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COMPLEX PRODUCT STRUCTURES ON HOM-LIE ALGEBRAS

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Abstract. In this paper, we introduce the notion of complex product structures on hom-Lie algebras and show that a hom-Lie algebra carrying a complex product structure is a double hom-Lie algebra and it is also endowed with a hom-left symmetric product. Moreover, we show that a complex product structure on a hom-Lie algebra determines uniquely a left symmetric product such that the complex and the product structures are invariant with respect to it. Finally, we introduce the notion of hyper-para-Kähler hom-Lie algebras and we present an example of hyper-para-Kähler hom-Lie algebras.

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1. Introduction. A complex product structure on a Lie algebra is a pair $\{J, K\}$ of a complex structure and a product structure on the Lie algebra that anticommute. This notion is an analogue of a hypercomplex structure on a Lie algebra, i.e., a pair of anticommuting complex structures.

Complex product structures on Lie algebras were introduced by Andrada and Salamon in [**3**]. Lie algebras carrying a complex product structure are closely related to many important fields in mathematics and mathematical physics, such as Rota–Baxter operators on pre-Lie algebras [**11**], geometric structures on compact complex surfaces that are related to the split quaternions [**7**], paraquaternionic Kähler structures [**5**] and nilpotent Lie algebras [**2**]. Recently, complex product structures have been extensively investigated in [**4, 6, 19**].

Hom-Lie algebras were introduced by Hartwig, Larsson, and Silvestrov in order to describe the structures on certain quantum deformations or q-deformations of the Witt and the Virasoro algebras [**8**]. A q-deformation of vector fields is achieved when replacing a derivation with a σ-derivation d_σ, where σ is an algebra endomorphism of a commutative associative algebra [**9**]. As this algebraic structure has a close relation with discrete and deformed vector fields and differential calculus, it plays an important role among some mathematicians and physicists [**8, 10**]. For example, some authors have studied cohomology and homology theories in [**1, 18**], representation theory in [**15**], and a matched pair of hom-Lie algebras [**16**].

The purpose of this paper is to introduce and study complex product structures on involutive hom-Lie algebras, which are natural generalizations of complex product structures on Lie algebras.

The paper is organized as follows. In Section 2, we review some definitions including hom-Lie algebra, hom-Lie subalgebra, double hom-Lie algebra, representation of a hom-Lie algebra, and pseudo-Riemannian hom-algebra. In Section 3, we give notions of Hermitian and para-Hermitian structures. Then, we introduce complex product structures on an involutive hom-Lie algebra. Also, we provide some properties of these structures on hom-Lie algebras. In the following, some examples of such structures are presented. In Section 4, we present the notions of a matched pair and hom-bicrossproduct of hom-Lie algebras. Also, it is shown that hom-Lie algebras carrying a complex product structure can be written in terms of double hom-Lie algebras endowed with a hom-left symmetric product. Moreover, we prove that under certain conditions a complex product structure on a hom-Lie algebra determines uniquely a hom-left symmetric product, such that the complex and the product structures are invariant with respect to it (see Proposition 4.6). In Section 5, we introduce a notion of a hyper-para-Kähler hom-Lie algebra and present an example of hyper-para-Kähler hom-Lie algebras.

2. Hom-algebras and pseudo-Riemannian metric on hom-Lie algebra. In this section, we present the definitions of hom-algebra, hom-left symmetric algebra, hom-Lie algebra and hom-Lie subalgebra. Then, we introduce a double hom-Lie algebra and a pseudo-Riemannian hom-algebra.

Let V be a linear space, $\cdot : V \times V \to V$ be a bilinear map (product) and $\phi_V : V \to V$ be an algebra morphism. Then, $(V, \cdot, \phi_V)$ is called a hom-algebra. For any $u \in V$, the left and the right multiplications by u are maps $L_u, R_u : V \to V$ given by $L_u(v) = u \cdot v$ and $R_u(v) = v \cdot u$, respectively. The *commutator* on V is given by $[u, v] = u \cdot v - v \cdot u$. If $(V, \cdot, \phi_V)$ is a hom-algebra and for any $u, v, w \in V$, we have

$$\phi_V(u) \cdot (v \cdot w) = (u \cdot v) \cdot \phi_V(w),$$

then we say $(V, \cdot, \phi_V)$ is a *hom-associative algebra*. A *hom-left symmetric algebra* is a hom-algebra $(V, \cdot, \phi_V)$ such that

$$ass_{\phi_V}(u, v, w) = ass_{\phi_V}(v, u, w),$$

where

$$ass_{\phi_V}(u, v, w) = (u \cdot v) \cdot \phi_V(w) - \phi_V(u) \cdot (v \cdot w).$$

Each hom-associative algebra is a hom-left symmetric algebra with $ass_{\phi_V}(u, v, w) = 0$, but the converse does not hold.

A hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a bilinear map (bracket) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and an algebra morphism $\phi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ satisfying the anti-symmetric property, i.e., $[u, v] = -[v, u]$ and the hom-Jacobi identity property, i.e.,

$$\circlearrowleft_{u,v,w} [\phi_{\mathfrak{g}}(u), [v, w]] = 0, \quad \forall u, v, w \in \mathfrak{g}. \tag{1}$$

Also, it is called regular (involutive), if $\phi_{\mathfrak{g}}$ is non-degenerate (satisfies ${\phi_{\mathfrak{g}}}^2 = 1$). A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a hom-Lie subalgebra of $\mathfrak{g}$ if $\phi_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{h}$ and $[u, v] \in \mathfrak{h}$, for any

$u, v \in \mathfrak{h}$. Also, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is said to be an ideal of $\mathfrak{g}$ if $\phi_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{h}$ and for $u \in \mathfrak{h}$ and $v \in \mathfrak{g}$ we have $[u, v] \in \mathfrak{h}$.

A homomorphism of hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, , \phi_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'}, \phi_{\mathfrak{g}'})$ is a linear map $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that

$$\psi \circ \phi_{\mathfrak{g}} = \phi_{\mathfrak{g}'} \circ \psi, \qquad \psi[u, v]_{\mathfrak{g}} = [\psi(u), \psi(v)]_{\mathfrak{g}'},$$

for any $u, v \in \mathfrak{g}$ [**16**].

Definition 2.1. A triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}')$ of hom-Lie algebras forms a double hom-Lie algebra if $\mathfrak{h}, \mathfrak{h}'$ are hom-Lie subalgebras of the hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ where, $\phi_{\mathfrak{g}} = \phi_{\mathfrak{g}|\mathfrak{h}} + \phi_{\mathfrak{g}|\mathfrak{h}'}$.

Let $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ be a hom-Lie algebra. A *representation* of $\mathfrak{g}$ is a triple (V, A, ρ) in which V is a vector space, $A \in gl(V)$ and $\rho : \mathfrak{g} \to gl(V)$ is a linear map satisfying

$$\begin{cases} \rho(\phi_{\mathfrak{g}}(u)) \circ A = A \circ \rho(u), \\ \rho([u, v]_{\mathfrak{g}}) \circ A = \rho(\phi_{\mathfrak{g}}(u)) \circ \rho(v) - \rho(\phi_{\mathfrak{g}}(v)) \circ \rho(u), \end{cases} \tag{2}$$

for any $u, v \in \mathfrak{g}$. If we consider V^* as the dual vector space of V, then we can define a linear map $\rho^* : \mathfrak{g} \to gl(V^*)$ by

$$\prec \rho^*(u)(\alpha), v \succ = - \prec \alpha, \rho(u)(v) \succ,$$

for any $u \in \mathfrak{g}, v \in V, \alpha \in V^*$, where $\prec \rho^*(u)(\alpha), v \succ$ is defined by $\rho^*(u)(\alpha)(v)$. A representation (V, A, ρ) is called *admissible* if (V^*, A^*, ρ^*) is also a representation of $\mathfrak{g}$ in which A^* is the transpose of the endomorphism A. It is known that the representation (V, A, ρ) is admissible if and only if [**16**]

$$\begin{cases} A \circ \rho(\phi_{\mathfrak{g}}(u)) = \rho(u) \circ A, \\ A \circ \rho([u, v]_{\mathfrak{g}}) = \rho(u) \circ \rho(\phi_{\mathfrak{g}}(v)) - \rho(v) \circ \rho(\phi_{\mathfrak{g}}(u)). \end{cases} \tag{3}$$

Example 2.2. Consider a 4-dimensional linear space $\mathfrak{g}$ with an arbitrary basis $\{e_1, e_2, e_3, e_4\}$. We define the bracket $[\cdot, \cdot]$ and linear map $\phi_{\mathfrak{g}}$ on $\mathfrak{g}$ as follows:

$$[e_1, e_3] = ae_4, \quad [e_2, e_4] = -ae_3,$$

and

$$\phi_{\mathfrak{g}}(e_1) = -e_2, \quad \phi_{\mathfrak{g}}(e_2) = -e_1, \quad \phi_{\mathfrak{g}}(e_3) = e_4, \quad \phi_{\mathfrak{g}}(e_4) = e_3.$$

The above bracket is not a Lie bracket on $\mathfrak{g}$ if $a \neq 0$, because

$$[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] = [e_2, -ae_4] = a^2 e_3.$$

It is easy to see that

$$\begin{aligned} [\phi_{\mathfrak{g}}(e_1), \phi_{\mathfrak{g}}(e_3)] &= ae_3 = \phi_{\mathfrak{g}}([e_1, e_3]), \\ [\phi_{\mathfrak{g}}(e_2), \phi_{\mathfrak{g}}(e_4)] &= -ae_4 = \phi_{\mathfrak{g}}([e_2, e_4]), \end{aligned}$$

i.e., $\phi_{\mathfrak{g}}$ is an algebra morphism. Also, we can deduce

$$[\phi_{\mathfrak{g}}(e_i), [e_j, e_k]] + [\phi_{\mathfrak{g}}(e_j), [e_k, e_i]] + [\phi_{\mathfrak{g}}(e_k), [e_i, e_j]] = 0, \quad i, j, k = 1, 2, 3, 4.$$

Thus, $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ is a hom-Lie algebra.

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A quadruple $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ is called a *pseudo-Riemannian hom-Lie algebra* if $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ is a finite-dimensional hom-Lie algebra and $\langle\cdot,\cdot\rangle$ is a bilinear symmetric non-degenerate form, such that for any $u, v \in \mathfrak{g}$, $\langle\phi_{\mathfrak{g}}(u), \phi_{\mathfrak{g}}(v)\rangle = \langle u, v\rangle$ or $\langle\phi_{\mathfrak{g}}(u), v\rangle = \langle u, \phi_{\mathfrak{g}}(v)\rangle$. In this case, we say that $\mathfrak{g}$ admits a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$. It is known that if $\phi_{\mathfrak{g}}$ is an isomorphism, then exists a unique product $\cdot$ (is called hom-Levi-Civita product) on it, which is given by Koszul's formula

$$2\langle u\cdot v, \phi_{\mathfrak{g}}(w)\rangle = \langle[u, v], \phi_{\mathfrak{g}}(w)\rangle + \langle[w, v], \phi_{\mathfrak{g}}(u)\rangle + \langle[w, u], \phi_{\mathfrak{g}}(v)\rangle, \tag{4}$$

which satisfies $[u, v] = u\cdot v - v\cdot u$ and $\langle u\cdot v, \phi_{\mathfrak{g}}(w)\rangle = -\langle\phi_{\mathfrak{g}}(v), u\cdot w\rangle$ (see **[13]**, for more details).

A quadruple $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \omega)$ is called a symplectic hom-Lie algebra if $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ is a regular hom-Lie algebra and ω is a bilinear skew-symmetric nondegenerate form (is called a symplectic structure), which is a 2-hom-cocycle, i.e.,

$$d\omega = 0, \quad \omega(\phi_{\mathfrak{g}}(u), \phi_{\mathfrak{g}}(v)) = \omega(u, v),$$

where, $d\omega \in \wedge^3\mathfrak{g}^*$ is given by

$$d\omega(u, v, w) = \omega(\phi_{\mathfrak{g}}(u), [v, w]) + \omega(\phi_{\mathfrak{g}}(v), [w, u]) + \omega(\phi_{\mathfrak{g}}(w), [u, v]), \tag{5}$$

for any $u, v, w \in \mathfrak{g}$.

3. Complex product structures on hom-Lie algebras. In this section, we introduce complex product structures on hom-Lie algebras. We also present an example of these structures (see **[13, 14]** for more details).

An isomorphism $K : \mathfrak{g} \to \mathfrak{g}$ is called an almost product structure on an involutive hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ if $K^2 = Id_{\mathfrak{g}}$ and $\phi_{\mathfrak{g}} \circ K = K \circ \phi_{\mathfrak{g}}$. Also, $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, K)$ is called an almost product hom-Lie algebra. In this case, we have $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, where

$$\mathfrak{g}^1 := ker(\phi_{\mathfrak{g}} \circ K - Id_{\mathfrak{g}}), \quad \mathfrak{g}^{-1} := ker(\phi_{\mathfrak{g}} \circ K + Id_{\mathfrak{g}}).$$

If $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ have the same dimension n, then K is called an *almost para-complex* structure on $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ (in this case the dimension of $\mathfrak{g}$ is even). An almost product (almost para-complex) structure is called a *product (para-complex)* structure if

$$[(\phi_{\mathfrak{g}} \circ K)u, (\phi_{\mathfrak{g}} \circ K)v] = \phi_{\mathfrak{g}} \circ K[(\phi_{\mathfrak{g}} \circ K)u, v] + \phi_{\mathfrak{g}} \circ K[u, (\phi_{\mathfrak{g}} \circ K)v] - [u, v], \quad \forall u, v \in \mathfrak{g}. \tag{6}$$

A quadruple $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, J)$ is called an almost complex hom-Lie algebra if $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ is an involutive hom-Lie algebra of even dimension $J : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism such that $J^2 = -Id_{\mathfrak{g}}$ and $\phi_{\mathfrak{g}} \circ J = J \circ \phi_{\mathfrak{g}}$ (J is called an almost complex structure). An almost complex structure is called a complex structure if

$$[(\phi_{\mathfrak{g}} \circ J)u, (\phi_{\mathfrak{g}} \circ J)v] = \phi_{\mathfrak{g}} \circ J[(\phi_{\mathfrak{g}} \circ J)u, v] + \phi_{\mathfrak{g}} \circ J[u, (\phi_{\mathfrak{g}} \circ J)v] + [u, v], \tag{7}$$

for all $u, v \in \mathfrak{g}$.

A Hermitian structure of a hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ is a pair $(J, \langle\cdot,\cdot\rangle)$ consisting of a complex structure and a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$, such that for each $u, v \in \mathfrak{g}$

$$\langle(\phi_{\mathfrak{g}} \circ J)u, (\phi_{\mathfrak{g}} \circ J)v\rangle = \langle u, v\rangle.$$

In this case, $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}}, J, \langle\cdot, \cdot\rangle)$ is called a Hermitian hom-Lie algebra. A Hermitian hom-Lie algebra has a natural bilinear skew-symmetric nondegenerate form ω, which is defined by

$$\omega(u, v) = \langle(\phi_{\mathfrak{g}} \circ J)u, v\rangle.$$

PROPOSITION 3.1. *Let $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}}, J, \langle\cdot, \cdot\rangle)$ be a Hermitian hom-Lie algebra. If we consider the product $\cdot$ as a hom-Levi-Civita product associated with metric $\langle\cdot, \cdot\rangle$ on $\mathfrak{g}$ given by (4), then*

$$2\langle u \cdot \phi_{\mathfrak{g}}(Jv) - (\phi_{\mathfrak{g}} \circ J)(u \cdot v), \phi_{\mathfrak{g}}(w)\rangle = d\omega(u, v, w) - d\omega(u, \phi_{\mathfrak{g}}(Jv), \phi_{\mathfrak{g}}(Jw)).$$

Proof. By Koszul's formula and the definition of ω, we get

$$\begin{aligned} 2\langle u \cdot \phi_{\mathfrak{g}}(Jv), \phi_{\mathfrak{g}}(w)\rangle =& \langle[u, \phi_{\mathfrak{g}}(Jv)], \phi_{\mathfrak{g}}(w)\rangle + \langle[w, \phi_{\mathfrak{g}}(Jv)], \phi_{\mathfrak{g}}(u)\rangle + \langle[w, u], Jv\rangle \\ =& \omega([u, \phi_{\mathfrak{g}}(Jv)], Jw) - \omega((\phi_{\mathfrak{g}} \circ J)[w, \phi_{\mathfrak{g}}(Jv)], \phi_{\mathfrak{g}}(u)) \\ & - \omega([w, u], \phi_{\mathfrak{g}}(v)), \end{aligned}$$

and

$$\begin{aligned} -2\langle(\phi_{\mathfrak{g}} \circ J)(u \cdot v), \phi_{\mathfrak{g}}(w)\rangle =& 2\langle u \cdot v, (\phi_{\mathfrak{g}} \circ J)\phi_{\mathfrak{g}}(w)\rangle = \langle[u, v], (\phi_{\mathfrak{g}} \circ J)\phi_{\mathfrak{g}}(w)\rangle \\ & + \langle[\phi_{\mathfrak{g}}(Jw), v], \phi_{\mathfrak{g}}(u)\rangle + \langle[\phi_{\mathfrak{g}}(Jw), u], \phi_{\mathfrak{g}}(v)\rangle \\ =& - \omega([u, v], \phi_{\mathfrak{g}}(w)) - \omega((\phi_{\mathfrak{g}} \circ J)[\phi_{\mathfrak{g}}(Jw), v], \phi_{\mathfrak{g}}(u)) \\ & - \omega(Jv, [\phi_{\mathfrak{g}}(Jw), u]). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d\omega(u, \phi_{\mathfrak{g}}(Jv), \phi_{\mathfrak{g}}(Jw)) =& \omega(\phi_{\mathfrak{g}}(u), [\phi_{\mathfrak{g}}(Jv), \phi_{\mathfrak{g}}(Jw)]) + \omega(Jv, [\phi_{\mathfrak{g}}(Jw), u]) \\ & + \omega(Jw, [u, \phi_{\mathfrak{g}}(Jv)]). \end{aligned}$$

From the above equations, (5) and (7), we conclude the assertion. □

DEFINITION 3.2. A para-Hermitian structure of a hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ is a pair $(K, \langle\cdot, \cdot\rangle)$ consisting of a para-complex structure and a pseudo-Riemannian metric $\langle\cdot, \cdot\rangle$ such that for each $u, v \in \mathfrak{g}$

$$\langle(\phi_{\mathfrak{g}} \circ K)u, (\phi_{\mathfrak{g}} \circ K)v\rangle = -\langle u, v\rangle.$$

In this case, $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}}, K, \langle\cdot, \cdot\rangle)$ is called a para-Hermitian hom-Lie algebra. Also, it defines a natural bilinear skew-symmetric nondegenerate form ω given by

$$\omega(u, v) = \langle(\phi_{\mathfrak{g}} \circ K)u, v\rangle.$$

Similar to the proof of Proposition 3.1, we can prove the following.

PROPOSITION 3.3. *Let $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, K, \langle\cdot,\cdot\rangle)$ be a para-Hermitian hom-Lie algebra. If we consider the product $\cdot$ as a hom-Levi-Civita product associated with metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ given by (4), then*

$$2\langle u\cdot\phi_{\mathfrak{g}}(Kv) - (\phi_{\mathfrak{g}}\circ K)(u\cdot v), \phi_{\mathfrak{g}}(w)\rangle = d\omega(u, v, w) + d\omega(u, \phi_{\mathfrak{g}}(Kv), \phi_{\mathfrak{g}}(Kw)).$$

DEFINITION 3.4. A complex product structure on the hom-Lie algebra $\mathfrak{g}$ is a pair $\{J, K\}$ of a complex structure J and a product structure K, such that $J\circ K = -K\circ J$ (note that $J\circ K = -K\circ J$ is equivalent to $\phi_{\mathfrak{g}}\circ J\circ K = -\phi_{\mathfrak{g}}\circ K\circ J$, because $\phi_{\mathfrak{g}}^2 = Id_{\mathfrak{g}}$).

We consider the vector spaces $\mathfrak{g}^1 = ker(\phi_{\mathfrak{g}}\circ K - Id_{\mathfrak{g}})$ and $\mathfrak{g}^{-1} = ker(\phi_{\mathfrak{g}}\circ K + Id_{\mathfrak{g}})$ as eigenspaces corresponding to the eigenvalues 1 and -1 of $\phi_{\mathfrak{g}}\circ K$, respectively.

THEOREM 3.5. *Let $\{J, K\}$ be a complex product structure on the hom-Lie algebra $\mathfrak{g}$. Then,*

(i) $\phi_{\mathfrak{g}}\circ J$ and J are isomorphisms between the eigenspaces $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$,
(ii) $\phi_{\mathfrak{g}}\circ K$ is a para-complex structure on $\mathfrak{g}$,
(iii) $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are hom-Lie subalgebras of $\mathfrak{g}$,
(iv) $(\mathfrak{g}, \mathfrak{g}^1, \mathfrak{g}^{-1})$ is a double hom-Lie algebra,
(v) $J\circ\phi_{\mathfrak{g}^1} = \phi_{\mathfrak{g}^{-1}}\circ J$ and $J\circ\phi_{\mathfrak{g}^{-1}} = \phi_{\mathfrak{g}^1}\circ J$.

Proof. Let $u\in\mathfrak{g}^1$. Then, the condition $J\circ\phi_{\mathfrak{g}}\circ K = -\phi_{\mathfrak{g}}\circ K\circ J$ implies $J(u)\in\mathfrak{g}^{-1}$. Thus, $J(\mathfrak{g}^1)\subset\mathfrak{g}^{-1}$. Similarly, we get $J(\mathfrak{g}^{-1})\subset\mathfrak{g}^1$. So $J^2 = -Id_{\mathfrak{g}}$ implies $J(\mathfrak{g}^1) = \mathfrak{g}^{-1}$. Also, if we consider $J(u) = J(v)$ for any $u, v\in\mathfrak{g}^1$, then $J^2 = -Id_{\mathfrak{g}}$ results in $u = v$. Thus, J is an isomorphism between $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$. Similarly, the condition $\phi_{\mathfrak{g}}\circ J\circ\phi_{\mathfrak{g}}\circ K = -\phi_{\mathfrak{g}}\circ K\circ\phi_{\mathfrak{g}}\circ J$ implies that $\phi_{\mathfrak{g}}\circ J$ is an isomorphism between $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$. Therefore, we have (i). From (i), we conclude that $dim\mathfrak{g}^1 = dim\mathfrak{g}^{-1}$ and so we have (ii). We now prove (iii). It is easy to see that (6) implies that $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are Lie subalgebras of $\mathfrak{g}$. Now, we let $u\in\mathfrak{g}^1$. Since $(K\circ\phi_{\mathfrak{g}})(u) = u$ and $K\circ\phi_{\mathfrak{g}} = \phi_{\mathfrak{g}}\circ K$, we imply that

$$(K\circ\phi_{\mathfrak{g}})(\phi_{\mathfrak{g}}(u)) = (\phi_{\mathfrak{g}}\circ K\circ\phi_{\mathfrak{g}})(u) = \phi_{\mathfrak{g}}(u),$$

which gives $\phi_{\mathfrak{g}}(u)\in\mathfrak{g}^1$. Similarly, we obtain $\phi_{\mathfrak{g}}(u')\in\mathfrak{g}^{-1}$, for any $u'\in\mathfrak{g}^{-1}$. Hence, it is easy to verify that $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are hom-Lie subalgebras. Therefore, we have (iii). Here, we prove (iv). According to (iii), we can write $\phi_{\mathfrak{g}}:\mathfrak{g}^1\oplus\mathfrak{g}^{-1}\to\mathfrak{g}^1\oplus\mathfrak{g}^{-1}$, as $\phi_{\mathfrak{g}}(u+u') = \phi_{\mathfrak{g}^1}(u) + \phi_{\mathfrak{g}^{-1}}(u')$ for any $u\in\mathfrak{g}^1, u'\in\mathfrak{g}^{-1}$. This shows that $(\mathfrak{g}, \mathfrak{g}^1, \mathfrak{g}^{-1})$ is a double hom-Lie algebra. To prove (v), let $u\in\mathfrak{g}^1$. Then, the conditions $J\mathfrak{g}^1 = \mathfrak{g}^{-1}$, $\phi_{\mathfrak{g}^1}\subset\mathfrak{g}^1, \phi_{\mathfrak{g}^{-1}}\subset\mathfrak{g}^{-1}$ and $J\circ\phi_{\mathfrak{g}} = \phi_{\mathfrak{g}}\circ J$, conclude $J(\phi_{\mathfrak{g}^1}(u)) = \phi_{\mathfrak{g}^{-1}}(Ju)$. Similarly, we have the second part. □

EXAMPLE 3.6. We consider the hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ introduced in Example 2.2. If isomorphisms J and K are determined as

$$J(e_1) = e_4,\quad J(e_2) = -e_3,\quad J(e_3) = e_2,\quad J(e_4) = -e_1,$$
$$K(e_1) = -e_2,\quad K(e_2) = -e_1,\quad K(e_3) = -e_4,\quad K(e_4) = -e_3,$$

then we have

$$J^2(e_i) = -K^2(e_i) = -\phi_{\mathfrak{g}}^2(e_i) = -e_i,\qquad i = 1, 2, 3, 4.$$

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Moreover, using the above equations, we get

$$(J\circ\phi_{\mathfrak g})e_1 = e_3 = (\phi_{\mathfrak g}\circ J)e_1, \quad (J\circ\phi_{\mathfrak g})e_2 = -e_4 = (\phi_{\mathfrak g}\circ J)e_2,$$
$$(J\circ\phi_{\mathfrak g})e_3 = -e_1 = (\phi_{\mathfrak g}\circ J)e_3, \quad (J\circ\phi_{\mathfrak g})e_4 = e_2 = (\phi_{\mathfrak g}\circ J)e_4,$$

and

$$(K\circ\phi_{\mathfrak g})e_1 = e_1 = (\phi_{\mathfrak g}\circ K)e_1, \quad (K\circ\phi_{\mathfrak g})e_2 = e_2 = (\phi_{\mathfrak g}\circ K)e_2,$$
$$(K\circ\phi_{\mathfrak g})e_3 = -e_3 = (\phi_{\mathfrak g}\circ K)e_3, \quad (K\circ\phi_{\mathfrak g})e_4 = -e_4 = (\phi_{\mathfrak g}\circ K)e_4.$$

Also, we have

$$(J\circ K)e_1 = e_3 = -(K\circ J)e_1, \qquad (J\circ K)e_2 = -e_4 = -(K\circ J)e_2,$$
$$(J\circ K)e_3 = e_1 = -(K\circ J)e_3, \qquad (J\circ K)e_4 = -e_2 = -(K\circ J)e_4.$$

Moreover, it follows that (6) and (7) hold. Therefore, $\{J, K\}$ is a complex product structure on $\mathfrak g = \mathfrak g^1 \oplus \mathfrak g^{-1}$, where $\mathfrak g^1 = \{e_1, e_2\}$ and $\mathfrak g^{-1} = \{e_3, e_4\}$.

LEMMA 3.7. *Let* $(\mathfrak g, [\cdot,\cdot], \phi_{\mathfrak g})$ *be a hom-Lie algebra with a complex product structure* $\{J, K\}$. *If we consider* $\mathfrak g^{-1}$ *as an ideal in* $\mathfrak g$, *then* $\mathfrak g^{-1}$ *is abelian. Moreover,* $\mathfrak g^1$ *carries a hom-left symmetric product given by*

$$u\cdot v = -(\phi_{\mathfrak g^1}\circ J)[u, \phi_{\mathfrak g^{-1}}(Jv)], \quad \forall u, v\in \mathfrak g^1. \tag{8}$$

Proof. Since $\mathfrak g^{-1}$ and $\mathfrak g^1$ are hom-Lie subalgebras of $\mathfrak g$, using (7), we get

$$[(\phi_{\mathfrak g^1}\circ J)u', (\phi_{\mathfrak g^1}\circ J)v'] - (\phi_{\mathfrak g^1}\circ J)[(\phi_{\mathfrak g^1}\circ J)u', v'] - (\phi_{\mathfrak g^1}\circ J)[u', (\phi_{\mathfrak g^1}\circ J)v'] = [u', v'],$$

for all $u', v'\in\mathfrak g^{-1}$. Since $\mathfrak g^{-1}$ is an ideal in $\mathfrak g$ and $\phi_{\mathfrak g^1}\circ J\subset\mathfrak g^1$, we conclude that the left-hand side of the above equation is in $\mathfrak g^1$ and the right-hand side of it is also in $\mathfrak g^{-1}$. Therefore, $\mathfrak g^{-1}$ is an abelian ideal. Now, if we consider $u, v, w\in\mathfrak g^1$, then using (7) and (8) we obtain

$$\begin{aligned} u\cdot v - v\cdot u =& -(\phi_{\mathfrak g^1}\circ J)[u, \phi_{\mathfrak g^{-1}}(Jv)] - (\phi_{\mathfrak g^1}\circ J)[\phi_{\mathfrak g^{-1}}(Ju), v]\\ =&[u, v] - [(\phi_{\mathfrak g^{-1}}\circ J)u, (\phi_{\mathfrak g^{-1}}\circ J)v].\end{aligned}$$

Since $\mathfrak g^{-1}$ is an abelian ideal, then from the above equation we obtain

$$u\cdot v - v\cdot u = [u, v]. \tag{9}$$

Also, using the hom-Jacobi identity and (8), we get

$$\begin{aligned}&\phi_{\mathfrak g^1}(u)\cdot(v\cdot w) - \phi_{\mathfrak g^1}(v)\cdot(u\cdot w)\\ &= -\phi_{\mathfrak g^1}(u)\cdot(\phi_{\mathfrak g^1}\circ J)[v, \phi_{\mathfrak g^{-1}}(Jw)] + \phi_{\mathfrak g^1}(v)\cdot(\phi_{\mathfrak g^1}\circ J)[u, \phi_{\mathfrak g^{-1}}(Jw)]\\ &= -(\phi_{\mathfrak g^1}\circ J)[\phi_{\mathfrak g^1}(u), [v, \phi_{\mathfrak g^{-1}}(Jw)]] + (\phi_{\mathfrak g^1}\circ J)[\phi_{\mathfrak g^1}(v), [u, \phi_{\mathfrak g^{-1}}(Jw)]]\\ &= (\phi_{\mathfrak g^1}\circ J)[(\phi_{\mathfrak g^{-1}}\circ J)\phi_{\mathfrak g^1}(w), [u, v]] = [u, v]\cdot\phi_{\mathfrak g^1}(w).\end{aligned}$$

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Moreover, (8) and part (v) of Theorem 3.5 yield

$$\begin{aligned}\phi_{\mathfrak{g}^1}(u)\cdot\phi_{\mathfrak{g}^1}(v) &= -(\phi_{\mathfrak{g}^1}\circ J)[\phi_{\mathfrak{g}^1}(u),(\phi_{\mathfrak{g}^{-1}}\circ J)\phi_{\mathfrak{g}^1}(v)] = -(\phi_{\mathfrak{g}^1}\circ J)[\phi_{\mathfrak{g}^1}(u),\phi^2_{\mathfrak{g}^{-1}}(Jv)]\\ &= -(\phi_{\mathfrak{g}^1}\circ J\circ\phi_{\mathfrak{g}^{-1}})[u,\phi_{\mathfrak{g}^{-1}}(Jv)]\\ &= -\phi_{\mathfrak{g}^1}((\phi_{\mathfrak{g}^1}\circ J)[u,\phi_{\mathfrak{g}^{-1}}(Jv)]) = \phi_{\mathfrak{g}^1}(u\cdot v).\end{aligned}$$

Consequently, (9) and the last equation imply $[\phi_{\mathfrak{g}^1}(u),\phi_{\mathfrak{g}^1}(v)] = \phi_{\mathfrak{g}^1}[u,v]$. Therefore, the product $\cdot$ is a hom-left symmetric product on $\mathfrak{g}^1$. □

EXAMPLE 3.8. We consider a 4-dimensional hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\phi_{\mathfrak{g}})$ with an arbitrary basis $\{e_1,e_2,e_3,e_4\}$, where

$$[e_1,e_2]=ae_1+ae_2,\quad [e_1,e_3]=ae_3,\quad [e_2,e_3]=ae_4,\quad [e_1,e_4]=-ae_3,\quad [e_2,e_4]=-ae_4,$$

and

$$\phi_{\mathfrak{g}}(e_1)=-e_2,\quad \phi_{\mathfrak{g}}(e_2)=-e_1,\quad \phi_{\mathfrak{g}}(e_3)=-e_4,\quad \phi_{\mathfrak{g}}(e_4)=-e_3.$$

If $a=0$, then the above bracket is a Lie bracket on $\mathfrak{g}$. Let isomorphisms J and K be given by

$$\begin{aligned}&J(e_1)=-e_3,\quad J(e_2)=-e_4,\quad J(e_3)=e_1,\quad J(e_4)=e_2,\\ &K(e_1)=-e_2,\quad K(e_2)=-e_1,\quad K(e_3)=e_4,\quad K(e_4)=e_3.\end{aligned}$$

Then, we have

$$J^2(e_i)=-K^2(e_i)=-\phi^2_{\mathfrak{g}}(e_i)=-e_i,\qquad i=1,2,3,4.$$

Also, using the above equations, we infer

$$\begin{aligned}&(J\circ\phi_{\mathfrak{g}})e_1=e_4=(\phi_{\mathfrak{g}}\circ J)e_1,\qquad (J\circ\phi_{\mathfrak{g}})e_2=e_3=(\phi_{\mathfrak{g}}\circ J)e_2,\\ &(J\circ\phi_{\mathfrak{g}})e_3=-e_2=(\phi_{\mathfrak{g}}\circ J)e_3,\quad (J\circ\phi_{\mathfrak{g}})e_4=-e_1=(\phi_{\mathfrak{g}}\circ J)e_4,\end{aligned}$$

and

$$\begin{aligned}&(K\circ\phi_{\mathfrak{g}})e_1=e_1=(\phi_{\mathfrak{g}}\circ K)e_1,\qquad (K\circ\phi_{\mathfrak{g}})e_2=e_2=(\phi_{\mathfrak{g}}\circ K)e_2,\\ &(K\circ\phi_{\mathfrak{g}})e_3=-e_3=(\phi_{\mathfrak{g}}\circ K)e_3,\quad (K\circ\phi_{\mathfrak{g}})e_4=-e_4=(\phi_{\mathfrak{g}}\circ K)e_4.\end{aligned}$$

It is easy to see that (6) and (7) hold, i.e., J and K are complex and product structures on $(\mathfrak{g},[\cdot,\cdot],\phi_{\mathfrak{g}})$, respectively. Also, we obtain

$$\begin{aligned}&(J\circ K)e_1=e_4=-(K\circ J)e_1,\qquad (J\circ K)e_2=e_3=-(K\circ J)e_2,\\ &(J\circ K)e_3=e_2=-(K\circ J)e_3,\qquad (J\circ K)e_4=e_1=-(K\circ J)e_4.\end{aligned}$$

Therefore, the pair $\{J,K\}$ is a complex product structure on $\mathfrak{g}$. Moreover, we can write $\mathfrak{g}$ as $\mathfrak{g}=\mathfrak{g}^1\oplus\mathfrak{g}^{-1}$, where $\mathfrak{g}^1=\{e_1,e_2\}$ and $\mathfrak{g}^{-1}=\{e_3,e_4\}$. Since $\mathfrak{g}^{-1}$ is an abelian ideal in $\mathfrak{g}$, $\mathfrak{g}^1$ carries a hom-left symmetric product. If we denote this product with $\cdot$, then using (8) we have

$$\begin{aligned} e_1 \cdot e_2 &= -\,(\phi_{\mathfrak{g}^1} \circ J)[e_1, \phi_{\mathfrak{g}^{-1}}(Je_2)] = -(\phi_{\mathfrak{g}^1} \circ J)[e_1, e_3] = -a(\phi_{\mathfrak{g}^1} \circ J)e_3 = ae_2, \\ e_2 \cdot e_1 &= -\,(\phi_{\mathfrak{g}^1} \circ J)[e_2, \phi_{\mathfrak{g}^{-1}}(Je_1)] = -(\phi_{\mathfrak{g}^1} \circ J)[e_2, e_4] = a(\phi_{\mathfrak{g}^1} \circ J)e_4 = -ae_1, \\ e_1 \cdot e_1 &= -\,ae_2, \qquad e_2 \cdot e_2 = ae_1. \end{aligned}$$

4. Matched pairs. In this section, we present the notions of a matched pair and hom-bicrossproduct of hom-Lie algebras. Also, it is shown that hom-Lie algebras carrying a complex product structure in terms of double hom-Lie algebras are endowed with a hom-left symmetric product.

DEFINITION 4.1 ([**16**]). A pair of hom-Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, \phi_{\mathfrak{g}'})$ with representations $\rho : \mathfrak{g} \to gl(\mathfrak{g}')$ and $\rho' : \mathfrak{g}' \to gl(\mathfrak{g})$ with respect to $\phi_{\mathfrak{g}'}$ and $\phi_{\mathfrak{g}}$, respectively, is called a matched pair of hom-Lie algebras if

$$\begin{aligned} \rho'(\phi_{\mathfrak{g}'}(u'))[u, v]_{\mathfrak{g}} &= [\rho'(u')(u), \phi_{\mathfrak{g}}(v)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(u), \rho'(u')(v)]_{\mathfrak{g}} + \rho'(\rho(v)(u'))(\phi_{\mathfrak{g}}(u)) \\ &\quad - \rho'(\rho(u)(u'))(\phi_{\mathfrak{g}}(v)), \\ \rho(\phi_{\mathfrak{g}}(u))[u', v']_{\mathfrak{g}'} &= [\rho(u)(u'), \phi_{\mathfrak{g}'}(v')]_{\mathfrak{g}'} + [\phi_{\mathfrak{g}'}(u'), \rho(u)(v')]_{\mathfrak{g}'} + \rho(\rho'(v')(u))(\phi_{\mathfrak{g}'}(u')) \\ &\quad - \rho(\rho'(u')(u))(\phi_{\mathfrak{g}'}(v')), \end{aligned}$$

for any $u, v \in \mathfrak{g}$, $u', v' \in \mathfrak{g}'$. We denote a matched pair of hom-Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ by $(\mathfrak{g}, \mathfrak{g}', \rho, \rho')$.

Given a matched pair $(\mathfrak{g}, \mathfrak{g}', \rho, \rho')$ of hom-Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, \phi_{\mathfrak{g}'})$, we can construct a new hom-Lie algebra $\mathfrak{g} \bowtie^{\rho}_{\rho'} \mathfrak{g}' = (\mathfrak{g} \oplus \mathfrak{g}', \Phi, [\cdot,\cdot])$, where

$$\begin{aligned} \Phi(u, u') &= (\phi_{\mathfrak{g}}(u), \phi_{\mathfrak{g}'}(u')), \\ [(u, u'), (v, v')] &= ([u, v]_{\mathfrak{g}} - \rho'(v')(u) + \rho'(u')(v), [u', v']_{\mathfrak{g}'} - \rho(v)(u') + \rho(u)(v')). \end{aligned}$$

We will call $\mathfrak{g} \bowtie^{\rho}_{\rho'} \mathfrak{g}'$ the hom-bicrossproduct of $\mathfrak{g}$ and $\mathfrak{g}'$ (see [**16**], for more details). Considering $\mathfrak{g} \equiv \mathfrak{g} \oplus \{0\}$ and $\mathfrak{g}' \equiv \{0\} \oplus \mathfrak{g}'$, we observe that $(\mathfrak{g} \oplus \mathfrak{g}', \mathfrak{g}, \mathfrak{g}')$ is a double hom-Lie algebra.

Conversely, if $(\mathfrak{g} \oplus \mathfrak{g}', \mathfrak{g}, \mathfrak{g}')$ is a double hom-Lie algebra, then $(\mathfrak{g}, \mathfrak{g}', \rho, \rho')$ forms a matched pair of hom-Lie algebras $\mathfrak{g}'$ and $\mathfrak{g}$ such that the representations $\rho : \mathfrak{g} \to gl(\mathfrak{g}')$ and $\rho' : \mathfrak{g}' \to gl(\mathfrak{g})$ are given by

$$[u, u'] = \rho(u)u' - \rho'(u')u, \quad \forall u \in \mathfrak{g}, u' \in \mathfrak{g}'. \tag{10}$$

From the above description, we can deduce the following.

COROLLARY 4.2. *Let* $(\mathfrak{g}, \phi_{\mathfrak{g}}, [\cdot,\cdot])$ *be a hom-Lie algebra with a complex product structure* $\{J, K\}$. *Then, there exist representations* $\rho : \mathfrak{g}^1 \to gl(\mathfrak{g}^{-1})$ *and* $\rho' : \mathfrak{g}^{-1} \to gl(\mathfrak{g}^1)$ *with respect to* $\phi_{\mathfrak{g}^{-1}}$ *and* $\phi_{\mathfrak{g}^1}$, *respectively, such that* $(\mathfrak{g}^1, \mathfrak{g}^{-1}, \rho, \rho')$ *forms a matched pair of hom-Lie algebras.*

PROPOSITION 4.3. *Let* $(\mathfrak{g}, \phi_{\mathfrak{g}}, [\cdot,\cdot])$ *be a hom-Lie algebra with a complex product structure* $\{J, K\}$. *Then, there exist representations* $\rho^* : \mathfrak{g}^1 \to gl(\mathfrak{g}^1)$ *and*

$\rho^{*\prime}: \mathfrak{g}^{-1} \to gl(\mathfrak{g}^{-1})$ *with respect to* $\phi_{\mathfrak{g}^1}$ *and* $\phi_{\mathfrak{g}^{-1}}$*, respectively, such that*

$$\rho^*(u) := -\phi_{\mathfrak{g}^1} \circ J \circ \rho(u) \circ \phi_{\mathfrak{g}^{-1}} \circ J, \qquad \rho^{*\prime}(u') := -\phi_{\mathfrak{g}^{-1}} \circ J \circ \rho'(u') \circ \phi_{\mathfrak{g}^1} \circ J. \tag{11}$$

Also, we have

$$[u, u'] = -\phi_{\mathfrak{g}^{-1}} \circ J \circ \rho^*(u) \circ \phi_{\mathfrak{g}^1} \circ J(u') + \phi_{\mathfrak{g}^1} \circ J \circ \rho^{*\prime}(u') \circ \phi_{\mathfrak{g}^{-1}} \circ J(u), \tag{12}$$

for any $u \in \mathfrak{g}^1$ *and* $u' \in \mathfrak{g}^{-1}$.

Proof. Using Corollary 4.2 and isomorphisms $\phi_{\mathfrak{g}^{-1}} \circ J : \mathfrak{g}^1 \to \mathfrak{g}^{-1}$ and $\phi_{\mathfrak{g}^1} \circ J : \mathfrak{g}^{-1} \to \mathfrak{g}^1$, we can consider ρ and ρ' as (11). Now, we show that ρ^* is a representation with respect to $\phi_{\mathfrak{g}^1}$. Using (11), we have

$$\rho^*(\phi_{\mathfrak{g}^1}(u)) \circ \phi_{\mathfrak{g}^1} = -\phi_{\mathfrak{g}^1} \circ J \circ \rho(\phi_{\mathfrak{g}^1}(u)) \circ \phi_{\mathfrak{g}^{-1}} \circ J \circ \phi_{\mathfrak{g}^1}.$$

Since ρ is a representation with respect to $\phi_{\mathfrak{g}^{-1}}$ and $\phi_{\mathfrak{g}^1} \circ J = J \circ \phi_{\mathfrak{g}^{-1}}$, the above equation implies

$$\begin{aligned}
\rho^*(\phi_{\mathfrak{g}^1}(u)) \circ \phi_{\mathfrak{g}^1} &= -\phi_{\mathfrak{g}^1} \circ J \circ \phi_{\mathfrak{g}^{-1}} \circ \rho(u) \circ J \circ \phi_{\mathfrak{g}^1} \\
&= -\phi_{\mathfrak{g}^1} \circ \phi_{\mathfrak{g}^1} \circ J \circ \rho(u) \circ \phi_{\mathfrak{g}^{-1}} \circ J = \phi_{\mathfrak{g}^1} \circ \rho^*(u).
\end{aligned}$$

Also, we get

$$\begin{aligned}
\rho^*([u, v]_{\mathfrak{g}^1}) \circ \phi_{\mathfrak{g}^1} &= -\phi_{\mathfrak{g}^1} \circ J \circ \rho([u, v]_{\mathfrak{g}^1}) \circ \phi_{\mathfrak{g}^{-1}} \circ J \circ \phi_{\mathfrak{g}^1} \\
&= -\phi_{\mathfrak{g}^1} \circ J \circ \rho(\phi_{\mathfrak{g}^1}(u)) \circ \rho(v) \circ J \circ \phi_{\mathfrak{g}^1} \\
&\quad + \phi_{\mathfrak{g}^1} \circ J \circ \rho(\phi_{\mathfrak{g}^1}(v)) \circ \rho(u) \circ J \circ \phi_{\mathfrak{g}^1}.
\end{aligned}$$

Applying $\phi_{\mathfrak{g}^{-1}} \circ J \circ \phi_{\mathfrak{g}^1} \circ J = -Id_{\mathfrak{g}}$ in the last equation, we obtain

$$\begin{aligned}
\rho^*([u, v]_{\mathfrak{g}^1}) \circ \phi_{\mathfrak{g}^1} &= \phi_{\mathfrak{g}^1} \circ J \circ \rho(\phi_{\mathfrak{g}^1}(u)) \circ \phi_{\mathfrak{g}^{-1}} \circ J \circ \phi_{\mathfrak{g}^1} \circ J \circ \rho(v) \circ \phi_{\mathfrak{g}^{-1}} \circ J \\
&\quad - \phi_{\mathfrak{g}^1} \circ J \circ \rho(\phi_{\mathfrak{g}^1}(v)) \circ \phi_{\mathfrak{g}^{-1}} \circ J \circ \phi_{\mathfrak{g}^1} \circ J \circ \rho(u) \circ \phi_{\mathfrak{g}^{-1}} \circ J \\
&= \rho^*(\phi_{\mathfrak{g}^1}(u)) \circ \rho^*(v) - \rho^*(\phi_{\mathfrak{g}^1}(v)) \circ \rho^*(u).
\end{aligned}$$

Similarly, we can see that $\rho^{*\prime}$ is a representation with respect to $\phi_{\mathfrak{g}^{-1}}$. Equations (10) and (11) imply (12). □

Applying (12), we can write ρ^* and $\rho^{*\prime}$ as follows:

$$\rho^*(u)v = -\pi^1(\phi_{\mathfrak{g}} \circ J[u, \phi_{\mathfrak{g}^{-1}}(Jv)]), \qquad \rho^{*\prime}(u')v' = -\pi^{-1}(\phi_{\mathfrak{g}} \circ J[u', \phi_{\mathfrak{g}^1}(Jv')]), \tag{13}$$

for any $u, v \in \mathfrak{g}^1$, $u', v' \in \mathfrak{g}^{-1}$ where $\pi^1 : \mathfrak{g} \to \mathfrak{g}^1$ and $\pi^{-1} : \mathfrak{g} \to \mathfrak{g}^{-1}$ are the projections.

THEOREM 4.4. *Let* $\{J, K\}$ *be a complex product structure on a hom-Lie algebra* $(\mathfrak{g}, \phi_{\mathfrak{g}}, [\cdot, \cdot])$. *Then,* $\mathfrak{g}^1$ *and* $\mathfrak{g}^{-1}$ *carry hom-left symmetric algebra structures.*

Proof. We consider $\cdot : \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^1$ as a bilinear product on $\mathfrak{g}^1$ given by $u \cdot v := \rho^*(u)v$, where ρ^* is determined in Proposition 4.3. Since ρ^* is a representation with respect to $\phi_{\mathfrak{g}^1}$, we obtain

$$\phi_{\mathfrak{g}^1}(u \cdot v) = \phi_{\mathfrak{g}^1}(\rho^*(u)v) = \rho^*(\phi_{\mathfrak{g}^1}(u))\phi_{\mathfrak{g}^1}(v) = \phi_{\mathfrak{g}^1}(u) \cdot \phi_{\mathfrak{g}^1}(v),$$

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and

$$\phi_{\mathfrak{g}^1}(u)\cdot(v\cdot w)-\phi_{\mathfrak{g}^1}(v)\cdot(u\cdot w)=\rho^*(\phi_{\mathfrak{g}^1}(u))\rho^*(v)w-\rho^*(\phi_{\mathfrak{g}^1}(v))\rho^*(u)w$$
$$=\rho^*([u,v]_{\mathfrak{g}^1})(\phi_{\mathfrak{g}^1}(w))=[u,v]\cdot\phi_{\mathfrak{g}^1}(w).$$

Also, (7) and (13) imply

$$u\cdot v-v\cdot u=\rho^*(u)v-\rho^*(v)u=-\pi^1\big(\phi_{\mathfrak{g}}\circ J([u,\phi_{\mathfrak{g}^{-1}}(Jv)]+[\phi_{\mathfrak{g}^{-1}}(Ju),v])\big)$$
$$=\pi^1([u,v]-[\phi_{\mathfrak{g}^{-1}}(Ju),\phi_{\mathfrak{g}^{-1}}(Jv)])=[u,v].$$

The two last equations imply

$$\phi_{\mathfrak{g}^1}(u)\cdot(v\cdot w)-\phi_{\mathfrak{g}^1}(v)\cdot(u\cdot w)=(u\cdot v)\cdot\phi_{\mathfrak{g}^1}(w)-(v\cdot u)\cdot\phi_{\mathfrak{g}^1}(w).$$

Therefore, $\mathfrak{g}^1$ carries a hom-left symmetric algebra structure. We define a bilinear product $\cdot:\mathfrak{g}^{-1}\times\mathfrak{g}^{-1}\to\mathfrak{g}^{-1}$ on $\mathfrak{g}^{-1}$ by $u'\cdot v':=\rho^{*\prime}(u')v'$. Similarly, it is shown that $\cdot$ is a hom-left symmetric product on $\mathfrak{g}^{-1}$. □

Let $(\mathfrak{g},\phi_{\mathfrak{g}},[\cdot,\cdot])$ be a hom-Lie algebra with a complex product structure $\{J,K\}$. We extend the hom-left symmetric products of $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ to $\mathfrak{g}$ by

$$(u+u')\cdot(v+v')=u\cdot v+\rho(u)v'+\rho'(u')v+u'\cdot v'. \tag{14}$$

We consider two bilinear maps $\Psi:\mathfrak{g}^1\times\mathfrak{g}^{-1}\to End(\mathfrak{g}^1)$ and $\Psi^*:\mathfrak{g}^{-1}\times\mathfrak{g}^1\to End(\mathfrak{g}^{-1})$ defined by

$$\Psi(u,u')w=\rho'(\phi_{\mathfrak{g}^{-1}}(u'))(u\cdot w)-\phi_{\mathfrak{g}^1}(u)\cdot\rho'(v')w$$
$$-\rho'(v')u\cdot\phi_{\mathfrak{g}^1}(w)+\rho'(\rho(u)u')(\phi_{\mathfrak{g}^1}(w)),$$
$$\Psi^*(u',u)w'=\rho(\phi_{\mathfrak{g}^1}(u))(u'\cdot w')-\phi_{\mathfrak{g}^{-1}}(u')\cdot\rho(v)w'$$
$$-\rho(v)u'\cdot\phi_{\mathfrak{g}^{-1}}(w')+\rho(\rho'(u')u)(\phi_{\mathfrak{g}^{-1}}(w')),$$

for any $u,w\in\mathfrak{g}^1$, $u',w'\in\mathfrak{g}^{-1}$.

PROPOSITION 4.5. *Let* $(\mathfrak{g},\phi_{\mathfrak{g}},[\cdot,\cdot])$ *be a hom-Lie algebra with a complex product structure* $\{J,K\}$*. Then, the product* $\cdot$ *on* $\mathfrak{g}$ *given by (14) is a hom-left symmetric product if and only if* $\Psi(u,u')w=\Psi^*(u',u)w'=0$*, for any* $u,w\in\mathfrak{g}^1$, $u',w'\in\mathfrak{g}^{-1}$.

Proof. Using (14), we get

$$\phi_{\mathfrak{g}}(u+u')\cdot\phi_{\mathfrak{g}}(v+v')=(\phi_{\mathfrak{g}^1}(u)+\phi_{\mathfrak{g}^{-1}}(u'))\cdot(\phi_{\mathfrak{g}^1}(v)+\phi_{\mathfrak{g}^{-1}}(v'))$$
$$=\phi_{\mathfrak{g}^1}(u)\cdot\phi_{\mathfrak{g}^1}(v)+\rho(\phi_{\mathfrak{g}^1}(u))(\phi_{\mathfrak{g}^{-1}}(v'))+\rho'(\phi_{\mathfrak{g}^{-1}}(u'))(\phi_{\mathfrak{g}^1}(v))+\phi_{\mathfrak{g}^{-1}}(u')\cdot\phi_{\mathfrak{g}^{-1}}(v')$$
$$=\phi_{\mathfrak{g}^1}(u\cdot v)+\phi_{\mathfrak{g}^{-1}}(\rho(u)v')+\phi_{\mathfrak{g}^1}(\rho'(u')v)+\phi_{\mathfrak{g}^{-1}}(u'\cdot v')=\phi_{\mathfrak{g}}((u+u')\cdot(v+v')).$$

Also, a direct computation yields

$$\begin{aligned}
&((u+u')\cdot(v+v'))\cdot\phi_{\mathfrak{g}}(w+w')-\phi_{\mathfrak{g}}(u+u')\cdot((v+v')\cdot(w+w'))\\
&\quad-((v+v')\cdot(u+u'))\cdot\phi_{\mathfrak{g}}(w+w')\\
&\quad+\phi_{\mathfrak{g}}(v+v')\cdot((u+u')\cdot(w+w'))=\Psi(u,v')w-\Psi(v,u')w\\
&\quad+\Psi^*(u',v)w'-\Psi^*(v',u)w'+\rho([u,v]_{\mathfrak{g}^1})(\phi_{\mathfrak{g}^{-1}}(w'))\\
&\quad-\ \rho(\phi_{\mathfrak{g}^1}(u))(\rho(v)w')+\rho(\phi_{\mathfrak{g}^1}(v))(\rho(u)w')\\
&\quad+\rho'([u',v']_{\mathfrak{g}^{-1}})(\phi_{\mathfrak{g}^1}(w))-\rho'(\phi_{\mathfrak{g}^{-1}}(u'))(\rho'(v')w)+\rho'(\phi_{\mathfrak{g}^{-1}}(v'))(\rho'(u')w).
\end{aligned}$$

Since ρ and ρ' are representations with respect to $\phi_{\mathfrak{g}^{-1}}$ and $\phi_{\mathfrak{g}^1}$, respectively, the above equation reduces to

$$\begin{aligned}
&((u+u')\cdot(v+v'))\cdot\phi_{\mathfrak{g}}(w+w')-\phi_{\mathfrak{g}}(u+u')\cdot((v+v')\cdot(w+w'))\\
&\quad-((v+v')\cdot(u+u'))\cdot\phi_{\mathfrak{g}}(w+w')\\
&\quad+\phi_{\mathfrak{g}}(v+v')\cdot((u+u')\cdot(w+w'))=\Psi(u,v')w-\Psi(v,u')w\\
&\quad+\Psi^*(u',v)w'-\Psi^*(v',u)w'.
\end{aligned}$$

Therefore, we conclude the assertion. □

Let $(\mathfrak{g},\phi_{\mathfrak{g}},[\cdot,\cdot])$ be a hom-Lie algebra. We consider

$$T(X,Y):=L_XY-L_YX-[X,Y],$$

and call it the tensor torsion of $\mathfrak{g}$. Also, we define the tensor curvature $\mathcal{K}$ of $\mathfrak{g}$ as follows:

$$\mathcal{K}(X,Y):=L_{\phi_{\mathfrak{g}}(X)}\circ L_Y-L_{\phi_{\mathfrak{g}}(Y)}\circ L_X-L_{[X,Y]}\circ\phi_{\mathfrak{g}}, \tag{15}$$

for any $X, Y\in\mathfrak{g}$.

Under the assumptions of Proposition 4.5, on a hom-Lie algebra $(\mathfrak{g},\phi_{\mathfrak{g}},[\cdot,\cdot])$ with a complex product structure $\{J,K\}$, we set

$$L_X^{CP}Y:=X\cdot Y,\qquad \forall X,Y\in\mathfrak{g},$$

where $\cdot$ is the hom-left symmetric product on $\mathfrak{g}$ that satisfies (14). Using (10), (14) and Proposition 4.5, we can write

$$\begin{aligned}
&[X,Y]=L_X^{CP}Y-L_Y^{CP}X,\\
&L^{CP}_{\phi_{\mathfrak{g}}(X)}\circ L^{CP}_Y-L^{CP}_{\phi_{\mathfrak{g}}(Y)}\circ L^{CP}_X=L^{CP}_{[X,Y]_{\mathfrak{g}}}\circ\phi_{\mathfrak{g}},
\end{aligned}$$

which are equivalent to the vanishing of the torsion and the curvature tensors of $(\mathfrak{g},\cdot)$.

PROPOSITION 4.6. *Let $(\mathfrak{g},\phi_{\mathfrak{g}},[\cdot,\cdot])$ be a hom-Lie algebra with a complex product structure $\{J,K\}$. Under the assumptions of Proposition 4.5, J and K are invariant with respect to hom-left symmetric product $\cdot$ given by (14), i.e.,*

$$\begin{aligned}
L_X^{CP}\circ\phi_{\mathfrak{g}}\circ J=&\phi_{\mathfrak{g}}\circ J\circ L_X^{CP},\\
L_X^{CP}\circ\phi_{\mathfrak{g}}\circ K=&\phi_{\mathfrak{g}}\circ K\circ L_X^{CP},
\end{aligned}$$

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for any $X \in \mathfrak{g}$. Moreover, the hom-left symmetric product $\cdot$ satisfying in two above equations is unique.

Proof. Let $u, v \in \mathfrak{g}^1$, $u', v' \in \mathfrak{g}^{-1}$. Then, (11) and (14) imply

$$\begin{aligned} L^{CP}_{(u+u')}((\phi_{\mathfrak{g}} \circ J)(v+v')) &= (u+u') \cdot (\phi_{\mathfrak{g}^{-1}}(Jv) + \phi_{\mathfrak{g}^1}(Jv')) \\ &= u \cdot \phi_{\mathfrak{g}^1}(Jv') + \rho(u)(\phi_{\mathfrak{g}^{-1}}(Jv)) + \rho'(u')\phi_{\mathfrak{g}^1}(Jv') + u' \cdot \phi_{\mathfrak{g}^{-1}}(Jv) \\ &= \phi_{\mathfrak{g}^1}(J\rho(u)v') + \phi_{\mathfrak{g}^{-1}}(J(u \cdot v)) + \phi_{\mathfrak{g}^1}(J(u' \cdot v')) + \phi_{\mathfrak{g}^{-1}}(J\rho'(u')v) \\ &= (\phi_{\mathfrak{g}} \circ J)((u+u') \cdot (v+v')) = (\phi_{\mathfrak{g}} \circ J)L^{CP}_{(u+u')}(v+v'). \end{aligned}$$

Also, we conclude

$$\begin{aligned} L^{CP}_{(u+u')}((\phi_{\mathfrak{g}} \circ K)(v+v')) &= (u+u') \cdot (\phi_{\mathfrak{g}}(Kv) + \phi_{\mathfrak{g}}(Kv')) \\ &= (u+u') \cdot (v - v') = u \cdot v - \rho(u)v' + \rho'(u')v - u' \cdot v' \\ &= (\phi_{\mathfrak{g}} \circ K)((u+u') \cdot (v+v')) = (\phi_{\mathfrak{g}} \circ K)L^{CP}_{(u+u')}(v+v'). \end{aligned}$$

Finally, we show the uniqueness of hom-left symmetric product. Let $\triangleright$ and $\bullet$ be two such products and A is $(1,2)$-tensor defined by $A_X := L^{\triangleright}_X - L^{\bullet}_X$. Since $L^{\triangleright}_X \circ \phi_{\mathfrak{g}} \circ K = \phi_{\mathfrak{g}} \circ K \circ L^{\triangleright}_X$ and $L^{\bullet}_X \circ \phi_{\mathfrak{g}} \circ K = \phi_{\mathfrak{g}} \circ K \circ L^{\bullet}_X$, we obtain

$$\begin{aligned} A_X \circ \phi_{\mathfrak{g}} \circ K &= L^{\triangleright}_X \circ \phi_{\mathfrak{g}} \circ K - L^{\bullet}_X \circ \phi_{\mathfrak{g}} \circ K = \phi_{\mathfrak{g}} \circ K \circ L^{\triangleright}_X - \phi_{\mathfrak{g}} \circ K \circ L^{\bullet}_X \\ &= \phi_{\mathfrak{g}} \circ K \circ (L^{\triangleright}_X - L^{\bullet}_X) = \phi_{\mathfrak{g}} \circ K \circ A_X. \end{aligned}$$

Similarly, we have $A_X \circ \phi_{\mathfrak{g}} \circ J = \phi_{\mathfrak{g}} \circ J \circ A_X$. Moreover, A is symmetric, i.e.,

$$A_X Y = L^{\triangleright}_X Y - L^{\bullet}_X Y = L^{\triangleright}_Y X + [X, Y]_{\mathfrak{g}} - L^{\bullet}_Y X + [Y, X]_{\mathfrak{g}} = A_Y X.$$

From the above equations, we deduce

$$\begin{aligned} A_{\phi_{\mathfrak{g}}(JX)}\phi_{\mathfrak{g}}(KY) &= (\phi_{\mathfrak{g}} \circ K)A_{\phi_{\mathfrak{g}}(JX)} Y = (\phi_{\mathfrak{g}} \circ K)A_Y \phi_{\mathfrak{g}}(JX) = (\phi_{\mathfrak{g}} \circ K)(\phi_{\mathfrak{g}} \circ J)A_Y X \\ &= -(\phi_{\mathfrak{g}} \circ J)(\phi_{\mathfrak{g}} \circ K)A_Y X = -(\phi_{\mathfrak{g}} \circ J)(\phi_{\mathfrak{g}} \circ K)A_X Y \\ &= -A_{\phi_{\mathfrak{g}}(JX)}\phi_{\mathfrak{g}}(KY), \end{aligned}$$

which gives $A = 0$. □

5. Hyper-para-Kähler hom-Lie algebra.

In this section, we introduce hyper-para-Kähler structures on hom-Lie algebras. Also, we present an example of these structures.

DEFINITION 5.1. An almost complex structure J on a symplectic hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \Omega)$ is called Ω-tame if

$$\Omega(X, \phi_{\mathfrak{g}}(JX)) > 0, \quad \forall X \neq 0.$$

Also, J is called Ω-compatible if it is Ω-tame and

$$\Omega(\phi_{\mathfrak{g}}(JX), \phi_{\mathfrak{g}}(JY)) = \Omega(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

Using the condition Ω-compatible of the structure J, we can define a Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ as follows:

$$\langle X, Y\rangle := \Omega(X, \phi_{\mathfrak{g}}(JY)).$$

From the above equations, we conclude $\langle\phi_{\mathfrak{g}}(JX), \phi_{\mathfrak{g}}(JY)\rangle := \langle X, Y\rangle$.

Definition 5.2. Let $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \Omega)$ be a symplectic hom-Lie algebra. An almost para-complex structure K on $\mathfrak{g}$ is called Ω-compatible if

$$\Omega(\phi_{\mathfrak{g}}(KX), \phi_{\mathfrak{g}}(KY)) = -\Omega(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

A pseudo-Riemannian metric associated with structure K is determined by $\ll X, Y \gg := \Omega(\phi_{\mathfrak{g}}(KX), Y)$ that satisfies

$$\ll \phi_{\mathfrak{g}}(KX), \phi_{\mathfrak{g}}(KY) \gg = - \ll X, Y \gg .$$

From Propositions 3.1 and 3.3, we deduce the following.

Corollary 5.3. *Let J and K be complex and para-complex structures on a symplectic hom-Lie algebra $(\mathfrak{g}, \Omega)$, respectively. If J and K are Ω-compatible structures, then we have*

$$X \cdot_J \phi_{\mathfrak{g}}(JY) = (\phi_{\mathfrak{g}} \circ J)(X \cdot_J Y),$$
$$X \cdot_K \phi_{\mathfrak{g}}(KY) = (\phi_{\mathfrak{g}} \circ K)(X \cdot_K Y),$$

where $\cdot_J$ and $\cdot_K$ denote the hom-Levi-Civita product associated with $\langle\cdot,\cdot\rangle$ and $\ll \cdot,\cdot \gg$, respectively.

Definition 5.4. A hyper-para-Kähler hom-Lie algebra is a symplectic hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \Omega)$ endowed with a complex product structure $\{J, K\}$, such that J, K are Ω-compatible.

Using $\langle\cdot,\cdot\rangle$ and $\ll,\gg$, we have

$$\langle\phi_{\mathfrak{g}}(KX), Y\rangle = \Omega(\phi_{\mathfrak{g}}(KX), \phi_{\mathfrak{g}}(JY)) = \ll X, \phi_{\mathfrak{g}}(JY) \gg .$$

By Theorem 3.5 and taking into account the above definition, we can easily conclude the following:

(i) $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are subalgebras isotropic with respect to $\ll,\gg$, and Lagrangian with respect to Ω,
(ii) $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle, J)$ is a Hermitian hom-Lie algebra,
(iii) $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, \ll,\gg, K)$ is a para-Hermitian hom-Lie algebra,
(v) for any $X \in \mathfrak{g}$, $X \cdot_K \mathfrak{g}^1 \subset \mathfrak{g}^1$ and $X \cdot_K \mathfrak{g}^{-1} \subset \mathfrak{g}^{-1}$ (see [**13,14**] for more details).

Example 5.5. We consider the hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ introduced in Example 2.2 endowed with complex product structure given in Example 3.6. We now consider the bilinear skew-symmetric nondegenerate form Ω as follows:

$$\begin{bmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & -A \\ -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \end{bmatrix}, \quad A \neq 0. \tag{16}$$

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Then, we get

$$\begin{aligned} \Omega(\phi_{\mathfrak{g}}(e_1), \phi_{\mathfrak{g}}(e_3)) &= A = \Omega(e_1, e_3), & \Omega(\phi_{\mathfrak{g}}(e_2), \phi_{\mathfrak{g}}(e_4)) &= -A = \Omega(e_2, e_4),\\ \Omega(\phi_{\mathfrak{g}}(e_1), \phi_{\mathfrak{g}}(e_2)) &= 0 = \Omega(e_1, e_2), & \Omega(\phi_{\mathfrak{g}}(e_1), \phi_{\mathfrak{g}}(e_4)) &= 0 = \Omega(e_1, e_4),\\ \Omega(\phi_{\mathfrak{g}}(e_2), \phi_{\mathfrak{g}}(e_3)) &= 0 = \Omega(e_2, e_3), & \Omega(\phi_{\mathfrak{g}}(e_3), \phi_{\mathfrak{g}}(e_4)) &= 0 = \Omega(e_3, e_4), \end{aligned}$$

and

$$\Omega([e_i, e_j], \phi_{\mathfrak{g}}(e_k)) + \Omega([e_j, e_k], \phi_{\mathfrak{g}}(e_i)) + \Omega([e_k, e_i], \phi_{\mathfrak{g}}(e_j)) = 0, \qquad i, j, k = 1, 2, 3, 4.$$

The above relations show that Ω is 2-hom-cocycle, and so $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}}, \Omega)$ is a symplectic hom-Lie algebra. Using the above equations, we obtain

$$\Omega(e_1, \phi_{\mathfrak{g}}(Je_1) = \Omega(e_2, \phi_{\mathfrak{g}}(Je_2) = \Omega(e_3, \phi_{\mathfrak{g}}(Je_3) = \Omega(e_4, \phi_{\mathfrak{g}}(Je_4) = A,$$

i.e., the complex structure J is a Ω-tame. Also, we get

$$\begin{aligned} \Omega(\phi_{\mathfrak{g}}(Je_i), \phi_{\mathfrak{g}}(Je_j)) &= \Omega(e_i, e_j), \quad i, j = 1, 2, 3, 4,\\ \Omega(\phi_{\mathfrak{g}}(Ke_i), \phi_{\mathfrak{g}}(Ke_j)) &= -\Omega(e_i, e_j), \quad i, j = 1, 2, 3, 4, \end{aligned}$$

$$\begin{aligned} \Omega(\phi_{\mathfrak{g}}(Je_1), \phi_{\mathfrak{g}}(Je_3)) &= A = \Omega(e_1, e_3),\\ \Omega(\phi_{\mathfrak{g}}(Je_2), \phi_{\mathfrak{g}}(Je_4)) &= -A = \Omega(e_2, e_4), \end{aligned}$$

and

$$\begin{aligned} \Omega(\phi_{\mathfrak{g}}(Ke_1), \phi_{\mathfrak{g}}(Ke_3)) &= -A = -\Omega(e_1, e_3),\\ \Omega(\phi_{\mathfrak{g}}(Ke_2), \phi_{\mathfrak{g}}(Ke_4)) &= A = -\Omega(e_2, e_4), \end{aligned}$$

i.e., the structures J and K are Ω-compatible. Therefore, $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}}, \Omega)$ is a hyper-para-Kähler hom-Lie algebra.

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