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A NEW IDENTITY AND SOME APPLICATIONS

BY

W. O. J. MOSER[1] AND RICHARD POLLACK[2]

ABSTRACT. Let $(n \mid k)$ denote the number of k-choices $1 \le x_1 < x_2 < \cdots < x_k \le n$ satisfying $x_i - x_{i-1} \ge 2$, $i = 2, \ldots, k$, $n + x_1 - x_k \ge 2$; let $(m, n \mid k) = \sum_{i+j=k} (m \mid i)(n \mid j)$. Several elementary proofs of the new identity $(m, n \mid k) = (m + n \mid k)$ if $0 \le k < m \le n$ and

$$(m, n \mid k) = (m + n \mid k) + (-1)^m (n - m \mid k - m)$$

if $0 \le m \le n$, $m \le k$, are given. Generalizations and applications are considered.

1. For non-negative integers n, k, w (n, k not both 0), let $(n \mid k)_w$ denote the number of k-choices (k-subsets)

$$1 \le x_1 < x_2 < \cdots < x_k \le n \tag{1}$$

satisfying the conditions

$$x_i - x_{i-1} \ge w + 1, \qquad i = 2, \ldots, k \tag{2}$$

and

$$x_1 + n - x_k \ge w + 1. \tag{3}$$

These conditions are best visualized by displaying $1, 2, \ldots, n$ in a circle (rising order clockwise) and conditions (2) and (3) are then: every chosen integer is followed (clockwise) by at least w non-chosen integers. Equivalently, such a k-choice can be described by a display of k 1's and $n - k$ 0's in a circle with one of the n entries capped. For example, the choice $\{2, 5, 9, 12\}$ counted in $(13 \mid 4)_2$ is described by

1
0 0
0 $\hat{0}$
1 1
0 0
0 0
0 1

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[2] While visiting McGill University.

(The cap tells you where to start!) It is well known [1, p. 222, problem 2] and easy to deduce [2, formula 17] that

$$(4)\quad (n \mid k)_w = \begin{cases} \dfrac{n}{n-wk}\dbinom{n-wk}{k}, & 0 \le k \le \dfrac{n}{w+1}, \quad (n, k) \ne (0, 0), \\ 0, & 0 \le \dfrac{n}{w+1} < k. \end{cases}$$

Taking $(0 \mid 0)_w = w+1$ permits $(n \mid k)_w$ to be determined by the recurrence

$$(5)\quad \begin{aligned} &(n \mid k)_w = (n-1 \mid k)_w + (n-w-1 \mid k-1)_w,\ n \ge w+1,\ k \ge 1, \\ &(0 \mid 0)_w = w+1,\ (n \mid 0)_w = 1 \text{ for } n \ge 1,\ (n \mid k)_w = 0 \text{ for } 0 \le n \le w,\ k \ge 1. \end{aligned}$$

Indeed, $(n-1 \mid k)_w$ counts the choices (1) (satisfying (2) and (3)) for which $x_k - x_{k-1} > w+1$, while $(n-w-1 \mid k-1)_w$ counts the choices for which $x_k - x_{k-1} = w+1$.

When $w=0$, $(n \mid k)_0$ is simply the number of k-subsets of a set of size n, i.e.,

$$(n \mid k)_0 = \begin{cases} \dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}, & 0 \le k \le n, \\ 0, & 0 \le n < k. \end{cases}$$

The obvious identity

$$\sum_{i=0}^{k} (m \mid i)_0 (n \mid k-i)_0 = (m+n \mid k)_0, \qquad m, n, k \ge 0,$$

or equivalently,

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}, \qquad m, n, k \ge 0,$$

is the well known Vandermonde Convolution.

When $w=1$, we conveniently suppress the subscript 1, so

$$(n \mid k) = (n \mid k)_1 = \begin{cases} 2, & \text{if } n = k = 0, \\ \dfrac{n}{n-k}\dbinom{n-k}{k}, & \text{if } 0 \le k \le \dfrac{n}{2}, \quad (n, k) \ne (0, 0), \\ 0, & \text{if } 0 \le \dfrac{n}{2} < k. \end{cases}$$

Defining

$$(m, n \mid k)_w = \sum_{i+j=k} (m \mid i)_w (n \mid j)_w,$$

the identity we establish is:

$$(6)\quad (m, n \mid k)=\begin{cases}(m+n \mid k), & \text{if } 0\le k<m\le n,\\ (m+n \mid k)+(-1)^m(n-m \mid k-m), & \text{if } 0\le m\le n,\ m\le k.\end{cases}$$

The combinatorial meaning of $(m, n \mid k)_w$ is this. It is the number of k-choices from $\{1, 2, \ldots, m+n\}$ such that in the display of $\{1, 2, \ldots, m\}$ in one circle and $\{m+1, \ldots, m+n\}$ in another circle each chosen integer is followed (in the circle it appears) by w non-chosen integers.

After providing several proofs of (6), thus illustrating different techniques, we will describe an application to counting $3\times n$ Latin rectangles, a generalization and its relation to an identity of Rothe ([9]; see [3] and [4]).

2. When $w=1$, recurrence (5) is

$$(n \mid k)=(n-1 \mid k)+(n-2 \mid k-1),\ n\ge 2,\ k\ge 1,$$
$$(0 \mid 0)=2,\ (n \mid 0)=1 \text{ for } n\ge 1,\ (n \mid k)=0 \text{ for } n=0,\ 1,\ k\ge 1,$$

and this leads to the generating function which we "partial fraction":

$$\sum_{n,k\ge 0}(n \mid k)x^nz^k=\frac{2-x}{1-x-x^2z}=\frac{1}{1-\alpha_1x}+\frac{1}{1-\alpha_2x}=\sum_{n\ge 0}(\alpha_1^n+\alpha_2^n)x^n,$$

where α_1, α_2 are power series in z satisfying

$$\alpha_1+\alpha_2=1, \qquad \alpha_1\alpha_2=-z, \qquad \alpha_1^n+\alpha_2^n=\sum_{k\ge 0}(n \mid k)z^k.$$

Now

$$(7)\qquad \sum_{m,n,k\ge 0}(m, n \mid k)x^my^nz^k=\frac{2-x}{1-x-x^2z}\cdot\frac{2-y}{1-y-y^2z}$$
$$=\sum_{m\ge 0}(\alpha_1^m+\alpha_2^m)x^m\sum_{n\ge 0}(\alpha_1^n+\alpha_2^n)y^n,$$

and equating coefficients of x^my^n yields for all $0\le m\le n$

$$\sum_{k\ge 0}(m, n \mid k)z^k=(\alpha_1^m+\alpha_2^m)(\alpha_1^n+\alpha_2^n)$$
$$=\alpha_1^{m+n}+\alpha_2^{m+n}+(\alpha_1\alpha_2)^m(\alpha_1^{n-m}+\alpha_2^{n-m})$$
$$=\sum_{k\ge 0}(m+n \mid k)z^k+(-z)^m\sum_{k\ge 0}(n-m \mid k)z^k$$
$$=\sum_{k\ge 0}(m+n \mid k)z^k+\sum_{k\ge m}(-1)^m(n-m \mid k-m)z^k.$$

Equating coefficients of z^k establishes identity (6).

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We can get by without partial fractioning. First note that

$$(8)\qquad \sum_{m,n,k\geq 0}(m+n \mid k)x^m y^n z^k = \sum_{r,k\geq 0}(r \mid k)z^k \sum_{m+n=r} x^m y^n$$
$$= \sum_{r,k\geq 0}(r \mid k)z^k \frac{x^{r+1}-y^{r+1}}{x-y}$$
$$= \frac{x}{x-y}\cdot\frac{2-x}{1-x-x^2z}+\frac{y}{y-x}\cdot\frac{2-y}{1-y-y^2z}$$

Furthermore, letting

$$a(m, n \mid k)=\begin{cases}(-1)^m(n-m \mid k-m), & \text{if } 0\leq m\leq n,\ m\leq k,\\ (-1)^n(m-n \mid k-n), & \text{if } 0\leq n<m,\ n\leq k,\\ 0 & \text{if } k<\min(m, n),\end{cases}$$

a little manipulation yields

$$(9)\qquad \sum_{m,n,k\geq 0} a(m, n \mid k)x^m y^n z^k = \frac{1}{1+xyz}\left\{\frac{2-x}{1-x-x^2x}+\frac{2-y}{1-y-y^2z}-2\right\}.$$

Identity (6) now follows because the sum of functions (8) and (9) is identically equal to function (7).

We proceed to outline an elementary proof of (6) which uses recurrences but not generating functions. Details are left to the reader. First note that

$$(m, n \mid k)=(m-1, n \mid k)+(m-2, n \mid k-1), \quad m\geq 2, \quad n\geq 0, \quad k\geq 1,$$
$$=(m, n-1 \mid k)+(m, n-2 \mid k-1), \quad m\geq 0, \quad n\geq 2, \quad k\geq 1,$$

(cf. (5) with $w=1$). Next, taking

$$g(m, n, k)=(m, n \mid k)-(m+n \mid k)+(m-1, n-1 \mid k-1)-(m+n-2 \mid k-1)$$

for $m, n, k\geq 1$, it follows that

$$(10)\qquad g(m, n, k)=g(m-1, n-1, k)+g(m-1, n-2, k-1)$$
$$+g(m-2, n-1, k-1)+g(m-2, n-2, k-2)$$

for $m, n, k\geq 3$. Now

$$(11)\qquad g(m, n, k)=0 \quad \text{if } m, n, k\geq 0,\ m, n, k \text{ not all } \geq 3,$$

easily follows, and induction (using (10) and (11)) implies

$$g(m, n, k)=0 \quad \text{for} \quad m, n, k\geq 1,$$

or

$$(12)\quad (m, n \mid k)-(m+n \mid k)=-\{(m-1, n-1 \mid k-1)-(m+n-2 \mid k-1)\}$$

for $m, n, k\geq 1$. Repeated application of (12) leads to (6).

The last proof we now give of (6) is strictly combinatorial, by means of one-to-one correspondences. Let $S(m, n \mid k)$ denote the set of k-choices counted by $(m, n \mid k)$. Each such choice can be represented by a display

$$\begin{matrix} \alpha_1\alpha_2\alpha_3\cdots\alpha_m \\ \beta_1\beta_2\beta_3\cdots\beta_m\beta_{m+1}\cdots\beta_n \end{matrix}, \qquad 0 \le m \le n, \tag{13}$$

of k 1's and $m+n-k$ 0's. In each of the two rows no two 1's are adjacent, with α_m, α_1 and β_n, β_1 adjacent pairs. (These displays really should be in two circles, but that would make the typesetting difficult.)

The identity (6) is easy to prove when $0 \le k < m \le n$. In this case there is a q, $1 \le q \le m$, such that $\alpha_q = \beta_q = 0$ while α_i, β_i are not both 0 when $i < q$. Consider the following mapping of a k-choice in $S(m, n \mid k)$ to a k-choice in $S(m+n \mid k)$ (the set counted by $(m+n \mid k)$):

$$\begin{aligned} &\alpha_1\alpha_2\cdots\alpha_{q-1}0\alpha_{q+1}\cdots\alpha_m \\ &\beta_1\beta_2\cdots\beta_{q-1}0\beta_{q+1}\cdots\beta_m\cdots\beta_n \\ &\qquad\qquad \to \alpha_1\alpha_2\cdots\alpha_{q-1}0\beta_{q+1}\cdots\beta_n\beta_1\cdots\beta_{q-1}0\alpha_{q+1}\cdots\alpha_m. \end{aligned} \tag{14}$$

This is a one-to-one mapping from $S(m, n \mid k)$ to $S(m+n \mid k)$ when $0 \le k < m \le n$. Furthermore, this map is onto $S(m+n \mid k)$. Indeed, if,

$$\gamma_1\gamma_2\cdots\gamma_{m+n} \tag{15}$$

is a sequence of k 1's and $m+n-k$ 0's representing a choice in $S\ (m+n \mid k)$ (so that no two 1's are adjacent, γ_{m+n} and γ_1 being adjacent) then, because $k < m$, there is a t, $1 \le t \le m$, such that $\gamma_t = \gamma_{t+n} = 0$, while γ_i, γ_{i+n} are not both 0 for $i < t$. Clearly the map takes the k-choice

$$\begin{matrix} \gamma_1\gamma_2\cdots\gamma_{t-1}0\gamma_{n+t+1}\cdots\gamma_{n+m} \\ \gamma_{1+n\ 2+n}\cdots\gamma_{t+n-1}0\gamma_{t+1}\cdots\gamma_n \end{matrix}$$

of $S(m, n \mid k)$ onto the k-choice (15), and hence $(m, n \mid k) = (m+n \mid k)$ if $0 \le k < m \le n$.

Several examples should make this correspondence clear. Thus for $k=3$, $m=4$, $n=6$:

$$\{5,7,9\} \text{ or } \begin{matrix} 0\,0\,0\,0 \\ 1\,0\,1\,0\,1\,0 \end{matrix} \leftrightarrow 0\,0\,1\,0\,1\,0\,1\,0\,0\,0 \text{ or } \{3,5,7\}$$

$$\{1,6,9\} \text{ or } \begin{matrix} 1\,0\,0\,0 \\ 0\,1\,0\,0\,1\,0 \end{matrix} \leftrightarrow 1\,0\,0\,0\,1\,0\,0\,1\,0\,0 \text{ or } \{1,5,8\}$$

$$\{2,5,7\} \text{ or } \begin{matrix} 0\,1\,0\,0 \\ 1\,0\,1\,0\,0\,0 \end{matrix} \leftrightarrow 0\,1\,0\,0\,0\,0\,1\,0\,1\,0 \text{ or } \{2,7,9\}$$

When $0 \le m \le n$ and $m \le k$, the situation is slightly complicated. If $m = n \le k$, (6) is obviously correct. Hence we may take $m < n$. Consider first the case m

even. Partition the set $S(m, n \mid k)$ into three disjoint subsets S_1, S_2, S_3 as follows. S_1 consists of choices (13) for which there is a $q \in \{1, 2, \ldots, m\}$ such that $\alpha_q = \beta_q = 0$ while α_i, β_i are not both 0 for $i < q$. If there is no such q then $\alpha_i + \beta_i = 1$, $1 \le i \le m$, and the k-choice (13) belongs to S_2 if $\beta_m = \beta_{m+1} = 0$, otherwise to S_3. The k-choices in S_1 are mapped by (14) to k-choices in $S(m+n \mid k)$; the k-choice (13) in S_2 is mapped to the k-choice

$$\alpha_1\alpha_2 \cdots \alpha_m\beta_{m+1}\beta_{m+2} \cdots \beta_n\beta_1\beta_2 \cdots \beta_m$$

in $S(m+n \mid k)$; the k-choice (13) in S_3 is mapped onto the $(k-m)$-choice

$$\beta_{m+1}\beta_{m+2} \cdots \beta_n$$

of $S(n-m \mid k-m)$. Illustrated below is a typical element of S_2 and two elements of S_3 and the mapping of each:

$$S_2: \begin{matrix} 0\,1\,0\,1 \cdots 0\,1\,0\,1 \\ 1\,0\,1\,0 \cdots 1\,0\,1\,0\,0\,\beta_{m+2} \cdots \beta_{n-1}\,0 \end{matrix}$$

$$\to 0\,1\,0\,1 \cdots 0\,1\,0\,1\,0\,\beta_{m+2} \cdots \beta_{n-1}\,0\,1\,0\,1\,0 \cdots 1\,0\,1\,0 \quad \text{in } S(m+n \mid k);$$

$$S_3: \begin{matrix} 0\,1\,0\,1 \cdots 0\,1\,0\,1 \\ 1\,0\,1\,0 \cdots 1\,0\,1\,0\,1\,\beta_{m+2} \cdots \beta_{n-1}\,0 \end{matrix}$$

$$\to 1\,\beta_{m+2} \cdots \beta_{n-1}\,0 \quad \text{in} \quad S(n-m \mid k-m);$$

$$S_3: \begin{matrix} 1\,0\,1\,0 \cdots 1\,0\,1\,0 \\ 0\,1\,0\,1 \cdots 0\,1\,0\,1\,0\,\beta_{m+2} \cdots \beta_n \end{matrix}$$

$$\to 0\,\beta_{m+2} \cdots \beta_n \quad \text{in} \quad S(n-m \mid k-m).$$

It is easy to see that $S_1 \cup S_2$ is mapped one-to-one *onto* $S(m+n \mid k)$ while S_3 is mapped one-to-one onto $S(n-m \mid k-m)$, thus proving (6) for $0 \le m \le n$, $m < k$, m even.

There remains the case $k \ge m$, m odd. Now split $S(m, n \mid k)$ into two sets, S_1 as already described in the case of even m, and S_2' consisting of the k-choices (13) for which

$$\alpha_1\alpha_2 \cdots \alpha_m = 0\,1\,0\,1\,0 \cdots 0\,1\,0$$

and

$$\beta_1\beta_2 \cdots \beta_m = 1\,0\,1\,0\,1 \cdots 1\,0\,1$$

As before, the k-choices of S_1 are mapped by (14) to elements of $S(m+n \mid k)$ and the k choice (13) in S_2' is mapped to the k-choice

$$\alpha_1\alpha_2 \cdots \alpha_m\beta_{m+1}\beta_{m+2} \cdots \beta_n\beta_1 \cdots \beta_m.$$

Note that $S_1 \cup S_2'$ is mapped one-to-one *into* $S(m+n \mid k)$, and the elements of

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$S(m+n \mid k)$ which are not images are the k-choices having one of the forms

$$0\,1\,0\,1\,0 \cdots 0\,1\,0\,1\,0\,\gamma_{m+3} \cdots \gamma_{n-1}\,0\,1\,0\,1\,0\,1 \cdots 1\,0\,1$$
$$1\,0\,1\,0 \cdots 1\,0\,1\,0\,\gamma_{m+2} \cdots \gamma_n\,0\,1\,0\,1\,0 \cdots 0\,1\,0.$$

The total number of these excluded k-choices in $S(m+n \mid k)$ is $(n-m \mid k-m)$, and the proof of (6) is complete.

There is an obvious generalization of (6) to any number of circles e.g.,

$$(10, 4, 17 \mid 15) = (10+4+17 \mid 11) + (-1)^{10}(-10+4+17 \mid 15-10)$$
$$+(-1)^4(10-4+17 \mid 15-4) + (-1)^{10+4}(-10-4+17 \mid 15-10-4).$$

3. Two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of $\{1, 2, \ldots, n\}$ are called discordant if $a_i \neq b_i$, $i = 1, \ldots, n$. The Problème des Ménages asks for the number u_n, $n \geq 2$, of permutations discordant with the two permutations

$$\begin{matrix} 1\,2\,3 \cdots n-1 & n \\ n\,1\,2 \cdots n-2 & n-1. \end{matrix}$$

Using $[i, j]$ to denote "the integer i is in the jth place", we seek permutations with none of the properties

$$[1, 1][1, 2][2, 2][2, 3] \cdots [n-1, n-1][n-1, n][n, n][n, 1].$$

Since two of these properties are consistent if and only if they are not adjacent when the $2n$ properties are arranged in a circle (so that $[1, 1]$ follows $[n, 1]$), the Principles of Inclusion and Exclusion yields

$$u_n = \sum_{k=0}^{n} (-1)^k (2n \mid k)(n-k)!, \qquad n \geq 2.$$

This is of course well known [5, p. 14].

Now let $u_{m,n}$, $2 \leq m \leq n$, denote the number of permutations of $1, 2, \ldots, m+n$ discordant with the two permutations

$$\begin{matrix} 1 & 2 & 3 & \cdots & m-1 & m & m+1 & m+2 & \cdots & m+n-1 & m+n \\ m & 1 & 2 & \cdots & m-2 & m-1 & m+n & m+1 & \cdots & m+n-2 & m+n-1. \end{matrix}$$

We seek permutations of degree $m+n$ having none of the properties

$$[1, 1][1, 2][2, 2][2, 3] \cdots [m-1, m-1][m-1, m][m, m][m, 1]$$
$$[m+1, m+1][m+1, m+2][m+2, m+2][m+2, m+3]$$
$$\cdots [m+n-1, m+n][m+n, m+n][m+n, 1].$$

Clearly there are $(2m, 2n \mid k)$ ways of choosing, from these $2(m+n)$ properties, k consistent ones. Hence

$$u_{m,n} = \sum_{k=0}^{m+n} (-1)^k (2m, 2n \mid k)(m+n-k)!$$

(and by (6))

$$=\sum_{k=0}^{m+n}(-1)^k(2m+2n\mid k)(m+n-k)!+$$

$$+\sum_{k=2m}^{m+n}(-1)^k(2n-2m\mid k-2m)(n+m-k)!$$

$$=u_{m+n}+\sum_{i=0}^{n-m}(-1)^i(2n-2m\mid i)(n-m-i)!$$

$$=u_{m+n}+u_{n-m}.$$

This formula also is well known ([1. p. 205], [8], [11, p. 15]), though our proof seems to be the first completely elementary one. It and its obvious generalization was used by Riordan to deduce a particularly elegant formula for the number of 3-line Latin rectangles [1, p. 205].

4. Although we have not been able to obtain a full generalization of (6) when $w\geq 1$ we can prove that

(16) $$(m,n\mid k)_w=(m+n\mid k)_w,\qquad 0\leq k<\frac{m}{w},\frac{n}{w}.$$

Let $S(m,n\mid k)_w$ denote the set of k-choices counted by $(m,n\mid k)_w$. Each such choice can be described by a display

(17) $$\begin{matrix}\alpha_1\,\alpha_2\cdots\alpha_m\\ \beta_1\,\beta_2\cdots\beta_m\,\beta_{m+1}\cdots\beta_n,\end{matrix}\qquad 0\leq m\leq n,$$

of k 1's and $m+n-k$ 0's and each 1 is followed by w 0's where we consider α_m, β_n to be followed by α_1, β_1 respectively.

We consider two cases according to whether or not one of the $m-w+1$ "rectangles"

$$R_i=\frac{\alpha_{i+1}\cdots\alpha_{i+w}}{\beta_{i+1}\cdots\beta_{i+w}},\qquad 0\leq i\leq m-w,$$

has all entries 0. We say such a rectangle is identically zero. If some R_i is identically zero, choose q so that R_q is identically zero while for each $0\leq i<q$ some entry in R_i is 1. Then

(18) $$\alpha_1\cdots\alpha_{q+w}\beta_{q+w+1}\cdots\beta_n\beta_1\cdots\beta_{q+w}\alpha_{q+w+1}\alpha_m$$

is in $S(n+m\mid k)_w$.

If none of the rectangles R_i, $0\leq i\leq m-w$, is identically zero, then none of the disjoint rectangles $R_{\ell w}$, $0\leq\ell\leq[m/w]-1$, is identically zero and therefore each such $R_{\ell w}$ contains at least one 1. Hence $[m/w]=k$ and each rectangle contains exactly one 1. Thus α_j, $\beta_j=0$ if $j>kw$. Because $k<(m/w)$ and no R_i, $i=1,\ldots,m-w$, is identically zero, a simple count shows that one of the w

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rectangles

$$\frac{\alpha_{j+1}\cdots\alpha_m\alpha_1\cdots\alpha_{j-m+w}}{\beta_{j+1}\cdots\beta_m\beta_1\cdots\beta_{j-m+w}}, \qquad m-w\le j<m,$$

is identically zero, and we see that (17) looks like

$$\begin{array}{l} 0\,0\cdots0\,\alpha_{t+1}\alpha_{t+2}\cdots\alpha_{t+m-w}\,0\,0\cdots0\,0 \\ 0\,0\cdots0\,\beta_{t+1}\beta_{t+2}\cdots\beta_{t+m-w}\,0\,0\cdots0\,0\,\beta_{m+1}\cdots\beta_n, \end{array}$$

for some $1\le t<w$. Thus

$$\alpha_1\alpha_2\cdots\alpha_m\beta_1\cdots\beta_n \tag{19}$$

is in $S(m+n\mid k)_w$.

Now we map display (17) to (18) in the first case and we map (17) to (19) in the second case. It is a simple matter to check that this mapping is one-to-one from $S(m,n\mid k)_w$ onto $S(m+n\mid k)_w$, and (16) is proved.

Several examples should make the above correspondence clear. Thus, for $w=3$, $m=10$, $n=12$, $k=3$:

$\{11,15,19\}$ or 0 0 0 0 0 0 0 0 0 0
1 0 0 0 1 0 0 0 1 0 0 0

$\leftrightarrow$ 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 or $\{5,9,13\}$;

$\{2,8,15\}$ or 0 1 0 0 0 0 0 1 0 0
0 0 0 0 1 0 0 0 0 0 0 0

$\leftrightarrow$ 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 or $\{2,8,15\}$;

$\{2,13,16\}$ or 0 1 0 0 0 0 0 0 0 0
0 0 1 0 0 1 0 0 0 0 0 0

$\leftrightarrow$ 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 or $\{2,15,18\}$.

5. For integral $k\ge 0$ we define the polynomials of degree k:

$$\binom{x}{k}=\begin{cases}1, & \text{if } k=0,\\ \dfrac{x(x-1)\cdots(x-k+1)}{k!}, & \text{if } k>0,\end{cases}$$

and

$$A_k(\alpha,\beta)=\frac{\alpha}{\alpha-k\beta}\binom{\alpha-k\beta}{k}.$$

The well known identity of Rothe [9] (see also [3], [4]) states:

$$\sum_{i+j=k} A_i(\alpha,\beta)A_j(\gamma,\beta)=A_k(\alpha+\gamma,\beta).$$

It is an immediate consequence of (16). Indeed if n, k, w are non-negative

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integers, $0 \leq k < (n/w)$, then by (4) $A_k(n, w) = (n \mid k)_w$. Thus the polynomial

$$\sum_{i+j=k} A_i(\alpha, \beta) A_j(\gamma, \beta) - A_k(\alpha+\gamma, \beta)$$

is 0 whenever α, γ, β are positive integers satisfying $1 \leq k < \min(\alpha/\beta+1, \gamma/\beta+1)$, and this surely implies Rothe's identity when $k \geq 1$. For related material in a much more general setting, see [10]; for this and other identities proved by counting lattice paths see [6] and [7].

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Department of Mathematics
Burnside Hall
McGill University
Montreal, Que H3A2K6

McGill University
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