R_2 + R_3$, and so there is an isotropic vector v in $(F_1x)^* \cap (F_1y)^*$

with v not in $R_1 + R_2 + R_3$. It follows that $y + v$ is isotropic, and $y + v$ is in neither $(F_1x)^*$ nor $R_1 + R_2 + R_3$. Let H be the plane spanned by the vectors x and $y + v$; then H is isotropic and regular and so H is a hyperbolic plane. Also $H \cap (R_1 + R_2 + R_3) = F_1 \cdot x$. By Case 1, $\Psi(R_1) \cap \Psi(R_2) \cap \Psi(H) = L$ and $\Psi(R_2) \cap \Psi(R_3) \cap \Psi(H) = K$ for certain lines L and K. Thus $L = \Psi(H) \cap \Psi(R_2) = K$. Hence $\Psi(R_1) \cap \Psi(R_2) \cap \Psi(R_3) = L$.

COROLLARY 3.2(a) *Let $\{R_\alpha\}$ be any family of hyperbolic planes in V. Then*

$$\bigcap_\alpha R_\alpha \neq 0 \Leftrightarrow \bigcap_\alpha \Psi(R_\alpha) \neq 0.$$

Now for any line L in V, let $\{R_\alpha\}$ be the family of all hyperbolic planes in V which contain L. By 3.2(a) we have that $\bigcap_\alpha \Psi(R_\alpha)$ is a line in W; we will call this line L'. It is easily verified that the map $L \to L'$ is an orthogonality-preserving bijection of the lines of V onto the lines of W. And for any line L in V, we see that L is isotropic if and only if L' is isotropic.

Now suppose we have two projective congruence groups G and H which are subgroups of $PU(V, f_1)$ and $PU(W, f_2)$ respectively. Suppose Λ is an isomorphism of G onto H. The proof of Theorem 2.1 shows that either f_1 and f_2 are both skew-symmetric, or else f_1 and f_2 are both skew-hermitian but not skew-symmetric. In the first case, dim V = dim W since the dimension of a symplectic space is twice the maximum number of pairwise orthogonal hyperbolic planes. In the second case, dim V = dim W because the dimension of a hermitian space equals the maximum number of pairwise orthogonal anisotropic lines. Now under the bijection $L \to L'$, two lines L_1 and L_2 are orthogonal if and only if L_1' and L_2' are orthogonal. So the images (under $L \to L'$) of all the lines contained in a fixed hyperplane of V again lie in a hyperplane of W. Thus the bijection $L \to L'$ satisfies the hypotheses of the Fundamental Theorem of Projective Geometry as given in [**1**, pp. 77-79]. Thus there is a semilinear isomorphism g of V onto W such that $gL = L'$ for all lines L of V. It follows that g preserves orthogonality; i.e.,

$$f_1(L_1, L_2) = 0 \Leftrightarrow f_2(fL_1, gL_2) = 0$$

for any two lines, L_1, L_2 in V. Thus by [**7**, Theorem 4.1] it follows that

(i) $$\phi(J_1(\lambda)) = J_2(\phi(\lambda)) \quad \text{for all } \lambda \in F_1,$$

where ϕ is the field isomorphism associated to g. (Here J_i denotes the involution of f_i for $i = 1, 2$.)

(ii) $$f_2(gx, gy) = \alpha \cdot \phi(f_1(x, y)) \quad \text{for all } x, y \in V,$$

where α is a scalar in F_2 such that $J_2(\alpha) = \alpha$, and

(iii) $$\sigma \in U(V, f_1) \text{ implies } g\sigma g^{-1} \in U(W, f_2).$$

A semilinear isomorphism g of V onto W satisfying (i) and (ii) above is called a unitary semi-similitude. By (iii) we can define a map Λ_g of $U(V, f_1)$ onto $U(W, f_2)$ by

$$\Lambda_g(\sigma) = g\sigma g^{-1}, \quad \text{for all } \sigma \in U(V, f_1).$$

Clearly Λ_g is an isomorphism. Therefore, Λ_g induces an isomorphism $\bar{\Lambda}_g$ of $PU(V, f_1)$ onto $PU(W, f_2)$ defined by

$$\bar{\Lambda}_g(\bar{\sigma}) = \overline{\Lambda_g(\sigma)}, \quad \text{for all } \bar{\sigma} \in PU(V, f_1).$$

Therefore, $\Lambda_g^{-1} \circ \Lambda$ is a monomorphism of G into $PU(V, f_1)$ which for any isotropic line L of V maps each projective transvection in G with proper line L to a projective transvection with proper line L again. An argument similar to the proof of [**9**, Proposition 4.4] shows that $\bar{\Lambda}_g^{-1} \circ \Lambda$ is the identity map on G, and so $\Lambda = \bar{\Lambda}_g | G$. Thus we have proved

THEOREM 3.3. *Let G and H be projective unitary congruence groups. Suppose $G \subseteq PU(V, f_1)$ and $H \subseteq PU(W, f_2)$ where V and W are finite-dimensional vector spaces over fields F_1 and F_2 respectively, each field of characteristic not* 2. *Suppose f_1 and f_2 both have Witt indices at least* 3, *and let Λ be an isomorphism of G onto H. Then there is a unitary semi-similitude g of V onto W such that*

$$\Lambda(\bar{\sigma}) = \bar{\Lambda}_g(\bar{\sigma}) \quad \textit{for all } \bar{\sigma} \in G.$$

COROLLARY. *Let S_1 and S_2 be unitary congruence groups such that $S_1 \subseteq U(V, f_1)$ and $S_2 \subseteq U(W, f_2)$, where the hypotheses on V, W, F_1, F_2, f_1, and f_2 are as in Theorem* 3.3. *Suppose Λ is an isomorphism of S_1 onto S_2. Then there is a unitary semi-similitude g of V onto W and a homomorphism χ of S_1 into the elements of F_2 of norm* 1 *such that*

$$\Lambda(\sigma) = \chi(\sigma) \cdot g\sigma g^{-1} \quad \textit{for all } \sigma \in S_1.$$

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