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ON DIAGONAL ACTS OF MONOIDS

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It is proved that the monoid $R_{\mathbb{N}}$ of all partial recursive functions of one variable is finitely generated, and that $R_{\mathbb{N}} \times R_{\mathbb{N}}$ is a cyclic (left and right) $R_{\mathbb{N}}$-act (under the natural diagonal actions $s(a,b) = (sa, sb)$, $(a,b)s = (as, bs)$). We also construct a finitely presented monoid S such that $S \times S$ is a cyclic left and right S-act, and study further interesting properties of diagonal acts and their relationship with power monoids.

1. INTRODUCTION

Let M be a monoid and let X be a set. We say that X is a *right M-act* if there is an action $(x,s) \mapsto xs$ from $X \times M$ into X with the property that $x(st) = (xs)t$ and $x1 = x$ where $x \in X$, $s,t \in M$ are arbitrary and 1 is the identity of M. We define the notion of a *left M-act* analogously. We say X is a *bi M-act* if it is both a right and a left M-act and these actions are linked by

$$s(xt) = (sx)t \quad (s,t \in M,\ x \in X).$$

A right M-act X is generated by a subset $U \subseteq X$ if $UM = X$. Similarly a left M-act X is generated by a subset $U \subseteq X$ if $MU = X$. A bi M-act X is generated by a subset $U \subseteq X$ if $MUM = X$. A (right, left or bi) M-act is *cyclic* if it is generated by a single element.

For any monoid M, the set $M \times M$ can be made into a right, left or bi M-act by defining

$$(x,y)s = (xs, ys), \quad s(x,y) = (sx, sy)$$

for all $x,y,s \in M$; we refer to these acts as the diagonal right, left, and bi M-acts respectively. In this paper we consider the question of finite generation of diagonal acts. If M is infinite, can $M \times M$ ever be finitely generated as a right, left or bi M-act? In the case of an infinite group G, we have that $G \times G$ is never a finitely generated right or left G-act; furthermore, $G \times G$ is a finitely generated bi G-act if and only if G has only

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finitely many conjugacy classes. However, the monoid case is different, and we start with a simple example, due to Bulman-Fleming and McDowell [**1**].

Let $T_{\mathbb{N}}$ be the monoid consisting of all mappings from $\mathbb{N}$ into $\mathbb{N}$ under composition. Now $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is both a cyclic left and a cyclic right $T_{\mathbb{N}}$-act. For, let α and β be mappings from $\mathbb{N}$ into $\mathbb{N}$ defined by $x\alpha = 2x$, and $x\beta = 2x+1$. Then for any $(f,g) \in T_{\mathbb{N}} \times T_{\mathbb{N}}$ we have $(f,g) = (\alpha,\beta)h$ where $h : \mathbb{N} \to \mathbb{N}$ is defined by $(2m)h = mf$, $(2m+1)h = mg$. To show that $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is a cyclic left $T_{\mathbb{N}}$-act, we first choose a bijection $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and let (λ,μ) be its inverse, where λ and μ are mappings from $\mathbb{N}$ to $\mathbb{N}$. Then for any $(f,g) \in T_{\mathbb{N}} \times T_{\mathbb{N}}$ we have $(f,g) = k(\lambda,\mu)$ where $k = (f,g)\psi \in T_{\mathbb{N}}$.

However, we note that $T_{\mathbb{N}}$ is not finitely generated, since it is uncountable. In Section 2 we consider a smaller monoid $R_{\mathbb{N}}$ consisting of all partial recursive functions from $\mathbb{N}$ into $\mathbb{N}$. We show that $R_{\mathbb{N}}$ retains the cyclic diagonal act properties of $T_{\mathbb{N}}$, and also that it is finitely generated (but not finitely presented). Based on this example, in Section 3 we construct further diagonal acts with interesting properties.

In Section 4 we explore connections between diagonal acts and finitary power monoids. Given two subsets A and B of a monoid M, we define their product AB to be the subset $\{ab : a \in A,\ b \in B\}$ of M. With this multiplication, the set of all finite subsets of M becomes a monoid, which we denote by $\mathcal{P}_f(M)$ and call the *power monoid* of M. In particular, we show that $\mathcal{P}_f(R_{\mathbb{N}})$ is finitely generated.

2. Partial recursive functions of one variable

Let $R_{\mathbb{N}}$ be the monoid of all partial recursive functions of one variable under composition. For various facts about the set of all partial recursive functions, see for example [**2**]. In the proof given in Section 1 that $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is both a cyclic right and a cyclic left $T_{\mathbb{N}}$-act, we see that α and β are recursive, while ψ, λ and μ may be chosen to be partial recursive. Furthermore, if f and g are themselves partial recursive functions, then both h and k are, and so the proof for $T_{\mathbb{N}}$ will also work for $R_{\mathbb{N}}$. So we have the following:

Proposition 2.1. *$R_{\mathbb{N}} \times R_{\mathbb{N}}$ is both a cyclic right and a cyclic left $R_{\mathbb{N}}$-act.*

This time however we have:

Theorem 2.2. *The monoid $R_{\mathbb{N}}$ is finitely generated.*

Proof: We use the fact that there exists a universal partial recursive function $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every partial recursive function $f : \mathbb{N} \to \mathbb{N}$ there is some $i \in \mathbb{N}$ such that

$$xf = (i,x)\phi \quad (x \in \mathbb{N}).$$

We let $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any partial recursive bijection, the standard enumeration of $\mathbb{N} \times \mathbb{N}$ will do. Let $\lambda : \mathbb{N} \to \mathbb{N}$ and $\mu : \mathbb{N} \to \mathbb{N}$ be the partial recursive functions such that (λ,μ) is the inverse of ψ. Thus $(x\lambda, x\mu)\psi = x$ and $(x,y)\psi\lambda = x$, $(x,y)\psi\mu = y$. We define $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$x\sigma = \big(x\lambda, (x\lambda, x\mu)\phi\big)\psi.$$

Then σ is also partial recursive. We define π and ρ as partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$ by

$$x\pi = (0, x)\psi$$
$$x\rho = (x\lambda + 1, x\mu)\psi.$$

We claim that $R_{\mathbb{N}} = \langle \pi, \rho, \sigma, \mu \rangle$. Given a function $f \in R_{\mathbb{N}}$ we find i such that $xf = (i, x)\phi$ for each x. We first note that

(1) $$(i, x)\psi\rho = \big((i, x)\psi\lambda + 1, (i, x)\psi\mu\big)\psi = (i + 1, x)\psi,$$

and that

(2) $$(i, x)\psi\sigma = \Big((i, x)\psi\lambda, \big((i, x)\psi\lambda, (i, x)\psi\mu\big)\phi\Big)\psi = \big(i, (i, x)\phi\big)\psi.$$

Then we have

$$\begin{aligned} x\pi\rho^i\sigma\mu &= (0, x)\psi\rho^i\sigma\mu \\ &= (i, x)\psi\sigma\mu \qquad \text{(by (1))} \\ &= \big(i, (i, x)\phi\big)\psi\mu \quad \text{(by (2))} \\ &= (i, x)\phi \\ &= xf, \qquad \text{(by choice of } i) \end{aligned}$$

and so $R_{\mathbb{N}}$ is finitely generated as required. ∎

THEOREM 2.3. *The monoid $R_{\mathbb{N}}$ is not finitely presented.*

PROOF: Suppose that $R_{\mathbb{N}}$ is finitely presented. Then it can be finitely presented in terms of the generators π, ρ, σ, μ, and so we may assume that $R_{\mathbb{N}} = \langle \pi, \rho, \sigma, \mu \mid Q \rangle$ for some finite set of relations Q. Let f be a partial recursive function that is not total, and let $m \in \mathbb{N}$ be such that $xf = (m, x)\phi$ for all x. For simplicity we define ϕ_n to be the function mapping x to $(n, x)\phi$. We let A be the singleton set containing f. By a corollary of the Rice-Shapiro Theorem (see [2, Theorem 2.8 and Corollary 1]) the set $\{n \in \mathbb{N} : \phi_n \in A\}$ is not recursively enumerable: indeed, if it was, then any extension of f would also be in A. Now $\phi_n = \pi\rho^n\sigma\mu$ as in the proof of Theorem 2.2. If $R_{\mathbb{N}}$ were finitely presented then there would exist an algorithmic procedure $\mathcal{P}$ that always answers yes if the two words $\pi\rho^m\sigma\mu$ and $\pi\rho^n\sigma\mu$ are equal in $R_{\mathbb{N}}$, but does not necessarily terminate if they are not equal. Indeed, one may start from $\pi\rho^m\sigma\mu$ and systematically apply defining relations until $\pi\rho^n\sigma\mu$ is obtained, which will happen if and only if the two words are equal. It now follows that the set $\{n \in \mathbb{N} : \phi_n = \phi_m\}$ is recursively enumerable: an algorithm for enumerating it consists of running $\mathcal{P}$ for all input pairs (ϕ_m, ϕ_n) $(n \in \mathbb{N})$ in parallel, and listing those n for which $\mathcal{P}$ terminates. We now conclude that the set $\{n \in \mathbb{N} : \phi_n \in A\}$ is recursively enumerable, which is a contradiction. ∎

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3. Further examples of finitely generated diagonal acts

Theorem 3.1. *There exists an infinite finitely presented monoid P such that $P \times P$ is a cyclic right P-act and a cyclic left P-act.*

Proof: We construct such a finitely presented monoid P which has $R_{\mathbb{N}}$ as a homomorphic image. Let $A = \{p, r, s, m\}$ be an alphabet, the letters p, r, s, m representing the generators π, ρ, σ, μ of $R_{\mathbb{N}}$ respectively, and let $f : A^* \to R_{\mathbb{N}}$ be the corresponding epimorphism. (As usual, A^* denotes the free monoid on A consisting of all words over A including the empty word 1.) Let $a, b \in A^*$ be such that $af = \alpha$, $bf = \beta$, where, as before, α and β are the mappings given by $x\alpha = 2x$ and $x\beta = 2x + 1$. For each $x \in A$ let $u_x, v_x \in A^*$ be such that $(\alpha, \beta)[u_x f] = \big((xf)\alpha, \beta\big)$ and $(\alpha, \beta)[v_x f] = \big(\alpha, (xf)\beta\big)$. Also, let $w \in A^*$ be such that $(\alpha, \beta)[wf] = (1_{\mathbb{N}}, 1_{\mathbb{N}})$, where $1_{\mathbb{N}}$ denotes the identity mapping on $\mathbb{N}$. We now define P_1 to be the monoid defined by the presentation

$$(3) \quad \big\langle\, p, r, s, m \mid au_x = xa, bu_x = b, av_x = a, bv_x = xb \quad \big(x \in \{p, r, s, m\}\big),\ aw = bw = 1 \,\big\rangle.$$

Clearly, P_1 is finitely presented and has $R_{\mathbb{N}}$ as a homomorphic image, so that it is infinite. We now prove that $P_1 \times P_1$ is a cyclic right P_1-act. Indeed, for any $w_1, w_2 \in A^*$, with $w_1 = x_1 x_2 \dots x_k$, $w_2 = y_1 y_2 \dots y_n$ ($x_i, y_j \in A$) we have

$$\begin{aligned} au_{x_1} u_{x_2} \dots u_{x_k} v_{y_1} \dots v_{y_n} w &= x_1 a u_{x_2} \dots u_{x_k} v_{y_1} \dots v_{y_n} w = \dots \\ &= x_1 \dots x_k a v_{y_1} \dots v_{y_n} w = x_1 \dots x_k a v_{y_2} \dots v_{y_n} w = \dots = x_1 \dots x_k a w \\ &= x_1 \dots x_k = w_1 \end{aligned}$$

as a consequence of defining relations, and similarly

$$bu_{x_1} \dots u_{x_k} v_{y_1} \dots v_{y_n} w = w_2.$$

Therefore $P_1 \times P_1$ is generated (as a right P_1-act) by (a, b).

One can now use the same technique and the fact that $R_{\mathbb{N}} \times R_{\mathbb{N}}$ is a cyclic left $R_{\mathbb{N}}$-act to add a further 18 relations to (3), obtaining a monoid P such that $R_{\mathbb{N}}$ is a homomorphic image of P and $P \times P$ is both a cyclic right P-act and a cyclic left P-act. ∎

Our next construction is aimed at demonstrating the independence of properties of $M \times M$ as a right M-act from those of $M \times M$ as a left M-act.

Given a monoid M we construct a new monoid $C(M)$ as follows. Let $M^{(1)}$ and $M^{(2)}$ be disjoint sets in 1-1 correspondence with M, where $s \leftrightarrow s^{(1)} \leftrightarrow s^{(2)}$ ($s \in M$) are bijections, and let $C(M) = M^{(1)} \cup M^{(2)}$. We define multiplication on $C(M)$ as follows:

$$\begin{aligned} s^{(1)} t^{(1)} &= (st)^{(1)}, & s^{(1)} t^{(2)} &= t^{(2)}, \\ s^{(2)} t^{(1)} &= (st)^{(2)}, & s^{(2)} t^{(2)} &= t^{(2)}. \end{aligned}$$

This turns $C(M)$ into a monoid with identity $1^{(1)}$; in [3], $C(M)$ is called the constant extension of M. Now we prove the following facts about $C(M)$.

Theorem 3.2. *Let M be any monoid, and let $C = C(M)$. Then*

(i) *C is finitely generated if and only if M is finitely generated;*

(ii) *$C \times C$ is a finitely generated right C-act if and only if $M \times M$ is a finitely generated right M-act;*

(iii) *if M is infinite, then $C \times C$ is not a finitely generated left C-act.*

Before proving this theorem, we make the following simple observation that we shall use frequently in what follows.

Lemma 3.3. *$M \times M$ is a finitely generated (right, left or bi) M-act if and only if it can be generated by $U \times U$ for some finite subset $U \subseteq M$.*

Proof: If $M \times M$ is generated by a finite set $Y \subseteq M \times M$ then take

$$U = \left\{ s \in M : (s,t) \in Y \text{ or } (t,s) \in Y \text{ for some } t \in M \right\}.$$

The converse is obvious. ∎

Proof of Theorem 3.2: (i) Suppose that $M = \langle X \rangle$, and let $X^{(1)}$ and $X^{(2)}$ be the copies of X in $M^{(1)}$ and $M^{(2)}$ respectively. We show that $C = \left\langle X^{(1)} \cup X^{(2)} \right\rangle$. Indeed, if $t \in C$ with $t = s^{(i)}$, and if $s = x_1 \dots x_n$ ($x_j \in X$) then

$$t = x_1^{(i)} x_2^{(1)} \dots x_n^{(1)}.$$

The converse follows from the fact that $M^{(1)} \cong M$ and that $C \setminus M^{(1)} = M^{(2)}$ is an ideal of C.

(ii) It is easy to check that if $M \times M = (U \times U)M$ then $C \times C = (V \times V)C$ where $V = U^{(1)} \cup U^{(2)}$. Thus if $M \times M$ is finitely generated as a right M-act then $C \times C$ is finitely generated as a right C-act. For the converse, we note that if the C-act $C \times C$ is generated by a set $V \times V$, then the M-act $M \times M$ is generated by $U \times U$ where $U = \left\{ u \in M : u^{(1)} \in V \right\}$.

(iii) Suppose $C \times C = C\left(\left(U^{(1)} \cup U^{(2)}\right) \times \left(U^{(1)} \cup U^{(2)}\right)\right)$ for some finite $U \subseteq M$. Let $q \in M$ be arbitrary. By hypothesis we can write $(q^{(1)}, q^{(2)}) = t^{(i)}(u^{(j)}, v^{(k)})$ for some $t \in M$, $u, v \in U$, $i, j, k \in \{1, 2\}$. From the way the multiplication between the elements of $M^{(1)}$ and $M^{(2)}$ in C is defined, we see that $i = j = 1$, and $k = 2$. But then $q^{(2)} = t^{(1)} v^{(2)} = v^{(2)}$, and so $q = v \in U$. Thus $U = M$ is infinite and therefore $C \times C$ is not a finitely generated left C-act. ∎

Corollary 3.4. *The monoid $C = C(R_{\mathbf{N}})$ is finitely generated. Furthermore, $C \times C$ is finitely generated as a right C-act, but is not finitely generated as a left C-act.*

We now describe another monoid construction. Given a monoid M we construct $D(M)$ to be the direct product of M with its opposite, M'. The elements of M' are in 1-1 correspondence $s \leftrightarrow s'$ ($s \in M$) with M, and multiplication is given by $s't' = (ts)'$. Then M' (and hence $D(M)$) is finitely generated (respectively finitely presented) if and only if M is finitely generated (finitely presented). We now prove the following facts about $D(M)$.

Theorem 3.5. *Let M be any monoid, and let $D = D(M)$.*

(i) *If $D \times D$ is a finitely generated right (or left) D-act then $M \times M$ is both a finitely generated right M-act and a finitely generated left M-act.*

(ii) *If $M \times M$ is a finitely generated right M-act then $D \times D$ is a finitely generated bi D-act.*

Proof: (i) Suppose $D \times D = (U \times U)D$, where U is finite. We may assume that $U = V \times V'$ where V is some finite subset of M, and V' is the corresponding finite subset in M'. We claim that $M \times M$ is finitely generated by the set $V \times V$ both as a right and a left M-act. Given $p, q \in M$, the hypothesis allows us to write

$$\big((p,p'),(q,q')\big) = \big((v_1,v_2'),(v_3,v_4')\big)(r,s')$$

for some $r, s \in M$, $v_i \in V$. Thus

$$(p,p') = (v_1,v_2')(r,s') \tag{4}$$

$$(q,q') = (v_3,v_4')(r,s'). \tag{5}$$

Equating first components in (4) and (5) we see that $p = v_1 r$, $q = v_3 r$, and so $(p,q) = (v_1,v_3)r \in (V \times V)M$. Equating second components in (4) we see that $p' = v_2' s'$ in M', and so $p = s v_2$. Similarly from (5) we obtain $q = s v_4$, and so $(p,q) = s(v_2,v_4) \in M(V \times V)$.

(ii) Suppose that $M \times M = (V \times V)M$, for some finite subset V of M. Then we claim that $D \times D = D(U \times U)D$, where $U = V \times V'$ is finite, and so $D \times D$ is a finitely generated bi D-act. To see this, we take two arbitrary elements (a,b') and (c,d') of D, where $a,b,c,d \in M$. Since $M \times M$ is finitely generated as a right M-act, we can find $s,t \in M$ and $v_1,v_2,v_3,v_4 \in V$ such that $(a,c) = (v_1,v_3)s$ and $(b,d) = (v_2,v_4)t$. Then $b' = t'v_2'$ and $d' = t'v_4'$ and so we have

$$(1,t')\big((v_1,v_2'),(v_3,v_4')\big)(s,1') = \big((a,b'),(c,d')\big)$$

as required. ∎

Corollary 3.6. *The monoid $D = D\big(C(R_{\mathbb{N}})\big)$ is finitely generated. Furthermore, $D \times D$ is a finitely generated bi D-act, but is not finitely generated as either a left or a right D-act.*

4. Power monoids

In this section we investigate links between diagonal acts and power monoids. First however we prove:

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Theorem 4.1. *If $\mathcal{P}_f(M)$ is finitely generated, then M must be finitely generated.*

Proof: Suppose that $\mathcal{P}_f(M)$ is finitely generated with generators the finite sets $A_1, \dots, A_n$. Then M is finitely generated by the set $\bigcup_{i=1}^{n} A_i$. Indeed, given $s \in M$, by hypothesis we have

$$\{s\} = A_{j_1} A_{j_2} \dots A_{j_r}$$

and so $s = a_1 \dots a_r$ for any $a_i \in A_{j_i}$. ∎

Theorem 4.2. *Let M be any monoid. If $\mathcal{P}_f(M)$ is finitely generated then $M \times M$ is a finitely generated bi M-act.*

Proof: Suppose that $\mathcal{P}_f(M)$ is finitely generated by the finite sets $A_1, \dots, A_n$. We shall show that the bi M-act $M \times M$ is generated by the (finite) set $U \times U$ where $U = \bigcup_{i=1}^{n} A_i$. Let $p, q \in M$ be arbitrary, and write

$$\{p, q\} = A_{j_1} A_{j_2} \dots A_{j_r}.$$

In particular, we have $p = x_1 x_2 \dots x_r$, $q = y_1 y_2 \dots y_r$ for some $x_i, y_i \in A_{j_i}$. Thus we have

$$\{p, q\} = B_1 B_2 \dots B_r \tag{6}$$

where $B_i = \{x_i, y_i\} \subseteq A_{j_i} \subseteq U$ has at most two elements. Clearly, there must exist at least one set, B_m say, with precisely two elements. Consider the sets

$$X = B_1 \dots B_{m-1}\{x_m\}B_{m+1} \dots B_r$$
$$Y = B_1 \dots B_{m-1}\{y_m\}B_{m+1} \dots B_r.$$

If $|X| = 2$ then B_m can be replaced by just $\{x_m\}$ with (6) remaining valid. Similarly if $|Y| = 2$ then B_m can be replaced by $\{y_m\}$. If $|X| = |Y| = 1$, then all B_i with $i \neq m$ can be replaced by one element sets $\{x_i\}$. Repeating this, if necessary, we obtain

$$\{p, q\} = \{z_1\} \dots \{z_{k-1}\}\{x_k, y_k\}\{z_{k+1}\} \dots \{z_r\}$$

for some k $(1 \leqslant k \leqslant r)$ and some $z_i \in B_i \subseteq U$. Thus we have either

$$(p, q) = z_1 \dots z_{k-1}(x_k, y_k) z_{k+1} \dots z_r \in M(U \times U)M$$

or

$$(p, q) = z_1 \dots z_{k-1}(y_k, x_k) z_{k+1} \dots z_r \in M(U \times U)M,$$

completing the proof. ∎

Theorem 4.3. *Let M be any finitely generated monoid such that $M \times M$ is a cyclic right (or left) M-act. Then $\mathcal{P}_f(M)$ is finitely generated.*

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PROOF: Suppose $M = \langle X\rangle$, and that $M \times M = (a,b)M$. We shall prove that $\mathcal{P}_f(M)$ is generated by the set $Y = \{\{a,b\}\} \cup \{\{x\} : x \in X\}$. Suppose $P \in \mathcal{P}_f(M)$. By induction on $|P|$ we prove that P can be written as a product of sets from Y. If $|P| = 1$ then P is easily seen to be a product of singleton sets. Suppose $P = \{p_1, \ldots, p_{n+1}\}$, and that all sets with at most n elements can be generated by Y. Since $M \times M = (a,b)M$ we may choose elements $q_1, \ldots, q_n$ such that $(a,b)q_i = (p_i, p_i)$ for $1 \leqslant i \leqslant n-1$ and $(a,b)q_n = (p_n, p_{n+1})$. Then we have

$$\{p_1, \ldots, p_{n+1}\} = \{a,b\}\{q_1, \ldots, q_n\}$$

and our proof by induction is completed. ∎

COROLLARY 4.4. *$\mathcal{P}_f(R_{\mathbf{N}})$ is finitely generated.*

We might hope that the converse to Theorem 4.3 held, that is, that if $\mathcal{P}_f(M)$ is finitely generated then $M \times M$ is a cyclic right or left M-act. In fact this is not the case, as our next example shows.

PROPOSITION 4.5. *Let $C = C(R_{\mathbf{N}})$ as in Corollary 3.4. Then $\mathcal{P}_f(C)$ is finitely generated, but $C \times C$ is not a cyclic right C-act, and is not even finitely generated as a left C-act.*

PROOF: Corollary 3.4 gives that $C \times C$ is not finitely generated as a left C-act, but is finitely generated as a right C-act. Suppose $C \times C$ were cyclic as a right C-act, that is, that $C \times C = (a^{(i)}, b^{(j)})C$ for $i, j \in \{1,2\}$, $a, b \in R_{\mathbf{N}}$. Then for generating $(p^{(1)}, q^{(1)})$ to be possible we would need $i = j = 1$, which would make generating $(p^{(1)}, q^{(2)})$ impossible. Thus $C \times C$ is not a cyclic right C-act.

We saw in Proposition 2.1 that $R_{\mathbf{N}} \times R_{\mathbf{N}}$ is a cyclic right $R_{\mathbf{N}}$-act, with generator (α, β), and also a cyclic left $R_{\mathbf{N}}$-act, with generator (λ, μ). By Theorem 2.2 $R_{\mathbf{N}}$ is finitely generated, by the set X say. We shall show that the finite set

$$\{\{\xi^{(i)}\} : \xi \in X, i \in \{1,2\}\} \cup \{\{\lambda^{(1)}, \mu^{(1)}\}, \{\alpha^{(2)}, \beta^{(1)}\}\}$$

generates $\mathcal{P}_f(C)$. Clearly we can generate all singleton sets – if $f = \xi_1 \ldots \xi_r \in R_{\mathbf{N}}$ then $\{f^{(1)}\} = \{\xi_1^{(1)}\} \ldots \{\xi_r^{(1)}\}$ and $\{f^{(2)}\} = \{\xi_1^{(2)}\}\{\xi_2^{(1)}\} \ldots \{\xi_r^{(1)}\}$. Since $R_{\mathbf{N}} \times R_{\mathbf{N}}$ is a cyclic left $R_{\mathbf{N}}$-act we may prove, as in the proof of Theorem 4.3, that any subset P of $R_{\mathbf{N}}$ of n elements may be written as $P = \{f\}\{\lambda, \mu\}^{n-1}$ where $f \in R_{\mathbf{N}}$. Also, if $P = \{p_1, \ldots, p_n\}$ then we may let $Q = \{q_1, \ldots, q_n\}$ where q_i is chosen so that $q_i(\lambda, \mu) = (p_i, p_i)$, and we see that $P = Q\{\lambda, \mu\}$. So we may actually write an n element subset as $\{f'\}\{\lambda, \mu\}^m$ for any $m \geqslant n-1$.

Now given a finite subset Z of C we may write $Z = U \cup V$ where $U \subseteq R_{\mathbf{N}}^{(1)}$ and $V \subseteq R_{\mathbf{N}}^{(2)}$ are both finite. By the above, we may write

$$U = \{f^{(1)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m$$
$$V = \{g^{(2)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m$$

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for some $f, g \in R_{\mathbb{N}}$, $m \in \mathbb{N}$. Then it is easy to see that

$$Z = \{f^{(1)}\}\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m.$$

Thus all that remains to check is that we may generate the sets $\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\}$ for $g \in R_{\mathbb{N}}$. Choosing $h \in R_{\mathbb{N}}$ such that $(\alpha, \beta)h = (g, 1_{\mathbb{N}})$ we see that $\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\} = \{\alpha^{(2)}, \beta^{(1)}\}\{h^{(1)}\}$. Thus $\mathcal{P}_f(C)$ is finitely generated as required. □

Various questions regarding the relationship between diagonal acts and power monoids remain unanswered. For example, does the following generalisation of Theorem 4.3 hold: if $M \times M$ is a finitely generated left or right M-act (or perhaps even a cyclic bi M-act) then $\mathcal{P}_f(M)$ is finitely generated? Theorem 4.2 tells us that if $P_f(M)$ is finitely generated then $M \times M$ must be finitely generated as a bi M-act, but does there exist a monoid M such that $P_f(M)$ is finitely generated, but $M \times M$ is not finitely generated as a left or right M-act? One thing we have not investigated here at all is the question of finite presentability of power monoids. If M is infinite, can $\mathcal{P}_f(M)$ ever be finitely presented?

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