APPROXIMATION OF FUNCTIONS BY A NEW CLASS OF LINEAR OPERATORS

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1. Introduction

Various extensions and generalizations of Bernstein polynomials have been considered among others by Szasz [13], Meyer-Konig and Zeller [8], Cheney and Sharma [1], Jakimovski and Leviatan [4], Stancu [12], Pethe and Jain [11]. Bernstein polynomials are based on binomial and negative binomial distributions. Szasz and Mirakyan [9] have defined another operator with the help of the Poisson distribution. The operator has approximation properties similar to those of Bernstein operators. Meir and Sharma [7] and Jain and Pethe [3] deal with generalizations of Szasz-Mirakyan operator. As another generalization, we define in this paper a new operator with the help of a Poisson type distribution, consider its convergence properties and give its degree of approximation. The results for the Szasz-Mirakyan operator can easily be obtained from our operator as a particular case.

2. The operator and its convergence

The operator and its convergence are based on the following two lemmas:

LEMMA 1. *For* $0 < \alpha < \infty$, $|\beta| < 1$, *let*

$$\omega_\beta(k, \alpha) = \alpha(\alpha + k\beta)^{k-1} e^{-(\alpha + k\beta)}/k!; \quad k = 0, 1, 2, \cdots \tag{2.1}$$

then

$$\sum_{k=0}^{\infty} \omega_\beta(k, \alpha) = 1. \tag{2.2}$$

It may be mentioned that (2.1) is a Poisson-type distribution which has been considered by Consul and Jain [2].

The proof of the lemma may be based upon results given by Jensen [5]. If we start with Lagrange's formula

(2.3) $$\phi(z) = \phi(0)+\sum_{k=1}^{\infty} \frac{1}{k!}\left[\frac{d^{k-1}}{dz^{k-1}}((f(z))^k)\phi'(z)\right]_{z=0}\left(\frac{z}{f(z)}\right)^k$$

and proceed by setting

$$\phi(z) = e^{\alpha z} \text{ and } f(z) = e^{\beta z}$$

we shall get

(2.4) $$e^{\alpha z} = \sum_{k=0}^{\infty} \alpha(\alpha+k\beta)^{k-1}u^k/k!,\ u = ze^{-\beta z},$$

where z and u are sufficiently small such that $|\beta u| < e^{-1}$ and $|\beta z| < 1$.

By taking $z = 1$, the lemma in (2.2) is obvious.

LEMMA 2. Let

(2.5) $$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} (\alpha+\beta k)^{k+r-1}e^{-(\alpha+\beta k)}/k!,\ r = 0, 1, 2, \cdots$$

and

(2.6) $$\alpha S(0, \alpha, \beta) = 1.$$

Then

(2.7) $$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k(\alpha+k\beta)S(r-1, \alpha+k\beta, \beta),$$

PROOF. It can easily be seen that the functions $S(r, \alpha, \beta)$ satisfy the reduction formula

(2.8) $$S(r, \alpha, \beta) = \alpha S(r-1, \alpha, \beta)+\beta S(r, \alpha+\beta, \beta).$$

By a repeated use of (2.8), the proof of the lemma is straightforward.

From (2.7) and (2.6) when $\beta < 1$ we get

(2.9) $$S(1, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k = 1/(1-\beta),$$

and

(2.10) $$S(2, \alpha, \beta) = \sum_{k=0}^{\infty} \frac{\beta^k(\alpha+k\beta)}{(1-\beta)} = \frac{\alpha}{(1-\beta)^2}+\frac{\beta^2}{(1-\beta)^3}.$$

We may now define the operator as

(2.11) $$P_n^{[\beta]}(f; x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx)f(k/n),$$

where $1 > \beta \geqq 0$ and $w_\beta(k, nx)$ has been defined in (2.1).

The parameter β may depend on the natural number n. It is easy to see that for $\beta = 0$, (2.11) reduces to Szasz--Mirakyan operator [9].

The convergence property of the operator $P_n^{[\beta]}(f: x)$ is proved in the following theorem:

THEOREM (2.1). *If* $f \in C[0, \infty)$ *and* $\beta \to 0$ *as* $n \to \infty$ *then the sequence* $\{P_n^{[\beta]}(f; x)\}$ *converges uniformly to* $f(x)$ *in* $[a, b]$, *where* $0 \leqq a < b < \infty$.

PROOF. Since $P_n^{[\beta]}(f; x)$ is a positive linear operator for $1 > \beta \geqq 0$, it is sufficient, by Korovkin's result, to verify the uniform convergence for test functions $f(t) = 1$, t and t^2.

It is clear from (2.2) that

$$P_n^{[\beta]}(1; x) = 1. \tag{2.12}$$

Going on to $f(t) = t$ and using (2.9) we have

$$\begin{aligned} P_n^{[\beta]}(t; x) &= xn \sum_{k=1}^{\infty} \frac{(nx+\beta k)^{k-1}}{k!} e^{-(nx+\beta k)} \left(\frac{k}{n}\right) \\ &= xS(1, nx+\beta, \beta) = \frac{x}{1-\beta}. \end{aligned} \tag{2.13}$$

Proceeding to the function $f(t) = t^2$, it can easily be shown that

$$\begin{aligned} P_n^{[\beta]}(t^2; x) &= xn \sum_{k=0}^{\infty} \frac{(nx+\beta k)^{k-1}}{k!} e^{-(nx+\beta k)} \frac{k^2}{n^2} \\ &= \frac{x}{n} [S(2, nx+2\beta, \beta)+S(1, nx+\beta, \beta)] \end{aligned}$$

and a use of (2.9) and (2.10) yields

$$P_n^{[\beta]}(t^2; x) = \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}. \tag{2.14}$$

Thus combining the results of (2.12), (2.13) and (2.14) we have

$$\operatorname*{Lim}_{n\to\infty} P_n^{[\beta]}(t^2; x) = x^r,\ r = 0, 1, 2, \text{ as } \beta \to 0$$

and hence by Korovkin's theorem the proof of theorem (2.1) is complete.

3. Order of approximation

THEOREM (3.1). *If* $f \in C[0, \lambda]$ *and* $1 > \beta'/n \geqq \beta \geqq 0$ *then*

$$|f(x)-P_n^{[\beta]}(f; x)| \leqq [1+\lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}] \cdot \omega(1/n^{\frac{1}{2}}),$$

where $w(\delta) = \sup |f(x'')-f(x')|;\ x', x'' \in [0, \lambda]$, δ *being a positive number such that* $|x''-x'| < \delta$.

PROOF. By using the properties of modulus of continuity

$$|f(x'')-f(x')| \leqq w(|x''-x'|); \tag{3.1}$$

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$$w(\gamma\delta) \leqq (\gamma+1)w(\delta), \gamma > 0 \tag{3.2}$$

and noting the fact that

$$\sum_{k=0}^{\infty} \omega_\beta(k, nx) = 1 \text{ and } \omega_\beta(k, nx) \geqq 0, \forall n, k$$

it can easily be seen, by the application of Cauchy's inequality, that

$$\begin{aligned} |f(x)-P_n^{[\beta]}(f;x)| &\leqq \left\{1+\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{nx(nx+\beta k)^{k-1}}{k!}e^{-(nx+\beta k)}\left|x-\frac{k}{n}\right|\right\}\omega(\delta) \\ &\leqq \left\{1+\frac{1}{\delta}\left[\sum_{k=0}^{\infty}\frac{nx(nx+\beta k)^{k-1}}{k!}e^{-(nx+\beta k)}\left(x-\frac{k}{n}\right)^2\right]^{\frac{1}{2}}\right\}\omega(\delta). \end{aligned} \tag{3.3}$$

Now by linearity of the operator and by using (2.12), (2.13) and (2.14) we have

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}\left(x-\frac{k}{n}\right)^2 &= x^2P_n^{[\beta]}(1;x)-2xP_n^{[\beta]}(t;x)+P_n^{[\beta]}(t^2;x) \\ &= x^2\beta^2/(1-\beta)^2+x/n(1-\beta)^3 \leqq \lambda[\lambda\beta\beta'/(1-\beta)^2+1/(1-\beta)^3]/n \\ &\leqq \lambda[1+\lambda\beta\beta']/n. \end{aligned} \tag{3.4}$$

Hence using (3.4) in (3.3) and choosing $\delta = 1/\sqrt{n}$ we prove

$$|f(x)-P_n^{[\beta]}(f;x)| \leqq [1+\lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}]\omega(1/\sqrt{n}). \tag{3.5}$$

For $\beta = 0$, the expression (3.5) reduces to an inequality for the Szasz-Mirakyan operator obtained earlier by Müller.

THEOREM (3.2). *If $f \in C'[0, \lambda]$, $1 > \beta'/n \geqq \beta \geqq 0$, then the following inequality holds*

$$|f(x)-P_n^{[\beta]}(f;x)| \leqq \lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}[1+\lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}]\omega_1(1/\sqrt{n})/\sqrt{n},$$

where $w_1(\delta)$ is the modulus of continuity of f'.

PROOF. For definiteness, we prove the theorem for $f'(x) \geqq 0$ but it also applies to $f'(x) < 0$. By the mean value theorem of differential calculus, it is known that

$$f(x)-f\left(\frac{k}{n}\right) = \left(x-\frac{k}{n}\right)f'(\xi),$$

where $\xi \equiv \xi_{n,k}(x)$ is an interior point of the interval determined by x and k/n. Now

$$f(x)-f\left(\frac{k}{n}\right) \leqq \left(x-\frac{k}{n}\right)[f'(\xi)-f'(x)]+\left[\frac{x}{1-\beta}-\frac{k}{n}\right]f'(x).$$

Multiplying both sides of the inequality by $xn(xn+\beta k)^{k-1}e^{-(nx+\beta k)}/k!$, summing over k and using (2.13) we get

$$(3.6)\quad |f(x)-P_n^{[\beta]}(f:x)| \leqq \sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|\frac{nx(nx+\beta k)^{k-1}e^{-(nx+\beta k)}}{k!}|f'(\xi)-f'(x)|.$$

But by (3.1) and (3.2)

$$|f'(\xi)-f'(x)| \leqq \omega_1(|\xi-x|) \leqq \left(1+\frac{1}{\delta}|\xi-x|\right)\omega_1(\delta)$$
$$\leqq \left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)\omega_1(\delta);$$

where δ is a positive number not depending on k.

A use of this in (3.6) gives

$$|f(x)-P_n^{[\beta]}(f;x)| \leqq \left\{\sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|\frac{nx(nx+\beta k)^{k-1}}{k!}e^{-(nx+\beta k)}\right.$$
$$\left.+\frac{1}{\delta}\sum_{k=0}^{\infty}\left(x-\frac{k}{n}\right)^2\frac{nx(nx+\beta k)^{k-1}}{k!}e^{-(nx+\beta k)}\right\}\omega_1(\delta).$$

Hence by applications of Cauchy's inequality and (3.4)

$$(3.7)\qquad |f(x)-P_n^{[\beta]}(f;x)| \leqq \frac{\lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}}{n^{\frac{1}{2}}}\left[1+\frac{\lambda^{\frac{1}{2}}(1+\lambda\beta\beta')^{\frac{1}{2}}}{\delta n^{\frac{1}{2}}}\right]\omega_1(\delta).$$

Choosing $\delta = 1/\sqrt{n}$, theorem (3.2) is proved.

We may put $\beta = 0, \delta = 1/\sqrt{n}$ in (3.7) to get the expression for Szasz-Mirakyan operator. The substitutions reduce (3.7) to

$$|f(x)-P_n^{[\beta]}(f;x)| \leqq \frac{1}{\sqrt{n}}(\lambda+\sqrt{\lambda})\omega_1(1/\sqrt{n});\ x\in[0\ \ \lambda]$$

in agreement with Stancu [12].

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