SPLITTING OF ALGEBRAS BY FUNCTION FIELDS OF ONE VARIABLE

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To the memory of TADASI NAKAYAMA

§ 1. Introduction

Let K be a field and $\mathfrak{B}(K)$ the Brauer group of K. It consists of the similarity classes of finite central simple algebras over K.[1] For any field extension F/K there is a natural mapping $\mathfrak{B}(K) \rightarrow \mathfrak{B}(F)$ which is obtained by assigning to each central simple algebra A/K the tensor product $A \underset{K}{\otimes} F$ which is a central simple algebra over F. The kernel of this map is the relative Brauer group $\mathfrak{B}(F/K)$, consisting of those $A \in \mathfrak{B}(K)$ which are split by F.

If F/K is finite algebraic, the investigation of $\mathfrak{B}(F/K)$ is part of the general theory of central simple algebras. In particular, if the ground field K is a number field or a local number field,[2] the relative Brauer group $\mathfrak{B}(F/K)$ can then explicitly be determined, using class field theory.

In this paper, we propose to investigate $\mathfrak{B}(F/K)$ in the case where F/K is a function field of one variable.[3] Our results will give a complete description of $\mathfrak{B}(F/K)$ if K is a local number field.

For any transcendental field extension F/K, let $\mathfrak{p}$ be a place of F/K such that the image field $F\mathfrak{p}$ is algebraic over K.[4] Any central simple algebra A/K which is split by F is also split by $F\mathfrak{p}$, as we have shown in an earlier

Received July 19, 1965.

1) For the general theory of central simple algebras see e.g. Deuring [6], chap. IV and V, or Artin-Nesbitt-Thrall [2], chap. V–VIII.

2) A field K is called a number field if it is a finite-dimensional extension of the field $\mathbf{Q}$ of rational numbers. A field K is called a local number field if it is the completion of a number field k with respect to a non-trivial valuation of K. This is the case if and only if K is either the field of real numbers or the field of complex numbers (archimedean case), or if K is a finite-dimensional extension of the rational p-adic field $\mathbf{Q}_p$, for some prime number p.

3) A field extension F/K is called a function field of one variable, if F/K is finitely generated, K is algebraically closed in F, and the degree of transcendency of F/K is 1.

4) For the general theory of places see e.g. Zariski-Samuel [15], vol. II, chap. VI.

https://doi.org/10.1017/S0027763000026441 Published online by Cambridge University Press

paper.[5] That is, we have $\mathfrak{B}(F/K) \subset \mathfrak{B}(F\mathfrak{p}/K)$. Let us put

$$\tilde{\mathfrak{B}}(F/K) = \bigcap_{\mathfrak{p}} \mathfrak{B}(F\mathfrak{p}/K),$$

$\mathfrak{p}$ ranging over the places of F/K such that $F\mathfrak{p}/K$ is algebraic. We then have

$$\mathfrak{B}(F/K) \subset \tilde{\mathfrak{B}}(F/K).$$

If F/K is a separable function field of one variable, we shall show in § 3 that the factor group $\dfrac{\tilde{\mathfrak{B}}(F/K)}{\mathfrak{B}(F/K)}$ can be described by a certain cohomological invariant $X(F/K)$ which is connected with the one-dimensional Galois cohomology of the idèle class group.

The interpretation of this result is as follows: As we have said above, the investigation of $\mathfrak{B}(L/K)$ for an algebraic field extension L of K is part of the classical theory of central simple algebras. Hence we may regard $\tilde{\mathfrak{B}}(F/K)$, which concerns only Brauer groups of algebraic extensions $F\mathfrak{p}/K$, as essentially known, in particular if the ground field is a number field or a local number field. Hence the invariant $X(F/K)$, which is explicitely defined in § 3, will describe the deviation of the group $\mathfrak{B}(F/K)$ from the (known) group $\tilde{\mathfrak{B}}(F/K)$.

In the special case where K is a local number field we shall see that $X(F/K) = 1$. In the archimedean case, this will follow from the results of Witt [12] while in the non-archimedean case we shall refer to the corresponding results of Tate [11]. This then shows that $\mathfrak{B}(F/K) = \tilde{\mathfrak{B}}(F/K)$. On the other hand, the known structure of $\mathfrak{B}(K)$ for local number fields permits to determine $\tilde{\mathfrak{B}}(F/K)$ explicitly. We then will obtain the following result which constitutes the main result of this paper:

THEOREM 1. *Let F/K be a function field of one variable over a local number field K. Let $d(F/K)$ be the smallest positive integer which is a degree of a divisor of F/K.*

Then:

The group $\mathfrak{B}(F/K)$ is cyclic of order $d(F/K)$; it consists of all $A \in \mathfrak{B}(K)$ whose Schur index divides $d(F/K)$.

As to global number fields K as ground fields, we shall show by examples

[5] [9], page 428, prop. 8.

that the equality $\mathfrak{B}(F/K) = \tilde{\mathfrak{B}}(F/K)$ is not true in general. This case has still to be investigated.

§ 2. The cohomological language

Let $\bar{K}/K$ be a finite Galois extension of K, with Galois group $G = G(\bar{K}/K)$. As shown in the theory of crossed product algebras, we have

$$\mathfrak{B}(\bar{K}/K) = H^2(G, \bar{K}^\times)^{6)} \tag{1}$$

where $\bar{K}^\times$ denotes the multiplicative group of the field K. Here, $H^2(G, \bar{K}^\times)$ denotes the second cohomology group of G in the multiplicative group of $\bar{K}$.[7]

If L/K is any field extension and $\bar{L} = L\bar{K}$ a field compositum of L with $\bar{K}$, then the subgroup

$$G_L = G(\bar{K}/L \cap \bar{K})$$

of G can be regarded as the Galois group of $\bar{L}/L$. The restriction from G to G_L together with the inclusion map $\bar{K}^\times \subset \bar{L}^\times$ gives a cohomology map

$$H^2(G, \bar{K}^\times) \to H^2(G_L, \bar{L}^\times). \tag{2}$$

On the other hand, the map $\mathfrak{B}(K) \to \mathfrak{B}(L)$ described in § 1 induces a map

$$\mathfrak{B}(\bar{K}/K) \to \mathfrak{B}(\bar{L}/L). \tag{3}$$

In the theory of crossed products it is shown that the two maps (2) and (3) coincide after the identification (1) and the corresponding identification for $\bar{L}/L$.[8] The kernel of (3) consists of those algebras over K, which are split by $\bar{K}$ and L. That is, this kernel is $\mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(L/K)$.
Hence:

$$\mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(L/K) = \text{kernel } H^2(G, \bar{K}^\times) \to H^2(G_L, \bar{L}^\times). \tag{4}$$

If $L = F$ is a separable function field of one variable, then F has a separable place $\mathfrak{p}$; hence we may choose $\bar{K}$ so as to contain $F\mathfrak{p}$. As said in § 1, $\mathfrak{B}(F/K) \subset \mathfrak{B}(F\mathfrak{p}/K) \subset \mathfrak{B}(\bar{K}/K)$. On the other hand, F is linearly disjoint to $\bar{K}$ over K and hence $G_F = G$ can be regarded as the Galois group of $\bar{F} = F \cdot \bar{K}$ over F.

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6) See e.g. the books mentioned in 1). For another approach see Serre [10], chap. X, § 5-6.

7) For the general cohomology theory we refer to [10] chap. VII, or Cartan-Eilenberg [3], or Artin [1].

8) Deuring [6], page 61, Satz 1.

Hence:

PROPOSITION 1. *Let F/K be a separable function field of one variable. Then there exists a finite Galois extension $\overline{K}/K$ such that $\mathfrak{B}(F/K)\subset\mathfrak{B}(\overline{K}/K)$. If this is so, then*

$$\mathfrak{B}(F/K)=\text{kernel } H^2(G,\ \overline{K}^{\times})\to H^2(G,\ \overline{F}^{\times}).$$

Next we shall give a cohomological interpretation of $\mathfrak{B}(F/K)$.

Let $\mathfrak{p}$ be a place of F/K and $F\mathfrak{p}$ its image field. Let $F\mathfrak{p}\cdot\overline{K}$ be a field compositum of $F\mathfrak{p}$ and $\overline{K}$ over K and denote by $G_{\mathfrak{p}}$ the group of $\overline{K}$ over $F\mathfrak{p}\cap\overline{K}$. From (4) we obtain:

$$\mathfrak{B}(\overline{K}/K)\cap\mathfrak{B}(F\mathfrak{p}/\overline{K})=\text{kernel } H^2(G,\ \overline{K}^{\times})\to H^2(G_{\mathfrak{p}},\ (F\mathfrak{p}\cdot\overline{K})^{\times}). \tag{5}$$

Let now $\mathfrak{p}$ range over all places of F/K and

$$H^2(G,\ \overline{K}^{\times})\to\prod_{\mathfrak{p}} H^2(G_{\mathfrak{p}},\ (F\mathfrak{p}\cdot\overline{K})^{\times})$$

be the map which in each component of the direct product induces the map mentioned in (5). Its kernel is the intersection of the kernels in (5). Hence

$$\mathfrak{B}(\overline{K}/K)\cap\mathfrak{B}(F/K)=\text{kernel } H^2(G,\overline{K}^{\times})\to\prod_{\mathfrak{p}}H^2(G_{\mathfrak{p}},\ (F\mathfrak{p}\cdot\overline{K})^{\times}) \tag{6}$$

If we choose $\overline{K}$ such that it contains $F\mathfrak{p}$ for some $\mathfrak{p}$, which is possible if F/K is separable, then $\mathfrak{B}(F/K)$ is contained in $\mathfrak{B}(\overline{K}/K)$ and hence we may replace the intersection on the left hand side of (6) by $\mathfrak{B}(F/K)$.

On the right hand side of (6), the image group is a direct product of cohomology groups with respect to various subgroups $G_{\mathfrak{p}}$ of G. However, this group can be interpreted as a cohomology group of G in a certain group $\overline{W}=\overline{W}(\overline{F}/\overline{K})$, as follows.

For a given prime $\mathfrak{p}$, the field compositum $F\mathfrak{p}\cdot\overline{K}$ is in general not uniquely determined. There may be several inequivalent field composita of $F\mathfrak{p}$ with $\overline{K}$ over K. Let $\overline{\mathfrak{p}}$ range over the primes of $\overline{F}/\overline{K}$ which lie above $\mathfrak{p}$ (we then write $\overline{\mathfrak{p}}|\mathfrak{p}$.). It is well known that the inequivalent field composita of $F\mathfrak{p}$ with $\overline{K}$ correspond $1-1$ to the $\overline{\mathfrak{p}}|\mathfrak{p}$. For any $\overline{\mathfrak{p}}|\mathfrak{p}$, the image field $\overline{F}\overline{\mathfrak{p}}$ contains $F\overline{\mathfrak{p}}$, which is K-isomorphic to $F\mathfrak{p}$ under the map

$$a\mathfrak{p}\to a\overline{\mathfrak{p}}\qquad(a\in F).$$

We have

https://doi.org/10.1017/S0027763000026441 Published online by Cambridge University Press

$$\bar{F}\bar{\mathfrak{p}} = F\bar{\mathfrak{p}} \cdot \bar{K},$$

and this is the field compositum belonging to $\bar{\mathfrak{p}}$.[9]

We now form the direct product $\prod_{\bar{\mathfrak{p}}|\mathfrak{p}} \bar{F}\bar{\mathfrak{p}}$. Since the $\bar{F}\bar{\mathfrak{p}}$ are all the inequivalent field composita of $F\mathfrak{p}$ and $\bar{K}$, we have a natural isomorphism

$$F\mathfrak{p} \otimes_K \bar{K} = \prod_{\bar{\mathfrak{p}}|\mathfrak{p}} \bar{F}\bar{\mathfrak{p}} \tag{7}$$

which is obtained by mapping $\bar{K}$ diagonally into $\prod_{\bar{\mathfrak{p}}|\mathfrak{p}} \bar{F}\bar{\mathfrak{p}}$ ($\bar{K}$ is contained in each $\bar{F}\bar{\mathfrak{p}}$) and by mapping

$$a\mathfrak{p} \to \prod_{\bar{\mathfrak{p}}|\mathfrak{p}} a\bar{\mathfrak{p}} \qquad (a \in F).$$

The Galois group G acts naturally on $F\mathfrak{p} \otimes \bar{K}$ (on the right factor) and hence on the direct product on the right hand side of (7), thereby permuting the factors $\bar{F}\bar{\mathfrak{p}}$ transitively. If $\bar{\mathfrak{p}} | \mathfrak{p}$ is fixed and $G_{\bar{\mathfrak{p}}}$ denotes the subgroup of G leaving the elements of $\bar{F}\bar{\mathfrak{p}}$ fixed, then we may write

$$F\mathfrak{p} \otimes_K \bar{K} = \prod_{\sigma \in G \bmod G_{\bar{\mathfrak{p}}}} (\bar{F}\bar{\mathfrak{p}})^{\sigma}. \tag{8}$$

Let $\bar{W}_{\mathfrak{p}}$ be the group of units of the algebra $F\mathfrak{p} \otimes \bar{K}$. We obtain

$$\bar{W}_{\mathfrak{p}} = \prod_{\sigma \bmod G_{\bar{\mathfrak{p}}}} (\bar{F}\bar{\mathfrak{p}})^{\times\sigma}. \tag{9}$$

Shapiros lemma from cohomology theory[10] now shows that

$$H^i(G, \bar{W}_{\mathfrak{p}}) = H^i(G_{\bar{\mathfrak{p}}}, (\bar{F}\bar{\mathfrak{p}})^{\times}) \qquad (i \geq 0). \tag{10}$$

This isomorphism is obtained by the restriction of G to the subgroup $G_{\bar{\mathfrak{p}}}$, followed by the projection $\bar{W}_{\mathfrak{p}} \to (\bar{F}\bar{\mathfrak{p}})^{\times}$.

Observe that on the right hand side in (10) we have one fixed compositum $\bar{F}\bar{\mathfrak{p}} = F\bar{\mathfrak{p}} \cdot \bar{K}$ of $F\mathfrak{p}$ and $\bar{K}$. This may take the place of what we have denoted by $F\mathfrak{p} \cdot \bar{K}$ in (5). The diagonal imbedding $\bar{K}^{\times} \to \bar{W}_{\mathfrak{p}}$ followed by the projection $\bar{W}_{\mathfrak{p}} \to (\bar{F}\bar{\mathfrak{p}})^{\times}$ is precisely the natural injection $\bar{K}^{\times} \to (\bar{F}\bar{\mathfrak{p}})^{\times} = (F\bar{\mathfrak{p}} \cdot \bar{K})^{\times}$. Hence we obtain from (5) and (10) (for $i = 2$) that

$$\mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(F\mathfrak{p}/K) = \text{kernel } H^2(G, \bar{K}^{\times}) \to H^2(G, \bar{W}_{\mathfrak{p}}). \tag{11}$$

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[9] Chevalley [5], page 92, theorem 3.

[10] [10], page 125, exercice.

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Now let us put all places $\mathfrak{p}$ of F/K together:

$$\overline{W} = \prod_{\mathfrak{p}} \overline{W}_{\mathfrak{p}} = \prod_{\bar{\mathfrak{p}}} (\overline{F}_{\bar{\mathfrak{p}}})^{\times}$$

G acts on $\overline{W}$ componentwise on each $\overline{W}_{\mathfrak{p}}$. We have

(12) $$H^i(G, \overline{W}) = \prod_{\mathfrak{p}} H^i(G, \overline{W}_{\mathfrak{p}}) \qquad (i \geqq 0)$$

and we obtain:

Proposition 2. *Let F/K be a separable function field of one variable. Then there is a finite Galois extension $\overline{K}/K$ such that $\mathfrak{B}(F/K) \subset \mathfrak{B}(\overline{K}/K)$. If this is so, we have*

$$\mathfrak{B}(F/K) = \text{kernel } H^2(G, \overline{K}^{\times}) \to H^2(G, \overline{W}),$$

where

$$\overline{W} = \prod_{\bar{\mathfrak{p}}} (\overline{F}_{\bar{\mathfrak{p}}})^{\times}$$

($\bar{\mathfrak{p}}$ ranging over the places of $\overline{F}/\overline{K}$), and G acts on $\overline{W}$ naturally as described above.

§ 3. The kernel theorem

Let F/K be a function field of one variable and $\overline{K}/K$ a finite Galois extension with group G. According to propositions 1 and 2, we shall study in this § 3 the maps

$$H^2(G, \overline{K}^{\times}) \to H^2(G, \overline{F}^{\times})$$

and

$$H^2(G, \overline{K}^{\times}) \to H^2(G, \overline{W})$$

described in § 2 and we shall compare their kernels.

We introduce the following notations:

D the divisor group of F/K

H the group of principal divisors in D

CD $= D/H$ the divisor class group

J the idèle group of F/K

CJ $= J/F^{\times}$ the group of idèle classes

U the group of idèle units in J

CU $= UF^{\times}/F^{\times} = U/K^{\times}$ the idèle unit classes.

As to the definitions, D is defined to be the free abelian multiplicative group generated by the prime divisors (places) $\mathfrak{p}$ of F/K. Hence every divisor $\mathfrak{a} \in D$ is a product

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{a(\mathfrak{p})}$$

with uniquely determined integers $a(\mathfrak{p})$ such that $a(\mathfrak{p}) = 0$ for all but a finite number of $\mathfrak{p}$.

Let $w_{\mathfrak{p}}$ be the additive normalized valuation of F belonging to $\mathfrak{p}$. The principal divisor for $a \in F^{\times}$ is

$$(a) = \prod_{\mathfrak{p}} \mathfrak{p}^{w_{\mathfrak{p}}(a)}.$$

H is defined to be the image of the map $a \to (a)$ from $F^{\times}$ into D. The kernel of this map is $K^{\times}$, so that the sequence

$$1 \to K^{\times} \to F^{\times} \to H \to 1$$

is exact.

J is defined to consist of all functions $\mathfrak{p} \to \alpha(\mathfrak{p})$, defined on the primes $\mathfrak{p}$ of F/K, with values $\alpha(\mathfrak{p})$ in the multiplicative group of $F_{\mathfrak{p}}$, the $\mathfrak{p}$-adic completion of F with respect to $\mathfrak{p}$.

These functions α have to satisfy the finiteness condition that $w_{\mathfrak{p}}(\alpha(\mathfrak{p})) \neq 0$ for all but a finite number of $\mathfrak{p}$.[11] There is a mapping $J \to D$ obtained by assigning to each $\alpha \in J$ its divisor $(\alpha) = \prod_{\mathfrak{p}} \mathfrak{p}^{w_{\mathfrak{p}}(\alpha(\mathfrak{p}))}$.

This mapping is epimorphic; its kernel is called U, so that the sequence

$$1 \to U \to J \to D \to 1$$

is exact.

There is a mapping $F^{\times} \to J$ obtained by assigning to each $a \in F^{\times}$ the idèle α_a given by $\alpha_a(\mathfrak{p}) = a$, for all $\mathfrak{p}$ (diagonal imbedding). This mapping is monomorphic and we identify $F^{\times}$ with its image in J. This identification is coherent with the mappings $F^{\times} \to D$ and $J \to D$, i.e. we have $(a) = (\alpha_a)$. In other words, the diagram

$$\begin{array}{ccc} F^{\times} & \longrightarrow & J \\ \downarrow & & \downarrow \\ H & \longrightarrow & D \end{array}$$

is commutative.

11) By continuity, the valuation $w_{\mathfrak{p}}$ of F extends uniquely to a valuation of the completion $F_{\mathfrak{p}}$, and this extension is again denoted by $w_{\mathfrak{p}}$.

From the above definitions and discussions it follows that the diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & K^\times & \to & U & \to & CU & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & F^\times & \to & J & \to & CJ & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H & \to & D & \to & CD & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

in which the arrows denote the natural maps in question, is commutative with exact rows and columns.

For the function field $\bar{F}/\bar{K}$ we have a similar diagram whose corresponding groups will be denoted by $\bar{D}$, $\bar{H}$, $\overline{CD}$, $\bar{J}$, etc:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \bar{K}^\times & \to & \bar{U} & \to & \overline{CU} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \bar{F}^\times & \to & \bar{J} & \to & \overline{CJ} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \bar{H} & \to & \bar{D} & \to & \overline{CD} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

As said in § 2, the Galois group $G = G(\bar{K}/K)$ can be regarded as the Galois group of $\bar{F}/F$, since F and $\bar{K}$ are linearly disjoint over K. Hence G acts on $\bar{F}^\times$. Also, G acts on all the other groups of our diagram, as follows.

G acts on the primes $\bar{\mathfrak{p}}$ of $\bar{F}$: If $w_{\bar{\mathfrak{p}}}$ is the additive normalized valuation of $\bar{F}$ belonging to $\bar{\mathfrak{p}}$ then $\bar{\mathfrak{p}}^\sigma$ is defined by

$$w_{\bar{\mathfrak{p}}^\sigma}(a^\sigma) = w_{\bar{\mathfrak{p}}}(a) \qquad (a \in F,\ \sigma \in G).$$

The map $\sigma : \bar{F} \to \bar{F}$ is continuous if $\bar{F}$ as the domain of this map is topologized by $w_{\bar{\mathfrak{p}}}$, and it is topologized by $w_{\bar{\mathfrak{p}}^\sigma}$ if considered as the range of σ. Hence σ extends, by continuity, uniquely to a map $\sigma : \bar{F}_{\bar{\mathfrak{p}}} \to \bar{F}_{\bar{\mathfrak{p}}^\sigma}$ of the corresponding completions. According to these maps, G acts on $\bar{J}$, namely:

$$\alpha^\sigma(\bar{\mathfrak{p}}^\sigma) = \alpha(\bar{\mathfrak{p}})^\sigma \qquad (\alpha \in \bar{J},\ \sigma \in G).$$

By definition, it is clear that the maps

$$\bar{F}^\times \to \bar{J}$$

(diagonal imbedding) and

$$\overline{J} \to \overline{D}$$

(divisor map) are G-permissible. Hence all the other maps of our diagram, being based on the two maps mentioned above, are G-permissible, G acting on the groups of the diagram in the natural way. In other words: our diagram is G-permissible.

In particular, for each group $\overline{M}$ of our diagram we can form the cohomology groups $H^i(G, \overline{M})$, and for each exact row or column $1 \to \overline{M}_1 \to \overline{M}_2 \to \overline{M}_3 \to 1$ of our diagram we obtain a cohomological connecting map $H^i(G, \overline{M}_3) \to H^{i+1}(G, \overline{M}_1)$.

From the lower horizontal sequence of the diagram we thus obtain a cohomology map

$$H^i(\overline{CD}) \to H^{i+1}(\overline{H}).$$

From the left vertical sequence we obtain also

$$H^{i+1}(\overline{H}) \to H^{i+2}(\overline{K}^{\times})$$

which combined with the map above yields a map

$$h^i : H^i(\overline{CD}) \to H^{i+2}(\overline{K}^{\times}).$$

Similarly, using first the right vertical sequence and then the upper horizontal sequence of the diagram we obtain another map

$$g^i : H^i(\overline{CD}) \to H^{i+2}(\overline{K}^{\times}).$$

It is known from general cohomology theory[12] *that both maps h^i and g^i differ only by a sign; in particular, both maps have the same kernel and the same image.*

Let us investigate these maps in the case $i = 0$.

Investigation of h^0:

By definition, h^0 is obtained by considering the left lower corner of the diagram, namely:

[12] Cartan-Eilenberg [3], page 56, prop. 2.1.

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$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ & & \overline{K}^{\times} & & & & & & \\ & & \downarrow & & & & & & \\ & & \overline{F}^{\times} & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \rightarrow & \overline{H} & \rightarrow & \overline{D} & \rightarrow & \overline{CD} & \rightarrow & 1 \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

This portion of our diagram gives the two maps

$$H^0(\overline{CD}) \rightarrow H^1(\overline{H})$$

and

$$H^1(\overline{H}) \rightarrow H^2(\overline{K}^{\times})$$

the composite of which is h^0.

We begin by observing that

$$H^1(\overline{F}^{\times}) = 1 \tag{13}$$

and

$$H^1(\overline{D}) = 1. \tag{14}$$

The first of these formulae is well known as the celebrated 'Hilbert theorem 90'. The second follows from the fact that $\overline{D}$ is the free abelian group generated by the primes $\bar{\mathfrak{p}}$ of $\overline{F}/\overline{K}$ which are only permuted under G.[13]

From (13) it follows, using the exactness of the column of our diagram portion, that

$$H^1(\overline{H}) \rightarrow H^2(\overline{K}^{\times}) \text{ is monomorphic.}$$

Similarly, from (14) it follows that

$$H^0(\overline{CD}) \rightarrow H^1(\overline{H}) \text{ is epimorphic.}$$

Putting both statements together we obtain

$$\begin{aligned} \text{image } (h_0) &= \text{image } H^1(\overline{H}) \rightarrow H^2(\overline{K}^{\times}) \\ &= H^1(\overline{H}). \end{aligned} \tag{15}$$

On the other hand,

$$\text{image } H^1(\overline{H}) \rightarrow H^2(\overline{K}^{\times}) = \text{kernel } H^2(\overline{K}^{\times}) \rightarrow H^2(\overline{F}^{\times})$$

13) See e.g. [9], page 437,

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so that we finally obtain

$$\text{image } (h^0) = \text{kernel } H^2(\overline{K}^\times) \to H^2(\overline{F}^\times). \tag{16}$$

Investigation of g^0:

Now we have to consider the right upper corner of our diagram:

$$\begin{array}{ccccccccc} & & & & 1 & & \\ & & & & \downarrow & & \\ 1 & \to & \overline{K}^\times & \to \overline{U} \to & \overline{CU} & \to & 1 \\ & & & & \downarrow & & \\ & & & & \overline{CJ} & & \\ & & & & \downarrow & & \\ & & & & \overline{CD} & & \\ & & & & \downarrow & & \\ & & & & 1 & & \end{array}$$

g^0 is the composite of the two maps

$$H^0(\overline{CD}) \to H^1(\overline{CU})$$

and

$$H^1(\overline{CU}) \to H^2(\overline{K}^\times).$$

First we have, in analogy to (13), the formula

$$H^1(\overline{U}) = 1. \tag{17}$$

Proof. Let $\overline{U}_{\bar{\mathfrak{p}}}$ be the group of $\bar{\mathfrak{p}}$-adic units in the $\bar{\mathfrak{p}}$-adic completion $\overline{F}_{\bar{\mathfrak{p}}}$ of $\overline{F}$. By definition, $\overline{U}$ is the direct product

$$\overline{U} = \prod_{\bar{\mathfrak{p}}} \overline{U}_{\bar{\mathfrak{p}}}.$$

Each place $\bar{\mathfrak{p}}$ induces an epimorphic map

$$\overline{U}_{\bar{\mathfrak{p}}} \to (\overline{F}_{\bar{\mathfrak{p}}})^\times.$$

These maps define an epimorphic map

$$\overline{U} \to \overline{W}$$

where $\overline{W}$ *is the direct product of the* $(\overline{F}_{\bar{\mathfrak{p}}})^\times$ *as in* §2. *By comparing the* definitions of the actions of G on $\overline{U}$ (as part of J) and on $\overline{W}$ (see §2) we see that this map is G-permissible.

Let $\overline{V}$ be the kernel, so that

$$1 \to \overline{V} \to \overline{U} \to \overline{W} \to 1$$

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is exact. We shall show in a moment that

(18) $$H^i(G, \overline{V}) = 1 \qquad (i \geqq 1).$$

This shows that $\overline{U} \to \overline{W}$ induces an isomorphism

(19) $$H^i(G, \overline{U}) = H^i(G, \overline{W}) \qquad (i \geqq 1).$$

Using (12) and (10) we obtain

$$H^i(G, \overline{U}) = \prod_{\mathfrak{p}} H^i(G_{\overline{\mathfrak{p}}}, (\overline{F}\overline{\mathfrak{p}})^{\times}) \qquad (i \geqq 1)$$

where $\mathfrak{p}$ ranges over the places of F/K and $\overline{\mathfrak{p}}$ denotes always a *fixed* extension of $\mathfrak{p}$ to $\overline{F}/\overline{K}$. For $i = 1$, the right hand side of (19) is 1, using Hilberts theorem 90 for each field $\overline{F}\overline{\mathfrak{p}}$. Hence (17).

Proof of (18). Let $\overline{V}_{\overline{\mathfrak{p}}}$ be the kernel of the map $\overline{U}_{\overline{\mathfrak{p}}} \to (\overline{F}\overline{\mathfrak{p}})^{\times}$, consisting of the elements $a \in \overline{F}_{\overline{\mathfrak{p}}}$ with $a\overline{\mathfrak{p}} = 1$. Then $\overline{V} = \prod_{\overline{\mathfrak{p}}} V_{\overline{\mathfrak{p}}}$. Put $\overline{V}_{\mathfrak{p}} = \prod_{\overline{\mathfrak{p}} | \mathfrak{p}} \overline{V}_{\overline{\mathfrak{p}}}$. Then $\overline{V} = \prod_{\mathfrak{p}} \overline{V}_{\mathfrak{p}}$ is a G-permissible direct product. Hence

$$H^i(G, \overline{V}) = \prod_{\mathfrak{p}} H^i(G, \overline{V}_{\mathfrak{p}}).$$

From Shapiros lemma[10] we infer that

$$H^i(G, \overline{V}_{\mathfrak{p}}) = H^i(G_{\overline{\mathfrak{p}}}, \overline{V}_{\overline{\mathfrak{p}}}),$$

$\overline{\mathfrak{p}}$ being a fixed extension of $\mathfrak{p}$. Hence we have to show that $H^i(G_{\overline{\mathfrak{p}}}, \overline{V}_{\overline{\mathfrak{p}}}) = 1$ for $i \geqq 1$. Changing notation, this amounts to show the following

LEMMA. *Let F be a complete field with respect to a non-archimedean, discrete valuation $w_{\mathfrak{p}}$ with corresponding prime $\mathfrak{p}$. Let V be the multiplicative subgroup of elements $a \in F$ with $a\mathfrak{p} = 1$* (i.e. $w_{\mathfrak{p}}(a-1) > 0$). *If G is a finite group of continuous automorphisms of F whose induced action on the image field $F\mathfrak{p}$ is faithful, then*

$$H^i(G, V) = 1 \qquad (i \geqq 1).$$

This lemma is well known from local class field theory. For the proof see e.g. Witt [14], page 154, no. 2 or Serre [10], page 193, lemma 2.

Let us return to our original notation. We now have proved (17) which is, for the map g^0, the analogue to (13). The analogue to (14) would be $H^1(G, \overline{CJ}) = 1$. This is not true in general (although we shall see later that it is true in the case where K is a local number field). We therefore introduce the group

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(20) $$X = X(\overline{F}/\overline{K}) = \text{kernel } H^1(G, \overline{CJ}) \to H^1(G, \overline{CD}).$$

From the exactness of the column of our diagram portion we infer that

(21) $$\begin{aligned} X &= \text{image } H^1(G, \overline{CU}) \to H^1(G, \overline{CJ}) \\ &= H^1(G, \overline{CU})/Y \end{aligned}$$

where

$$Y = \text{image } H^0(G, \overline{CD}) \to H^1(G, \overline{CU}).$$

From (17) it follows that

$$H^1(G, \overline{CU}) \to H^2(G, \overline{K}^\times) \text{ is monomorphic.}$$

Its image is the kernel of $H^2(G, \overline{K}^\times) \to H^2(G, \overline{U})$. Hence

$$H^1(G, \overline{CU}) \approx \text{kernel } H^2(G, \overline{K}^\times) \to H^2(G, \overline{U}).$$

In this isomorphism, the image Y of $H^0(G, \overline{CD}) \to H^1(G, \overline{CU})$ corresponds to the image of g^0 (by definition of g^0). Hence we obtain from (21):

(22) *The image of g^0 is contained in the kernel of $H^2(G, \overline{K}^\times) \to H^2(G, \overline{U})$ and the corresponding factor group is isomorphic to X.*

Finally, we claim:

(23) $$\text{kernel } H^2(G, \overline{K}^\times) \to H^2(G, \overline{U}) = \text{kernel } H^2(G, \overline{K}^\times) \to H^2(G, \overline{W}).$$

Proof. As shown in (19), the map $\overline{U} \to \overline{W}$ induces an isomorphism of cohomology groups. Hence the map $\overline{K}^\times \to \overline{U} \to \overline{W}$ induces a map $H^2(G, \overline{K}^\times) \to H^2(G, \overline{W})$ which has the same kernel as $H^2(G, \overline{K}^\times) \to H^2(G, \overline{U})$. Q.e.d.

Observe that the map $\overline{K}^\times \to \overline{U} \to \overline{W}$ is the diagonal imbedding of $\overline{K}^\times$ in $\overline{W}$ which we have considered in § 2.

Now remember that the maps h^0 and g^0 have the same image, as mentioned above. Comparing (16), (22) and (23) we obtain therefore the following 'kernel theorem':

THEOREM 2. *Let F/K be a function field of one variable and $\overline{K}/K$ a finite Galois extension with Galois group G. Then the kernel of $H^2(G, \overline{K}^\times) \to H^2(G, \overline{F}^\times)$ is contained in the kernel of $H^2(G, \overline{K}^\times) \to H^2(G, \overline{W})$ and the corresponding factor group is isomorphic to X, where X is defined to be the kernel of $H^1(G, \overline{CJ}) \to H^1(G, \overline{CD})$.*

In particular, if $H^1(G, \overline{CJ}) = 1$ then $X = 1$ and therefore kernel $H^2(G, \overline{K}^\times) \to H^2(G, \overline{F}^\times) = \text{kernel } H^2(G, \overline{K}^\times) \to H^2(G, \overline{W})$.

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Using propositions 1 and 2 of § 2 we obtain as an immediate consequence:

THEOREM 3. *Let F/K be a separable function field of one variable. Then the factor group $\tilde{\mathfrak{B}}(F/K)$ modulo $\mathfrak{B}(F/K)$ can be cohomologically described as the group X of theorem 2, where $\bar{K}/K$ has to be chosen such that $\tilde{\mathfrak{B}}(F/K) \subset \mathfrak{B}(\bar{K}/K)$.*

(As mentioned in § 2, the latter inclusion is true if $\bar{K}$ contains the image field $F\mathfrak{p}$ of a separable place $\mathfrak{p}$ of F/K.)

In particular, if the Galois cohomology of the idèle classes $\overline{CJ}$ vanishes in dimension 1, *then $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(F/K)$.*

§ 4. Proof of theorem 1

Now let F/K be a function field of one variable over a local number field K. If the valuation of K is non-archimedean, then there is a theorem of Tate which says that the Galois cohomology of the idèle classes vanishes in dimension 1.[14] Hence $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(F/K)$.

Now let the valuation of K be archimedean. Then K is either the field of complex numbers, or the field of real numbers. In the first case K is algebraically closed and hence $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(F/K) = 1$. In the second case, assume first that F/K has a real place $\mathfrak{p}$. Then $F\mathfrak{p} = K$, $\tilde{\mathfrak{B}}(F/K) \subset \mathfrak{B}(K/K) = 1$, hence $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(F/K) = 1$. Secondly, if all places $\mathfrak{p}$ of F/K are complex, then $F\mathfrak{p} = \bar{K}$ is the field of complex numbers for all $\mathfrak{p}$. Hence $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(\bar{K}/K) = \mathfrak{B}(K)$ is of order two, the only non-trivial element of $\mathfrak{B}(K)$ corresponding to the quaternion algebra over K. On the other hand, Witt has shown that if F/K has no real places, then -1 is a sum of two squares in F, hence -1 is a norm of $F\bar{K}/F$, i.e. the quaternion algebra splits over F.[15] This implies that $\mathfrak{B}(F/K)$ contains the quaternion algebra and is therefore equal to $\mathfrak{B}(K)$.

Hence, in any case, $\tilde{\mathfrak{B}}(F/K) = \mathfrak{B}(F/K)$.[16]

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[14] Tate [11], page 156-02, line 2-5.

[15] Witt [12], page 7 Satz 2.

[16] Using Witts results, it can be shown that Tates relation $H^1(G, \overline{CJ}) = 1$ holds also if K is real and $\bar{K}$ complex. For, if one interprets Witts statement I' ([12], page 5) cohomologically, it says that the map $H^2(G, \bar{F}^{\times}) \to H^2(G, \bar{J})$ is injective. On the other hand, from our diagram in § 3 we obtain an exact sequence $H^1(G, \bar{J}) \to H^1(G, \overline{CJ}) \to H^2(G, \bar{F}^{\times}) \to H^2(G, \bar{J})$ and we have $H^1(G, \bar{J}) = 1$ from Hilberts theorem 90 for the completions $\bar{F}_{\bar{\mathfrak{p}}}$. Hence $H^1(G, \overline{CJ}) = 1$.

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In order to complete the proof of theorem 1 we have to describe the group $\mathfrak{B}(F/K)$.

Consider first the non-archimedean case. As is well known from local class field theory, the Brauer group $\mathfrak{B}(K)$ is isomorphic to the additive group $\mathbf{Q}/\mathbf{Z}$ of rational numbers modulo integers.[17] The isomorphism

$$\mathfrak{B}(K) \approx \mathbf{Q}/\mathbf{Z}$$

is obtained by assigning to each central simple algebra A/K its Hasse invariant $\mathrm{inv}_K(A)$. If L/K is a finite algebraic extension field, then $\mathrm{inv}_L(A \underset{K}{\otimes} L) = (L:K)\cdot\mathrm{inv}_K(A)$. In particular, L splits A if and only if $(L:K)$ is a multiple of the order of A in $\mathfrak{B}(K)$. In other words $\mathfrak{B}(L/K)$ consists of all those $A \in \mathfrak{B}(K)$ for which $A^{(L:K)} = 1$. The group structure of $\mathbf{Q}/\mathbf{Z}$ implies moreover that $\mathfrak{B}(L/K)$ is cyclic of order $(L : K)$.

In particular, $\mathfrak{B}(F\mathfrak{p}/K)$ is cyclic of order $(F\mathfrak{p} : K) = \deg(\mathfrak{p})$, and $\mathfrak{B}(F\mathfrak{p}/K)$ consists of all elements $A \in \mathfrak{B}(K)$ with $A^{\deg(\mathfrak{p})} = 1$. Taking the intersection for all $\mathfrak{p}$, we see that if

$$0 < d(F/K) = \underset{\mathfrak{p}}{\gcd}\, \deg(\mathfrak{p})$$

then $\mathfrak{B}(F/K)$ is cyclic of order $d(F/K)$ and consists of all elements $A \in \mathfrak{B}(K)$ with $A^{d(F/K)} = 1$.

If $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{a(\mathfrak{p})}$ is a divisor of F/K then $\deg(\mathfrak{a}) = \sum_{\mathfrak{p}} a(\mathfrak{p})\cdot\deg(\mathfrak{p})$ is a linear combination of the degrees $\deg(\mathfrak{p})$, hence a multiple of $d(F/K)$, and conversely. Hence $d(F/K)$ can be characterized as the least positive degrees of divisors of F/K.

This proves theorem 3 in the non-archimedean case, if one uses the fact (proved by studying the Hasse invariant as above) that the Schur index of any $A \in \mathfrak{B}(K)$ is equal to its order in $\mathfrak{B}(K)$.

In the archimedean real case, we have

$$\mathfrak{B}(K) \approx \frac{1}{2}\mathbf{Z}/\mathbf{Z}.$$

If one defines the Hasse invariant of the quaternion algebra to be $\frac{1}{2}$ modulo $\mathbf{Z}$, then the above considerations carry over verbally in order to prove theorem 1.

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[17] See e.g. Deuring [6], page 112, Satz 3.

In the archimedean complex case, there is nothing to prove.

§ 5. Some additional remarks

(a) *Examples of function fields F/K over a number field K for which $\mathfrak{B}(F/K) \neq \tilde{\mathfrak{B}}(F/K)$*:

Let K be a number field and F/K be a function field of one variable which is of genus 0 but not rational. It has been shown by Witt [13] that $F = F(A)$ is a generic splitting field of a certain quaternion algebra A over K which is uniquely determined by F. Then $\mathfrak{B}(F/K)$ is of order 2, the only non-trivial element of $\mathfrak{B}(F/K)$ being A; this follows also from our general theory of generic splitting fields[18]. Let $\mathfrak{q}$ range over the primes of K including the primes at infinity. Let M be the set of primes $\mathfrak{q}$ at which A is ramified, i.e. for which the $\mathfrak{q}$-adic Hasse invariant $\mathrm{inv}_{\mathfrak{q}}(A) \equiv \frac{1}{2} \bmod \mathbf{Z}$. According to the Hasse sum formula $\sum_{\mathfrak{q}} \mathrm{inv}_{\mathfrak{q}}(A) \equiv 0 \bmod \mathbf{Z}$[19] the number m of primes $\mathfrak{q} \in M$ is even. To every non-empty subset $N \subset M$ which consists of an even number of primes $\mathfrak{q}$ there exists one and only one quaternion algebra $A(N)$ with the primes in N as its ramification primes.[19] In particular, $A = A(M)$. These quaternion algebras generate a subgroup of $\mathfrak{B}(K)$ of order 2^{m-1}. We claim that this subgroup coincides with $\tilde{\mathfrak{B}}(F/K)$. Let $\mathfrak{p}$ be a prime of F/K. Then $F\mathfrak{p}$ splits A, hence the $\mathfrak{q}$-adic completion $(F\mathfrak{p})_{\mathfrak{q}}$ splits $A_{\mathfrak{q}} = A \otimes_K K_{\mathfrak{q}}$ for every $\mathfrak{q}$. Hence $((F\mathfrak{p})_{\mathfrak{q}} : K_{\mathfrak{q}}) \equiv 0 \bmod 2$ for $\mathfrak{q} \in M$. In particular, this holds for $\mathfrak{q} \in N$. Hence $(F\mathfrak{p})_{\mathfrak{q}}$ splits $A(N)_{\mathfrak{q}}$. Since this is true for all $\mathfrak{q}$, it follows[20] that $F\mathfrak{p}$ splits $A(N)$. Hence $A(N) \in \tilde{\mathfrak{B}}(F/K)$ for all N. Conversely, let $B \in \tilde{\mathfrak{B}}(F/K)$, $B \neq 1$. Let L/K be a finite algebraic splitting field of A. Since $F = F(A)$ is a generic splitting field for A, there is a place $\mathfrak{p}$ of F/K such that $F\mathfrak{p} \subset L$.[21] Since $F\mathfrak{p}$ splits B it follows that L splits B. Hence every finite algebraic splitting field L of A is also a splitting field for B. According to the existence theorem of Grunwald[22] there exists a finite algebraic extension field L/K such that $(L_{\mathfrak{q}} : K_{\mathfrak{q}}) = 2$ for $\mathfrak{q} \in M$ and $L_{\mathfrak{q}_0} = K_{\mathfrak{q}_0}$ if $\mathfrak{q}_0 \notin M$ is arbitrarily chosen. This field L splits A by construction and hence B.

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18) [9], page 414, theorem 5.
19) Deuring [6], page 119, Satz 9.
20) Deuring [6], page 117, Satz 1.
21) [9] page 413, theorem 2.
22) Hasse [7], page 40, Ganz schwacher Existenzsatz.

It follows $\mathrm{inv}_{\mathfrak{q}}(B) \equiv 0 \bmod \frac{1}{2}$ for $\mathfrak{q} \in M$ and $\mathrm{inv}_{\mathfrak{q}_0}(B) = 1$. Since $\mathfrak{q}_0 \notin M$ is arbitrary, we see that B is unramified outside of M. For $\mathfrak{q} \in M$, the invariant $\mathrm{inv}_{\mathfrak{q}}(B)$ is either 0 or $\frac{1}{2}$ mod $\mathbf{Z}$. Hence $B = A(N)$ is a quaternion algebra belonging to some subset $N \subset M$.

We have now shown that $\mathfrak{B}(F/K)$ is of order 2 while $\tilde{\mathfrak{B}}(F/K)$ is of order 2^{m-1}. If we choose A such that the number m of ramification points of A is $m > 2$, which is possible[19], then for the field $F = F(A)$ we have $\mathfrak{B}(F/K) \neq \tilde{\mathfrak{B}}(F/K)$.

(b) *Examples of function fields F/K over a number field K such that $d(F/K) \neq 1$ and* $\mathfrak{B}(F/K) = 1$.

Let K be a number field and F/K a function field of one variable and genus 1 with the property that $d(F/K) > 1$ but $d(F_{\mathfrak{q}}/K_{\mathfrak{q}}) = 1$ for all primes $\mathfrak{q}$ of K, where $F_{\mathfrak{q}} = FK_{\mathfrak{q}}$ is the constant extension of F/K with respect to the completion $K_{\mathfrak{q}}$ of $\mathfrak{q}$.[23] If $A \in \mathfrak{B}(F/K)$, then for every $\mathfrak{q}$ the completion $A_{\mathfrak{q}}$ is split by $F_{\mathfrak{q}}/K_{\mathfrak{q}}$. Since $d(F_{\mathfrak{q}}/K_{\mathfrak{q}}) = 1$ it follows from theorem 1 that $A_{\mathfrak{q}}$ splits too. Hence $\mathrm{inv}_{\mathfrak{q}}(A) \equiv 0 \bmod \mathbf{Z}$ for all $\mathfrak{q}$, i.e. $A = 1$.[19]

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23) The existence of such function fields has been proved by Reichardt [8]. See also Cassels [4], page 65, theorem.

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