# ON THE QUOTIENTS OF INDECOMPOSABLE INJECTIVE MODULES

A.K. Tiwary

(received October 5, 1965)

It is well known that $Z(p^{\infty})$ is isomorphic to each of its non-zero homomorphic images [3]. The aim of the present note is to generalize this fact about $Z(p^{\infty})$ to indecomposable injective modules over rings more general than the ring of integers which will include Dedekind domains as a special case.

I should like to thank Professor B. Banaschewski for his help and guidance.

Throughout this paper we consider $R$ to be an integral domain and $P \subseteq R$ a maximal ideal.

Let $R_P$ denote the ring of quotients of $R$ with respect to $P$ and define $\varphi: R/P \longrightarrow R_P/R_PP$ by $\varphi(x+P) = x + R_PP$. The mapping $\varphi$ is clearly an R-module homomorphism and is one-to-one since $x + R_PP = 0$ implies $x \in R_PP \cap R = P$. To show that it is an epimorphism, let $(r/s) \in R_P$. Then, since $R = Rs + P$, there exist $a \in R$, $b \in P$ such that $1 = as + b$. Hence $(r/s) + R_PP = ra + (r/s)b + R_PP = ra + R_PP = \varphi(ra + P)$. Thus $\varphi$ is an R-isomorphism. Since $R_P/R_PP$ is an $R_P$-module, this shows that $R/P$ can be made into an $R_P$-module by $(1/s)(r+P) = ra + P$ if $s \in R \setminus P$ and $1 = as + b$ as above, such that $\varphi$ is an $R_P$-isomorphism. This fact extends as follows to $E$, the R-injective hull of $R/P$:

LEMMA By extending the $R_P$-module structure of $R/P$, $E$ can be made into an $R_P$-module such that it is isomorphic to the $R_P$-injective hull of $R_P/R_PP$.

Canad. Math. Bull. vol. 9, no. 2, 1966



https://doi.org/10.4153/CMB-1966-024-1 Published online by Cambridge University Press

Proof. We first show that for $s \in R \setminus P$, the $R$-module homomorphism $f: x \longrightarrow sx$, is an automorphism of $E$. Let $0 \neq x \in E$, then since $E$ is an essential extension of $R/P$, there exists a non-zero element $rx \in Rx \cap R/P$ and since $f$ is one-to-one on $R/P$ we have $0 \neq f(rx) = srx$ which implies that $sx \neq 0$ and so $f$ is one-to-one on $E$. The fact that $E$ is indecomposable [2] and $f(E)$ is isomorphic to $E$ and hence injective, gives $f(E) = E$. $f$ is, therefore, an automorphism of $E$. Thus for any $x \in E$, $s \in R \setminus P$, there exists a unique element $y \in E$ such that $x = sy$ and we can define $(1/s)x = y$ which makes $E$ into an $R_P$-module.

Finally, to prove the required isomorphism, let $E'$ be an $R_P$-injective hull of $R_P / R_P P$ and $j: R/P \longrightarrow E$ and $i: R_P / R_P P \longrightarrow E'$ the natural injections. From the $R$-injectivity of $E$ it follows that there exists an $R$-homomorphism $\psi: E' \longrightarrow E$ such that $j \circ \varphi^{-1} = \psi \circ i$ with $\varphi$ as defined above. Take $x' \in \ker \psi$ and suppose $x' \neq 0$. Then $R_P x' \cap R_P / R_P P \neq 0$ since $E'$ is an $R_P$-essential extension of $R_P / R_P P$. Hence there exists a non-zero element $rx' \in Rx' \cap R_P / R_P P$. As $\psi \circ i$ is a monomorphism, we have $0 \neq \psi \circ i\,(rx') = \psi(rx') = r\psi(x') = 0$, a contradiction. Hence $\psi$ is a monomorphism. Now, if $s \in R \setminus P$, $x' \in E'$, then there exists a unique element $y \in E$ such that $sy = \psi(x') = \psi(s(1/s)x') = s(\psi(1/s)x)$ whence $(1/s)\psi(x') = \psi((1/s)x')$. This shows that $\psi$ is $R_P$-linear. It follows that $\psi(E')$ is $R_P$-injective and therefore an $R_P$-direct summand of $E$. In particular $\psi(E')$ is an $R$-direct summand of $E$. Hence $\psi(\;') = E$. Thus $\psi: E' \longrightarrow E$ is the desired $R_P$-isomorphism.

Proposition. Let $R_P$ be a principal ideal ring. Then the injective hull of $R/P$ is isomorphic to any of its quotients by a proper submodule.

Proof. Here $R_P P = R_P \pi$ for some $\pi \in R_P$, and $R_P / R_P \pi$ has $E = R_P[\pi^{-1}]/R_P \pi$ as its injective hull [1] where $R_P[\pi^{-1}]$ is generated by $\pi^{-1}$ as a ring extension of $R_P$ in the quotient field of $R$. By the lemma it suffices to consider this $R$-module $E$.



https://doi.org/10.4153/CMB-1966-024-1 Published online by Cambridge University Press

We first show that every R-submodule of E is also an $R_P$-submodule which will imply that the $R_P$-submodules are the same as the R-submodules. For this, it is sufficient to prove that if $S \subseteq E$ is any R-submodule and $s_o \in R/P$, then $(1/s_o)S \subseteq S$. Now, $R[\pi^{-1}] = \bigcup_{k \geq 0} R_P \pi^{-k}$ implies that any element in S is of the form $x = (a/s)\pi^{-k} + R_P \pi$ where $a \in R$, $s \in R \setminus P$ and k an integer. From $R = Rs_o + P^{k+1}$ [4], we get $1 = s_o t + u$ with $t \in R$, $u \in P^{k+1}$ and, therefore, $(1/s_o)x = tx + (1/s_o)ux = tx + (u/s_o)((a/s)\pi^{-k} + R_P\pi) = tx \in S$. Hence $(1/s_o)S \subseteq S$ and we can talk about the submodules of E without reference to R or $R_P$.

We next show that every submodule of E is of the form $R_P\pi^{-n}/R_P\pi$. The lattice of all submodules of E is isomorphic to the lattice of $R_P$-submodules of $R_P[\pi^{-1}]$ which contain $R_P\pi$. Hence any submodule of E corresponds to exactly one fractional ideal S of $R_P$ with $R_P\pi \subseteq S \subseteq R_P[\pi^{-1}]$. Let $S_k = S \cap R_P\pi^{-k}$ then $R_P\pi \subseteq S_k \subseteq R_P\pi^{-k}$ which implies $R_P\pi^{k+1} \subseteq S_k\pi^k \subseteq R_P$. By the fact that $R_P$ is a principal ideal ring, one has $S_k\pi^k = R_P\pi^{\ell_k}$ for some $\ell_k$ with $0 \leq \ell_k \leq k+1$. Therefore, $S_k = R_P\pi^{\ell_k - k}$. If S corresponds to a proper submodule of E, then $S \subset R_P[\pi^{-1}]$ and since $S = \bigcup_{k \geq 0} S_k$ and the $S_k$'s form an ascending sequence, one has $S = R_P\pi^{-n}$ for some integer n. Thus every proper submodule of E is of the form $R_P\pi^{-n}/R_P\pi$, and any quotient of E by such a submodule may be expressed as $R_P[\pi^{-1}]/R_P\pi^{-n}$.

Now, if we compose the homomorphism $x \longrightarrow \pi^{-(n+1)}x$ of $R_P[\pi^{-1}]$ into itself, with the natural homomorphism $y \longrightarrow y + R_P\pi^{-n}$ from $R_P[\pi^{-1}]$ to $R_P[\pi^{-1}]/R_P\pi^{-n}$, we get an epimorphism $R_P[\pi^{-1}] \longrightarrow R_P[\pi^{-1}]/R_P\pi^{-n}$ whose kernel is $R_P\pi$. This shows that $E$ is isomorphic to $R_P[\pi^{-1}]/R_P\pi^{-n}$.

Remark. If $R$ is a Dedekind domain then each proper prime ideal $P$ of $R$ is maximal, and $R_P$ is a principal ideal ring [4]; therefore, the Proposition then applies to any $R/P$. It follows from this that the indecomposable injective torsion modules over a Dedekind domain all have the property that they are isomorphic to any of their non-zero homomorphic images.

In conclusion we provide an example where an indecomposable injective module has a quotient module which is neither zero nor isomorphic to itself:

Let $R$ be a Noetherian domain, $P$ a non-zero, non-maximal prime ideal in $R$ and $E$ an injective hull of $R/P$. Then there exists a maximal ideal $M$ such that $O \subset P \subset M \subset R$ and so $E \supseteq R/P \supset M/P \neq 0$. Hence $E/(M/P) \neq 0$. We will show that $E$ is not isomorphic to $E/(M/P)$. Assume the contrary. Then $E/(M/P)$ is indecomposable injective and contains $(R/P)/(M/P)$ which is isomorphic to $R/M \neq 0$, and hence $E/(M/P)$ is isomorphic to the injective hull of $R/M$. This implies that $R/M$ and $R/P$ have isomorphic injective hulls which leads to a contradiction since $P$ and $M$ are different prime ideals [2]. Thus the quotient module $E/(M/P)$ is neither zero nor isomorphic to $E$ and we have a counter-example where the above proposition fails to be true.

## REFERENCES


1. B. Banaschewski, On Coverings of Modules, Math. Nachr. (to appear).

2. E. Matlis, Injective Modules Over Noetherian Rings, Pacific J. Math., volume 8, (1958), pp. 511-528.

3. I. Kaplansky, Infinite Abelian Groups, University of Michigan Press, (1954).

4. P. Samuel and O. Zarisky, Commutative Algebra, volume 1, Princeton (1958).


McMaster University