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A GENERAL TAUBERIAN CONDITION THAT IMPLIES EULER SUMMABILITY

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ABSTRACT. Let V be any summability method (whether linear or conservative or not), $0 < p < 1$ and s a real or complex sequence. Let E_p denote the matrix of the Euler method. A theorem is proved, giving a condition under which the V-summability of $E_p s$ will imply the E_p-summability of s. This extends, in generalized form, an earlier result of N. H. Bingham who considered the case where s is a real sequence and $V = B$ (Borel's method). It is also proved that even for real sequences, the condition given in the theorem cannot be replaced by the condition used by Bingham.

For $0 < p < 1$, the sequence $s = \{s_n\}$ of real or complex numbers is said to be *summable by the Euler method* E_p if $E_p s = \{t_n\} \in c$ (the convergent sequences), where

$$t_n = \sum_{k=0}^{n} h_{nk} s_k \quad (n = 0, 1, \ldots)$$

and $h_{nk} = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le k \le n$ and $= 0$ for $k > n$.

The sequence is said to be *summable by the Borel method* B if $\lim_{x\to\infty} e^{-x} \sum_{k=0}^{\infty} s_k x^k / k!$ exists.

For basic properties of the methods E_p and B, and relations between them, see [3], [9]. The methods E_p, B and certain related summability methods are important in probability and analytic number theory (see [1] for some references). It is well known that

$$s \text{ is } E_p\text{-summable} \Rightarrow s \text{ is } B\text{-summable} \quad (0 < p < 1).$$

The major (Tauberian) result in the reverse direction was proved by Meyer-König [4]:

THEOREM 1. *Let s be a real or complex sequence that is B-summable and let $s_n = O(1)$. Then s is E_p-summable for every* $0 < p < 1$.

It is also well known ([3], Theorems 156 and 157) that if s is any real or complex sequence and $V = B$ or E_p $(0 < p < 1)$ or one of certain related summability methods, then $\sqrt{n}a_n := \sqrt{n}(s_n - s_{n-1}) = O(1)$ is a Tauberian condition for the method V (that is, any V-summable sequence s with $\sqrt{n}a_n = O(1)$ must be convergent).

Theorem 1 was generalized in [6] by the present author as follows.

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THEOREM 2. *Let V be any "summability method" (whether conservative or linear or not), applicable to some sequences and such that*

(2) $\sqrt{n}a_n := \sqrt{n}(s_n - s_{n-1}) = O(1)$ *is a Tauberian condition for* V.

Then for any real or complex sequence s and $0 < p < 1$,

(3) $E_p s$ *is* V*-summable and* $s_n = O(1) \Rightarrow s$ *is* E_p*-summable.*

For *real sequences*, N. H. Bingham has generalized Theorem 1 in a different direction, replacing the condition $s_n = O(1)$ by a condition that is even more general than $s_n = O_L(1)$.

THEOREM 3 (BINGHAM [1]). *Let* $s = \{s_n\}$ *be a real sequence such that*

(4. 1) *s is Borel-summable*

and

(4. 2) $$\lim_{h\to 0}\liminf_{x\to\infty}\inf_{0\le u\le h}\left(\frac{1}{h\sqrt{x}}\right)\sum_{x\le n<x+u\sqrt{x}} s_n > -\infty.$$

Then s is E_p*-summable for every* $p \in (0,1)$.

If we wished to get a result for *complex* sequences, similar to Theorem 3, we would have to consider the real and imaginary parts separately, involving *two* relations of the type (4.2), or, what is more natural, replace (4.2) by the appropriate two-sided condition obtained from (4.2) by replacing the sum in (4.2) by the absolute value of the sum. But we prove rather more in the following theorem.

THEOREM 4. *Let V be any summability method (whether conservative or linear or not), satisfying (2). Let s be a real or complex sequence such that for some p in* $(0,1)$,

(5. 1) $E_p s$ *is* V*-summable*

and let s satisfy the condition (5.2) or the weaker condition (6) given below:

(5.2) $$\lim_{h\to 0+}\limsup_{x\to\infty}\sup_{0\le u\le h}\left(\frac{1}{h\sqrt{x}}\right)\left|\sum_{x\le n<x+u\sqrt{x}} s_n\right| < \infty$$

(6) $$\limsup_{x\to\infty}\sup_{0\le u\le h}\frac{1}{\sqrt{x}}\left|\sum_{x\le n<x+u\sqrt{x}} s_n\right| < \infty \quad \text{for some } h > 0.$$

Then s is E_p*-summable.*

PROOF. Let $E_p s = \{t_n\}$ be defined by (1) and

(7) $$d_n = t_{n+1} - t_n = \sum_{k=0}^{n+1} a_{nk} s_k \text{ (say).}$$

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Since V satisfies (2), to prove the theorem it is enough to prove that

$$d_n = O(n^{-1/2}). \tag{8}$$

We note that the condition (5.2) is equivalent to the assertion that

$$\limsup_{x\to\infty} \sup_{0\le u\le h} \frac{1}{\sqrt{x}} \Big| \sum_{x\le n<x+u\sqrt{x}} s_n \Big| = O(h) \quad \text{as } h \to 0+. \tag{9}$$

Since (9) implies (6), we may assume that s is a real or complex sequence such that $E_p s = \{t_n\}$ is V-summable for a certain p in $(0, 1)$ and that (6) holds. {We remark that if (6) holds for some $h > 0$, then it holds for every fixed $h > 0$.}

Now (6) implies that $s_n = O(\sqrt{n})$ and hence it follows (from Theorem 138 of [3]) that if ζ is a constant with $1/2 < \zeta < 2/3$, then the contribution to the sum (7) of values of k outside the range

$$pn - n^{\zeta} \le k \le pn + n^{\zeta} \tag{$*$}$$

is of the order $O\big(\exp(-n^{\eta})\big)$ for some constant $\eta > 0$. Hence, to prove that $d_n = O(n^{-1/2})$, it is enough to prove that

$$\sum\nolimits^{*} a_{nk} s_k = O(n^{-1/2}) \tag{10}$$

where the symbol $\sum^{*}$ denotes summation over the range in $(*)$. We now write

$$S_k = \sum_{i=0}^{k} s_i \quad \text{and} \quad T_k(n) = S_k - S_{[pn]}$$

where $[pn]$ denotes the integral part of pn. Then, writing $k = pn + t$, it follows from (6) that

$$T_k(n) = O(n^{1/2}) \quad \text{if } |t| \le n^{1/2} \text{ and } T_k(n) = O(|t|) \quad \text{if } |t| > n^{1/2}. \tag{11}$$

So, for the whole range in $(*)$ we get

$$T_k(n) = O(n^{1/2}) + O(|t|).$$

Now the sum

$$\begin{aligned} \sum\nolimits^{*} a_{nk} s_k &= \sum\nolimits^{*} a_{nk}[T_k(n) - T_{k-1}(n)] \\ &= \sum\nolimits^{*} (a_{nk} a_{n,k+1}) T_k(n) + \text{ (two end terms)}. \end{aligned}$$

But the two end terms are again $O\big(\exp(-n^{\eta})\big)$. Hence, to prove (10) (and the theorem), it is enough to prove that

$$\sum\nolimits^{*} (a_{nk} - a_{n,k+1}) T_k(n) = O(n^{-1/2}). \tag{12}$$

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Using the fact that

$$a_{nk} = h_{n+1,k} - h_{n,k} = h_{nk}\Big(\frac{n+1}{n+1-k}(1-p) - 1\Big) \\ = h_{nk}\big(k-(n+1)p\big)/(n+1-k), \tag{13}$$

it is easy to verify that (with the operator Δ acting on k),

$$\Delta a_{nk} = a_{nk} - a_{n,k+1} = \frac{k+1-(n+1)p}{n-k}\Delta h_{nk} - \frac{(1-p)(n+1)}{(n+1-k)(n-k)}h_{nk}.$$

But $\Delta h_{nk} = h_{nk}(1 - \frac{n-k}{k+1}\cdot\frac{p}{1-p}) = \frac{(k+1)-(n+1)p}{(k+1)(1-p)}h_{nk}$. Hence, in the range $(*)$ under consideration, we see from (11) that

$$\Delta a_{nk} = a_{nk} - a_{n,k+1} = O\Big[h_{nk}\Big(\frac{t^2}{n^2} + \frac{1}{n}\Big)\Big]$$

and hence also that

$$(a_{nk} - a_{n,k+1})T_k(n) = O\Big[h_{nk}\Big(\frac{1}{n^{1/2}} + \frac{|t^3|}{n^2}\Big)\Big].$$

But $h_{nk} = O(n^{-1/2}e^{-t^2/n})$, so that the sum considered in (12) is

$$O\Big[\sum e^{-t^2/n}\Big(\frac{1}{n} + \frac{|t^3|}{n^{5/2}}\Big)\Big], \tag{14}$$

where the sum is taken over those values of t with $|t| \leq n^{\zeta}$ for which $pn + t$ is an integer. The quantity in (14) is

$$O\Big(\int_0^\infty \frac{1}{n}e^{-t^2/n}\,dt\Big) + O\Big(\int_0^\infty \frac{t^3}{n^{5/2}}e^{-t^2/n}\,dt\Big)$$

and we see that this is $O(n^{-1/2})$, by making the substitution $t = un^{1/2}$. Thus (12), and the theorem, are proved.

REMARKS. (1) Since the condition (5.2) holds whenever $s_n = O(1)$, Theorem 4 is clearly a generalization of Theorem 2.

(2) When $V = B$, the Borel method, the conditions (4.1) and (5.1) are equivalent (see for instance [3], proof of Theorem 128). However, in Theorem 4 we cannot replace (5.1) by the condition that "s is V-summable" and change the conclusion to (even) "s is Borel-summable". To see this, let $s = \{s_k\}$ where $s_k = \sum_{j=0}^{k} a_j$ and $a_j = 1$ if $j \in \{n^2\}$ and $a_j = 0$ otherwise. Then, taking $h = 2$ and $n_k = k^2$ for all k, we see that the series $\sum a_n$ satisfies the conditions of the 'Gap Tauberian Theorem' for the Borel method due to Meyer-König and Zeller ([5], Satz 1.5): (i) $a_n = 0$ for $n \notin \{n_k\}$ where $n_{k+1} - n_k \geq h\sqrt{n_k}$ for some $h > 0$ and (ii) $a_n = O(K^n)$ for some constant K. (Indeed Gaier ([2], Satz 1) has shown that the condition (ii) can be omitted.) Hence the divergent sequence s is not B-summable. But, since s is unbounded, there exists (by [8]; [9] Satz 26.X) a normal, regular matrix method V which sums only those sequence of the form $\{\lambda s_n + u_n\}$ where $\{u_n\}$ is convergent. Then V satisfies (2) and s satisfies (4.2) and (5.2), but s is not Borel-summable.

(3) It is shown in Theorem 5 below that, *even for real sequences*, the condition (4.2) cannot replace the condition (5.2) in Theorem 4, and hence Theorem 4 is, in a sense, a best possible one; indeed, we prove somewhat more.

THEOREM 5. *For any real sequence s, let* (BSD) *denote the condition*
(BSD)*:* $\liminf(s_m - s_n) \geq 0$ *as* $m > n \longrightarrow \infty$, $(m-n)/\sqrt{n} \longrightarrow 0$.
For arbitrary given $p \in (0,1)$, *there exists a regular, row-finite matrix V and a real sequence s such that*

(i) *the condition* (BSD) *is a Tauberian condition for V [and hence (2) holds];*
(ii) *s and* $E_p s$ *are V-summable;*
(iii) $s_n \geq 0$ *for all n [and hence (4.2) is satisfied trivially];*
(iv) *s is not Borel-summable [and hence is not* E_q*-summable for any* $q \in (0,1)$*].*

PROOF. We use the same notation as in the proof of Theorem 4. We shall also write $x(i)$ for x_i if i is a symbol containing a subscript. Let $p \in (0,1)$ be given. From the relation (13) we see that for each fixed k, $a_{nk} < 0$ if $n > k/p$. We now define sequences $\{k_r\}$, $\{n_r\}$ of integers inductively as follows. Choose any nonnegative integer as k_0. When k_r has been chosen, choose $n_r > k_r + 2$ so that $a(n_r, k_r) < 0$. Having chosen n_r, choose any integer greater than $n_r + 2$ as k_{r+1}. Now define the sequence $s = \{s_k\}$ as follows:

$$s_k = M_r \quad \text{if } k = k_r \text{ for some } r, \text{ and } s_k = 0 \text{ otherwise},$$

where the numbers M_r will be defined inductively as described below. We have

$$d(n_r) = t(n_r + 1) - t(n_r) = \sum_{i=0}^{r} a(n_r, k_i) M_i \quad (r = 0, 1, 2, \ldots),$$

since the other terms in the expression for d_{n_r} will vanish. We can choose an increasing sequence $\{M_i\}$ of positive integers such that, for $r = 0, 1, 2, \ldots,$

$$d(n_r) = \sum_{i=0}^{r} a(n_r, k_i) M_i \leq -r. \tag{15}$$

For, since $a_{n_r}, k_r < 0$, if $M_0, M_1, \ldots, M_{r-1}$ have been chosen, we can ensure that (15) holds, by taking M_r sufficiently large. Now the relation (15) implies that

$$\liminf(t_{n+1} - t_n) = -\infty.$$

Thus the sequence $t = E_p s$ does not satisfy the Tauberian condition (BSD), and, in particular, s is not E_p-summable. Since s satisfies (4.2), it follows from Theorem 3 that it is not B-summable.

We note also that the definitions of $\{s_k\}$ and $\{M_r\}$ ensure that the sequence $\lambda s + \mu t$ will be unbounded for all real λ and μ, unless $\lambda = \mu = 0$. Now, by a result of Wilansky and Zeller ([7], Theorem 3), there exists a regular, row-finite matrix V which will sum precisely those sequences x of the form $x = z + \lambda s + \mu t$ where $z \in c$ (the convergent sequences) and λ, μ are real constants. It is easy to see that such a sequence will not satisfy the condition (BSD) unless $\lambda = \mu = 0$, that is, unless x is a convergent sequence. Hence (BSD) is a Tauberian condition for the method V. Since V sums s and $E_p s$, and $E_p s$ is not convergent, the theorem is proved.

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