Proceedings of the Edinburgh Mathematical Society (2007) **50**, 87–114 ©
DOI:10.1017/S001309150500074X   Printed in the United Kingdom

AN ASYMPTOTIC MAJORANT FOR SOLUTIONS OF STURM–LIOUVILLE EQUATIONS IN $L_P(\mathbb{R})$

N. A. CHERNYAVSKAYA[1] AND L. A. SHUSTER[2]

[1] *Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, PO Box 653, Beer-Sheva 84105, Israel* (nina@math.bgu.ac.il)
[2] *Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, Israel* (miriam@macs.biu.ac.il)

(Received 18 May 2005)

Abstract Under certain assumptions on $g(x)$, we obtain an asymptotic formula for computing integrals of the form

$$F(x,\alpha)=\int_{-\infty}^{\infty} g(t)^{\alpha}\exp\biggl(-\biggl|\int_x^t g(\xi)\,d\xi\biggr|\biggr)\,dt,\quad \alpha\in\mathbb{R},$$

as $|x|\to\infty$. We use this formula to study the properties (as $|x|\to\infty$) of the solutions of the correctly solvable equations in $L_p(\mathbb{R})$, $p\in[1,\infty]$,

$$-y''(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R},\tag{1}$$

where $0\leqslant q\in L_1^{\mathrm{loc}}(\mathbb{R})$, and $f\in L_p(\mathbb{R})$. (Equation (1) is called correctly solvable in a given space $L_p(\mathbb{R})$ if for any function $f\in L_p(\mathbb{R})$ it has a unique solution $y\in L_p(\mathbb{R})$ and if the following inequality holds with an absolute constraint $c_p\in(0,\infty)$: $\|y\|_{L_p(\mathbb{R})}\leqslant c(p)\|f\|_{L_p(\mathbb{R})}$, $\forall f\in L_p(\mathbb{R})$.)

Keywords: Sturm–Liouville equation; asymptotic majorant for solutions; asymptotic estimates

2000 *Mathematics subject classification:* Primary 34B27; 34L20

1. Introduction

This paper continues the authors' work in [**2, 3, 5**]. We consider the equation

$$-y''(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R},\tag{1.1}$$

where $f\in L_p(\mathbb{R})$, $p\in[1,\infty]$ ($L_\infty(\mathbb{R})\overset{\mathrm{def}}{=}C(\mathbb{R})$) and

$$0\leqslant q\in L_1^{\mathrm{loc}}(\mathbb{R}).\tag{1.2}$$

By a solution of equation (1.1), we mean any function y such that $y,\ y'\in AC^{\mathrm{loc}}(\mathbb{R})$ and equality (1.1) hold almost everywhere in $\mathbb{R}$. We also assume that (1.1) is correctly solvable in $L_p(\mathbb{R})$. This means (see [**6**, Chapter III, § 6, no. 2]) that the following assumptions both hold:

(I) for a fixed $p \in [1, \infty]$, for any function $f \in L_p(\mathbb{R})$ there is a unique solution $y \in L_p(\mathbb{R})$ of equation (1.1);

(II) there is an absolute positive constant $c(p)$ such that the solution $y \in L_p(\mathbb{R})$ of equation (1.1) satisfies the inequality

$$\|y\|_p \leqslant c(p)\|f\|_p, \quad \forall f \in L_p(\mathbb{R}). \tag{1.3}$$

Throughout the paper we assume that the above conditions hold. We always denote by f an arbitrary function from $L_p(\mathbb{R})$, $p \in [1, \infty]$, and by y the solution of (1.1) mentioned in (I) and (II). We denote by c, $c(\cdot)$ absolute positive constants which are not essential for exposition and may differ even within a single chain of computations. Finally, see § 2 for criteria under which (I) and (II) hold.

Our general goal is to study bounds for the solutions of (1.1) as $|x| \to \infty$. To state the problem more precisely, denote by $\mathcal{D}_p$ the set of solutions of (1.1) with f running over the surface of the unit sphere $S_p = \{f : \|f\|_p = 1\}$ in $L_p(\mathbb{R})$, $p \in [1, \infty]$, and introduce the following definition.

Definition 1.1. Suppose that equation (1.1) is correctly solvable in $L_p(\mathbb{R})$, $p \in [1, \infty]$. A continuous, positive function $\varkappa_p(x)$ for $x \in \mathbb{R}$ is called an asymptotic majorant for the set $\mathcal{D}_p$ of the solutions of (1.1) if the following conditions hold:

(1) for any $\gamma > 1$ there exists a $c(\gamma)$ such that for all $|x| \geqslant c(\gamma)$ the inequality

$$|y(x)| \leqslant \gamma \varkappa_p(x) \tag{1.4}$$

holds regardless of the choice of a solution $y \in \mathcal{D}_p$;

(2) for any $\gamma \in (0, 1)$ and any c, as large as we wish, there is a solution $y \in \mathcal{D}_p$ and a point $|x_0| > c$ such that $|y(x_0)| > \gamma \varkappa_p(x_0)$.

Thus, our goal is as follows: given $p \in [1, \infty]$, for a correctly solvable equation (1.1) in $L_p(\mathbb{R})$, find an asymptotic majorant $\varkappa_p(x)$ for the set $\mathcal{D}_p$ of the solutions of (1.1). For brevity, the function $\varkappa_p(x)$ is called an asymptotic majorant for the solutions of (1.1).

Note that this problem seems to be new. It can be viewed as a next step after the study of a natural problem of estimating the solutions $y \in \mathcal{D}_p$ in the uniform metric on the whole real axis (see § 2). The difference between the two problems is clear: the new setting requires a more detailed study of the uniform estimates of the solutions of (1.1) at infinity. The ultimate goal of such a study of the asymptotic majorant $\varkappa_p(x)$ allows one to find bounds, as sharp as possible (in the sense of Definition 1.1), containing all integral curves of the set $\mathcal{D}_p$ as $|x| \to \infty$. We believe that such *a priori* information on the asymptotic behaviour at infinity of all solutions from the set $\mathcal{D}_p$ may be useful, for example, for the analysis of computational algorithms for the numerical solution of equation (1.1).

We now give a general description of the results of the paper (see § 3 for the precise statements). Let $\tilde{H}$ be a given set of coefficients q of equation (1.1). The set $\tilde{H}$ is characterized by the condition that its elements satisfy certain local requirements of integral type together with (1.2) (see § 3). We also need the Otelbaev average q^* of the function q (this is a special case of the Steklov average (see § 2)). With this notation, our main result is the following equality (see Theorem 3.3):

$$\lim_{|x|\to\infty} \sup_{y\in\mathcal{D}_p} q^*(x)^{1-1/2p}|y(x)| = \ell(p), \quad p \in [1,\infty], \tag{1.5}$$

which holds for all $q \in \tilde{H}$. Here

$$\ell(p) = \begin{cases} \frac{1}{2} & \text{if } p = 1, \\ \dfrac{1}{(p')^{1/p'}2^{1/p}} & \text{if } p \in (1,\infty),\ p' = \dfrac{p}{p-1}, \\ 1 & \text{if } p = \infty. \end{cases} \tag{1.6}$$

We emphasize that the relations (1.5) and (1.6) remain true regardless of the actual choice of $q \in \tilde{H}$, although the class of coefficients $\tilde{H}$ is sufficiently large and contains, for example, non-differentiable, slowly and rapidly increasing, oscillating functions. From (1.5) and (1.6) it immediately follows that the asymptotic majorant $\varkappa_p(x)$ is given by the following equality:

$$\varkappa_p(x) = \frac{\ell(p)}{q^*(x)^{1-1/2p}}, \quad x \in \mathbb{R},\ p \in [1,\infty], \tag{1.7}$$

which represents a full solution of the initial problem for equation (1.1) with $q \in \tilde{H}$.

This solution can be viewed in the following as the final one, or as a subject for further investigation, depending on the goal of the application of (1.7). This uncertainty arises because q^* in (1.7) is given as an implicit function (see § 2). Therefore, if one needs more detailed information on $\varkappa_p(x)$, one must carry out a separate study of the function q^* which assumes the conditions on q be independent of the initial problem on an asymptotic majorant for the solutions of (1.1). For example, for $q \in \tilde{H}$ one can apply formula (1.7), without any additional restrictions on q, to the study of general properties of the solutions of (1.1), because usually one does not need any explicit expression for q^* (see [**5**]). However, if, given equation (1.1) with $q \in \tilde{H}$, one has to find a concrete expression for $\varkappa_p(x)$, then formula (1.7) is not very helpful. It only works in particular cases where, given the initial function q, one can find the explicit value of q^*. Note that a continuous and positive function $\tilde{\varkappa}_p(x)$ for $x \in \mathbb{R}$ is an asymptotic majorant of the solutions of (1.1) provided

$$\lim_{|x|\to\infty} \frac{\tilde{\varkappa}_p(x)}{\varkappa_p(x)} = 1, \tag{1.8}$$

and $\varkappa_p(x)$ is an asymptotic majorant of the solutions of (1.1) (see Lemma 7.3).

The above conclusion allows us to rehabilitate formula (1.7). Indeed, according to (1.8), in order to find an asymptotic majorant $\tilde{\varkappa}_p(x)$, it suffices to replace $q^*(x)$ in (1.7) with the principal part of its asymptotic expansion at infinity. Thus, we can find an explicit form of an asymptotic majorant for the solutions of (1.1) under some requirements on q (in addition to the condition $q \in \tilde{H}$). These requirements arise from any conditions under which one can solve the technical problem of the proof of the asymptotic formula for $q^*(x)$ as $|x| \to \infty$. Such a problem was considered in [**3**] for the case $1 \leqslant q \in L_1^{\mathrm{loc}}(\mathbb{R})$. In § 6 we extend its solution given in [**3**] to the case (1.2) under new conditions that are more convenient for practical verification (see Theorem 3.4). See § 7 for technical details concerning application of Theorems 3.3 and 3.4. In particular, in § 7 we show that equation (1.1) with the coefficient

$$q(x) = \exp(x^2) + \exp(x^2)\cos(\exp(x^2)) \tag{1.9}$$

is correctly solvable in $L_p(\mathbb{R})$ for $p \in [1, \infty]$ and we find an explicit form for the asymptotic majorant $\varkappa_p(x)$ of its solutions for all $p \in [1, \infty]$.

Let us now briefly describe the methods of the present paper. The main roles in the proof of equality (1.5) are played by

(1) the Davies–Harrell representation for the Green function $G(x, t)$ of equation (1.1) via its diagonal values $\rho(x) \overset{\text{def}}{=} G(x, t)|_{x=t}$, $x \in \mathbb{R}$ (see § 2),

(2) *a priori*, sharp-by-order, two-sided estimates for $\rho(x)$ due to Otelbaev (see § 2),

(3) an asymptotic formula for $\rho(x)$ as $|x| \to \infty$ (see § 2),

(4) a new asymptotic formula for computing integrals of the form

$$G_\alpha(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} G(x, t)^\alpha \, \mathrm{d}t, \quad \alpha > 0, \ x \in \mathbb{R}, \tag{1.10}$$

as $|x| \to \infty$ (see Theorem 3.1); this formula, which plays an auxiliary role in the present paper, is stated as a separate assertion which is a result that may be of independent interest.

2. Preliminaries

In this section, we give a summary of results used in the proofs.

Theorem 2.1 (Chernyavskaya and Shuster [5]). *Let $p \in [1, \infty]$ be given. Equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ if and only if there is $a \in (0, \infty)$ such that*

$$q_0(a) > 0, \quad q_0(a) \overset{\text{def}}{=} \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) \, \mathrm{d}t. \tag{2.1}$$

In particular, one of the following assertions holds:

(A) *for all $p \in [1, \infty]$ equation (1.1) is correctly solvable in $L_p(\mathbb{R})$;*

(B) *for any $p \in [1, \infty]$ equation (1.1) is not correctly solvable in $L_p(\mathbb{R})$.*

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Let us introduce Otelbaev's functions d and q^* (see [**10**]). Assume that, together with (1.2), the following assertion holds:

$$\int_{-\infty}^{x} q(t)\,\mathrm{d}t > 0, \quad \int_{x}^{\infty} q(t)\,\mathrm{d}t > 0 \quad \text{for any } x \in \mathbb{R}. \tag{2.2}$$

For a given $x \in \mathbb{R}$, consider an equation in $d \geqslant 0$:

$$d \int_{x-d}^{x+d} q(t)\,\mathrm{d}t = 2. \tag{2.3}$$

Equation (2.3) has a unique positive solution $d(x)$ (see [**2**]). Set $q^*(x) \stackrel{\text{def}}{=} d(x)^{-2}$. The equalities

$$q^*(x) = \frac{1}{d(x)^2} = \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t = \frac{1}{2h} \int_{x-h}^{x+h} q(t)\,\mathrm{d}t\bigg|_{h=d(x)}$$

show that $q^*(x)$ is Steklov's average (see [**12**]) of q with special average step $h = d(x)$.

Theorem 2.2 (Chernyavskaya and Shuster [5]). *Let $p \in [1, \infty]$ be given. Equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ if and only if conditions (2.2) and*

$$q_0^* > 0, \quad q_0^* \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} q^*(x) \tag{2.4}$$

both hold.

Remark 2.3. From the definition of $q^*(x)$ it follows that the condition $q_0^* > 0$ is equivalent to the condition

$$d_0 < \infty, \quad d_0 \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} d(x). \tag{2.5}$$

Since, under condition (2.1), requirement (2.2) holds automatically, we obtain the following conclusion (see [**5**]).

Lemma 2.4. *Under conditions (1.2) and (2.1), we have $d_0 < \infty$.*

Lemma 2.5 (Chernyavskaya and Shuster [2]). *Suppose that conditions (1.2) and (2.2) hold. Then there is a fundamental system of solutions (FSS) $\{u, v\}$ of equation (2.6),*

$$z''(x) = q(x)z(x), \quad x \in \mathbb{R}, \tag{2.6}$$

for which the following relations hold:

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0 \quad \textit{for } x \in \mathbb{R}, \tag{2.7}$$

$$v'(x)u(x) - u'(x)v(x) = 1 \quad \textit{for } x \in \mathbb{R}, \tag{2.8}$$

$$\lim_{x \to -\infty} \frac{v(x)}{u(x)} = \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0. \tag{2.9}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Below, the symbols $\{u, v\}$ stand for an FSS of (2.6) with properties (2.7)–(2.9). Introduce the Green function $G(x, t)$ of equation (1.1):

$$G(x,t) = \begin{cases} u(x)v(t) & \text{if } x \geqslant t, \\ u(t)v(x) & \text{if } x \leqslant t. \end{cases} \tag{2.10}$$

Definition 2.6 (Chernyavskaya and Shuster [4]). Let $p \in [1, \infty]$ be given. The inversion problem for equation (1.1) in $L_p(\mathbb{R})$ is called regular if, together with requirements (I) and (II) (see § 1), the following holds:

(III) for any $f \in L_p(\mathbb{R})$ the solution $y \in L_p(\mathbb{R})$ of equation (1.1) is of the form

$$y(x) = \int_{-\infty}^{\infty} G(x,t) f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{2.11}$$

Theorem 2.7 (Chernyavskaya and Shuster [4]). *Suppose that conditions (1.2) and (2.2) hold. Then, for any $p \in [1, \infty]$, the inversion problem for (1.1) is regular in $L_p(\mathbb{R})$ if and only if $d_0 < \infty$ (see (2.5)).*

Thus, we obtain the following conclusion.

Lemma 2.8. *Let $p \in [1, \infty]$. If equation (1.1) is correctly solvable in $L_p(\mathbb{R})$, then, for any $f \in L_p(\mathbb{R})$, the solution $y \in L_p(\mathbb{R})$ of (1.1) is of the form (2.11).*

In connection with Lemma 2.8, note that the following two assertions play the main role in the study of the properties of the solutions of equation (1.1).

Theorem 2.9 (Davies and Harrell [7]). *Suppose that conditions (1.2) and (2.2) hold. Then, for $x, t \in \mathbb{R}$, the Green function $G(x, t)$ of equation (1.1) admits the Davies–Harrell representation*

$$G(x,t) = \sqrt{\rho(x)\rho(t)} \exp\left(-\frac{1}{2}\left|\int_x^t \frac{\mathrm{d}\xi}{\rho(\xi)}\right|\right), \quad \rho(x) \overset{\text{def}}{=} G(x,t)|_{x=t} = u(x)v(x). \tag{2.12}$$

Theorem 2.10 (Chernyavskaya and Shuster [2]). *Suppose that conditions (1.2) and (2.2) hold. Then the diagonal value $\rho(x)$ of the Green function $G(x, t)$ of equation (1.1) satisfies Otelbaev's inequalities:*

$$\tfrac{1}{4} d(x) \leqslant \rho(x) \leqslant \tfrac{3}{2} d(x), \quad x \in \mathbb{R}. \tag{2.13}$$

Remark 2.11. We call relations (2.13) 'Otelbaev's inequalities' because the estimates of this type were first obtained in [**11**]. In [**11**] stronger requirements than (1.2) and (2.2) were imposed on q, and another more complicated auxiliary function was used instead of d.

Let us give an example of a direct application of Theorems 2.9 and 2.10.

Lemma 2.12. *Suppose that equation (1.1) is correctly solvable in $L_p(\mathbb{R})$, $p \in [1,\infty]$. Then the set $\mathcal{D}_p$ of solutions of (1.1) is uniformly bounded on the whole axis:*

$$\sup_{x\in\mathbb{R}} |y(x)| \leqslant c(p) < \infty \quad \textit{for all } y \in \mathcal{D}_p. \tag{2.14}$$

Proof. From (2.5) and (2.13) it follows that

$$\rho(x) \leqslant \tfrac{3}{2} d(x) \leqslant \tfrac{3}{2} d_0, \quad x \in \mathbb{R} \quad \Longrightarrow \quad \frac{1}{\rho(x)} \geqslant \frac{2}{3d_0} \quad \text{for } x \in \mathbb{R}.$$

Hence, from (2.12), (2.5) and (2.13) we get

$$G(x,t) = \sqrt{\rho(x)\rho(t)}\exp\biggl(-\frac{1}{2}\biggl|\int_x^t \frac{\mathrm{d}\xi}{\rho(\xi)}\biggr|\biggr) \leqslant \tfrac{3}{2} d_0 \exp\biggl(-\frac{|t-x|}{3d_0}\biggr), \qquad x,t \in \mathbb{R}. \tag{2.15}$$

By Lemma 2.8, estimate (2.15) and Hölder's inequality for $p \in (1,\infty)$, $p = 1$ and $p = \infty$, we now obtain, respectively,

$$\begin{aligned}
|y(x)| &\leqslant \int_{-\infty}^{\infty} G(x,t)|f(t)|\,\mathrm{d}t \leqslant \biggl(\int_{-\infty}^{\infty} G(x,t)^{p'}\,\mathrm{d}t\biggr)^{1/p'} \|f\|_p \\
&\leqslant c\biggl(\int_{-\infty}^{\infty} \exp\biggl(-\frac{p'|t-x|}{3d_0}\biggr)\mathrm{d}t\biggr)^{1/p'} = c(p) \quad \Longrightarrow \quad (2.14); \\
|y(x)| &\leqslant \int_{-\infty}^{\infty} G(x,t)|f(t)|\,\mathrm{d}t \leqslant c\|f\|_1 \quad \Longrightarrow \quad (2.14); \\
|y(x)| &\leqslant \int_{-\infty}^{\infty} G(x,t)|f(t)|\,\mathrm{d}t \leqslant \int_{-\infty}^{\infty} G(x,t)\,\mathrm{d}t \|f\|_{C(\mathbb{R})} \leqslant c\int_{-\infty}^{\infty} \exp\biggl(-\frac{|t-x|}{3d_0}\biggr)\mathrm{d}t \\
&= c \quad \Longrightarrow \quad (2.14).
\end{aligned}$$

□

Definition 2.13 (Chernyavskaya and Shuster [3]). Suppose that conditions (1.2) and (2.1) hold. We say that a function q belongs to the class H (and write $q \in H$) if there is a continuous function $k(x)$ such that, for $x \in \mathbb{R}$, the following relations hold:

(1) $k(x) \geqslant 2$ for $x \in \mathbb{R}$, $k(x) \to \infty$ as $|x| \to \infty$;

(2) there is an absolute constant $a \in [1,\infty)$ such that, for $x \in \mathbb{R}$, the inequalities

$$a^{-1}k(x) \leqslant k(t) \leqslant ak(x) \quad \text{for } t \in [x - k(x)d(x), x + k(x)d(x)] \tag{2.16}$$

hold;

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

(3) for $x \in \mathbb{R}$, the inequality

$$\Phi(x) \overset{\text{def}}{=} k(x)d(x) \sup_{z\in\omega(x)} \left| \int_0^z [q(x+t) - q(x-t)]\,\mathrm{d}t \right| \leqslant c < \infty \tag{2.17}$$

holds; here $\omega(x) = [0, k(x)d(x)]$, $x \in \mathbb{R}$.

Theorem 2.14 (Chernyavskaya and Shuster [3]). *Suppose that conditions (1.2) and (2.1) hold, $q \in H$, and $k(x)$ is the function from Definition 2.13. Then, for all $|x| \gg 1$, we have*

$$\rho(x) = (1+\varepsilon(x))\tfrac{1}{2}d(x), \quad |\varepsilon(x)| \leqslant \frac{c}{\sqrt{k(x)}}, \tag{2.18}$$

$$|\rho'(x)| \leqslant \frac{c}{\sqrt{k(x)}}. \tag{2.19}$$

Remark 2.15. In [3], Definition 2.13 and Theorem 2.14 are given for the case

$$1 \leqslant q \in L_1^{\text{loc}}(\mathbb{R}). \tag{2.20}$$

Minor technical changes in the proof allow one to keep the results of [3] when condition (2.20) is replaced by conditions (1.2) and (2.1). These changes arise when the inequality $d_0 \leqslant 1$, which follows from (2.20) (see [3]), is replaced with a more general inequality, (2.5) (see Lemma 2.4).

Lemma 2.16 (Chernyavskaya and Shuster [5]). *Suppose that (1.2) and (2.2) hold. Then the functions $d(x)$ and $q^*(x)$ are continuous for $x \in \mathbb{R}$.*

Theorem 2.17. *Assume that one can represent q in the form $q = q_1 + q_2$, where $q_1(x)$ is continuous for $x \in \mathbb{R}$, $q_1(x) > 0$ for $x \in \mathbb{R}$, $q_2 \in L_1^{\text{loc}}(\mathbb{R})$. Let $A(x) = [0, 2q_1(x)^{-1/2}]$. Consider the functions*

$$h_1(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t\in A(x)} \left| \int_0^t [q_1(x+s) - 2q_1(x) + q_1(x-s)]\,\mathrm{d}s \right|, \tag{2.21}$$

$$h_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t\in A(x)} \left| \int_{x-t}^{x+t} q_2(s)\,\mathrm{d}s \right|. \tag{2.22}$$

If $h_1(x) \to 0$, $h_2(x) \to 0$ as $|x| \to \infty$, then

$$d(x) = \frac{1+\varepsilon(x)}{\sqrt{q_1(x)}}, \quad |\varepsilon(x)| \leqslant c(h_1(x) + h_2(x)). \tag{2.23}$$

Remark 2.18. This result has been obtained in [3] under the additional condition $q_1 \geqslant 1$. The same proof given there (see [3]) is also valid here for Theorem 2.17.

3. Statement of results

In this section, we present the main results of this paper. The following theorem contains an asymptotic formula for computing integrals of the form

$$F(x,\alpha) \overset{\text{def}}{=} \int_{-\infty}^{\infty} g(t)^{\alpha} \exp\biggl(-\biggl|\int_x^t g(\xi)\,\mathrm{d}\xi\biggr|\biggr)\,\mathrm{d}t, \quad \alpha \in \mathbb{R}, \tag{3.1}$$

as $|x| \to \infty$.

Theorem 3.1. *Let $g(x)$ be positive and differentiable for $x \in \mathbb{R}$, and suppose that there is a function $s(x)$ with the following properties:*

(a) *$s(x)$ is continuous and positive for $x \in \mathbb{R}$, and $s(x) \to \infty$ as $|x| \to \infty$;*

(b) *the equality*

$$\lim_{|x|\to\infty} \frac{s(x)}{xg(x)} = 0 \tag{3.2}$$

holds;

(c) *for all $|x| \gg 1$, we have*

$$\frac{1}{s(x)} \geqslant \frac{|g'(x)|}{g(x)^2}; \tag{3.3}$$

(d) *there is $\nu \in [1,\infty)$ such that, for all $|x| \gg 1$, we have*

$$\frac{1}{\nu} \leqslant \frac{s(t)}{s(x)} \leqslant \nu \quad \text{for } t \in \Delta(x) = [\Delta^-(x), \Delta^+(x)] \overset{\text{def}}{=} \biggl[x - \frac{s(x)}{g(x)}, x + \frac{s(x)}{g(x)}\biggr]. \tag{3.4}$$

Then the following assertions hold:

(A)

$$\int_{-\infty}^{0} g(t)\,\mathrm{d}t = \infty, \qquad \int_0^{\infty} g(t)\,\mathrm{d}t = \infty; \tag{3.5}$$

(B) *for all $|x| \gg 1$ and $\alpha \in \mathbb{R}$, we have*

$$F(x,\alpha) = 2g(x)^{\alpha-1}(1 + \varepsilon(x)), \quad |\varepsilon(x)| \leqslant \frac{c(\nu,\alpha)}{s(x)}, \tag{3.6}$$

$$|c(\nu,\alpha)| \leqslant 5 + 2\nu|\alpha - 1|; \tag{3.7}$$

(C) *for $x \in \mathbb{R}$, we have*

$$F(x,1) = 2. \tag{3.8}$$

Definition 3.2. Suppose that (1.2) and (2.1) hold, and $q \in H$. Let K be the set of functions $k(x)$ each of which satisfies all the hypotheses of Definition 2.13. We say that q belongs to the class $\tilde{H}$ (and write $q \in \tilde{H}$) if there is at least one function $k(x) \in K$ such that

$$\lim_{|x|\to\infty} \frac{\sqrt{k(x)}d(x)}{x} = 0. \tag{3.9}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Theorem 3.3. *Suppose that (1.2) and (2.1) hold, and $q \in \tilde{H}$. Then for all $p \in [1,\infty]$, equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ and, in addition, equality (1.5) holds. For each $p \in [1,\infty]$, an asymptotic majorant $\varkappa_p(x)$ for the set $\mathcal{D}_p$ of the solutions of (1.1) is given by equality (1.7).*

The following technical assertion gives a practical device which simplifies the verification of condition (2.1) for correct solvability of equation (1.1) in $L_p(\mathbb{R})$, $p \in [1,\infty]$, and allows one to find an asymptotic for the function $d(x)$ as $|x| \to \infty$. This information is usually sufficient to check whether $q \in \tilde{H}$ and, if this is the case, to find an asymptotic majorant for the solutions of (1.1) in an explicit form (see Example 7.2).

Theorem 3.4. *Suppose that condition (1.2) holds and that q can be represented in the form*

$$q = q_1 + q_2, \tag{3.10}$$

where $q_1(x)$ is a positive doubly differentiable function for $x \in \mathbb{R}$, and $q_2 \in L_1^{\mathrm{loc}}(\mathbb{R})$. Define

$$A(x) \overset{\mathrm{def}}{=} \left[0, \frac{2}{\sqrt{q_1(x)}}\right], \qquad x \in \mathbb{R}, \tag{3.11}$$

$$\hat{h}_1(x) \overset{\mathrm{def}}{=} \frac{1}{q_1(x)^{3/2}} \sup_{t\in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi)\,\mathrm{d}\xi \right|, \quad x \in \mathbb{R}, \tag{3.12}$$

$$\hat{h}_2(x) \overset{\mathrm{def}}{=} \frac{1}{\sqrt{q_1(x)}} \sup_{t\in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi)\,\mathrm{d}\xi \right|, \quad x \in \mathbb{R}. \tag{3.13}$$

Then, if

$$\hat{h}_1(x) \to 0, \quad \hat{h}_2(x) \to 0 \quad \text{as } |x| \to \infty, \tag{3.14}$$

then for every $x \in \mathbb{R}$ equation (2.3) has a unique positive solution $d(x)$. Moreover, we have

$$d(x) = \frac{1+\varepsilon(x)}{\sqrt{q_1(x)}}, \quad |\varepsilon(x)| \leqslant 2(\hat{h}_1(x) + \hat{h}_2(x)) \quad \text{for } |x| \gg 1, \tag{3.15}$$

$$\frac{c^{-1}}{\sqrt{q_1(x)}} \leqslant d(x) \leqslant \frac{c}{\sqrt{q_1(x)}} \quad \text{for } x \in \mathbb{R}. \tag{3.16}$$

In addition, for $p \in [1,\infty]$ equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ if and only if

$$\inf_{x\in\mathbb{R}} q_1(x) > 0. \tag{3.17}$$

4. Proof of the asymptotic formula for computing integrals of special type at infinity

In this section we prove Theorem 3.1. In the following we assume that its conditions are satisfied and do not include them in the statements of the auxiliary assertions.

Proof of assertion (A) of Theorem 3.1. Both equalities of (3.5) are checked in a similar way. Let us prove, for example, the second one. We need the following lemmas.

Lemma 4.1. *Consider the segments*

$$\sigma(x)=[x,\sigma^+(x)]\overset{\text{def}}{=}\left[x,x+\gamma\frac{\sqrt{s(x)}}{g(x)}\right],\quad \gamma\in(0,\infty),\ x\in\mathbb{R}. \tag{4.1}$$

There exists $\gamma\in(0,\infty)$ such that, for all $|x|\gg 1$, the following relations hold:

$$\sigma(x)\subseteq\varDelta(x)\quad(\textit{see } (3.4)), \tag{4.2}$$

$$2^{-1}g(x)\leqslant g(t)\leqslant 2g(x)\quad \textit{for } t\in\sigma(x). \tag{4.3}$$

Proof. Condition (a) of the theorem implies that

$$s_0>0,\quad s_0=\inf_{x\in\mathbb{R}}s(x). \tag{4.4}$$

Set $\gamma=(2\nu)^{-1}\sqrt{s_0}$ (see (3.4)). Then inclusion (4.2) can be checked directly. Furthermore, using (3.3), (4.2) and (3.4) for $|x|\gg 1$ and $t\in\sigma(x)$, we get

$$\begin{aligned}\left|\frac{1}{g(t)}-\frac{1}{g(x)}\right|&=\left|\int_x^t\frac{g'(\xi)}{g(\xi)^2}\,\mathrm{d}\xi\right|\leqslant\int_x^t\frac{|g'(\xi)|}{g(\xi)^2}\,\mathrm{d}\xi\\&\leqslant\int_x^t\frac{\mathrm{d}\xi}{s(\xi)}\leqslant\frac{\nu(t-x)}{s(x)}\leqslant\frac{\nu}{s(x)}\gamma\frac{\sqrt{s(x)}}{g(x)}\\&=\frac12\sqrt{\frac{s_0}{s(x)}}\frac{1}{g(x)}\leqslant\frac{1}{2g(x)}\quad\Longrightarrow\quad(4.3).\end{aligned}$$

□

Let γ be chosen as in Lemma 4.1. For a given $x\in\mathbb{R}$, we construct the sequence of points $\{x_k\}_{k=1}^\infty$ and segments $\{\sigma_k\}_{k=1}^\infty$ as follows:

$$x_1=x,\quad x_{k+1}=\sigma^+(x_k)=x_k+\gamma\frac{\sqrt{s(x_k)}}{g(x_k)},\quad \sigma_k=\sigma(x_k),\quad k\in N. \tag{4.5}$$

Lemma 4.2. *Let $x\in\mathbb{R}$ and let $\{\sigma_k\}_{k=1}^\infty$ be the segments constructed by (4.5). Then the following relations hold:*

$$\sigma_k\cap\sigma_{k+1}=x_{k+1},\quad k\in N;\quad [x,\infty)=\bigcup_{k=1}^\infty\sigma_k. \tag{4.6}$$

Proof. The first equality of (4.6) immediately follows from (4.5). We prove the second assertion of (4.6) *ad absurdum*. Assume that it does not hold. Then there is $x_0\in(x,\infty)$ such that $x_k<x_0$ for all $k\in N$. Since, by construction, the sequence $\{x_k\}_{k=1}^\infty$ is monotone increasing, it converges to some $z\leqslant x_0$. Hence,

$$\infty<x_0-x\geqslant\sum_{k=1}^\infty(x_{k+1}-x_k)=\sum_{k=1}^\infty\gamma\frac{\sqrt{s(x_k)}}{g(x_k)}\quad\Longrightarrow\quad\lim_{k\to\infty}\frac{\sqrt{s(x_k)}}{g(x_k)}=0.$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

But since the functions $s(z)$ and $g(z)$ are continuous for $z \in \mathbb{R}$, we conclude that

$$0 = \lim_{k\to\infty} \frac{\sqrt{s(x_k)}}{g(x_k)} = \frac{\sqrt{s(z)}}{g(z)} \neq 0 \quad \Longrightarrow \quad \text{a contradiction.}$$

□

Lemma 4.3. *We have*

$$\int_0^\infty \frac{g(\xi)}{\sqrt{s(\xi)}}\,\mathrm{d}\xi = \infty. \tag{4.7}$$

Proof. Let $x_0 \gg 1$, suppose that for $x \geqslant x_0$ the assertion of Lemma 4.1 holds, and let $\{\sigma_k\}_{k=1}^\infty$ be the segments constructed by (4.5) for $x_1 = x_0$. Then, using Lemma 4.2, (4.3) and (3.4), we get

$$\begin{aligned}\int_{x_0}^\infty \frac{g(\xi)\,\mathrm{d}\xi}{\sqrt{s(\xi)}} &= \sum_{k=1}^\infty \int_{\sigma_k} \frac{g(\xi)\,\mathrm{d}\xi}{\sqrt{s(\xi)}} = \sum_{k=1}^\infty \int_{\sigma_k} \frac{g(\xi)}{g(x_k)} \cdot \frac{g(x_k)}{\sqrt{s(x_k)}} \cdot \sqrt{\frac{s(x_k)}{s(\xi)}}\,\mathrm{d}\xi \\ &\geqslant \sum_{k=1}^\infty \frac{1}{2\sqrt{\nu}} \cdot \frac{g(x_k)}{\sqrt{s(x_k)}} \int_{\sigma_k} \mathrm{d}\xi = \sum_{k=1}^\infty \frac{\gamma}{2\sqrt{\nu}} = \infty \quad \Longrightarrow \quad (4.7).\end{aligned}$$

□

We now obtain (3.5) from (4.7) and (4.4):

$$\infty \geqslant \frac{1}{\sqrt{s_0}} \int_0^\infty g(\xi)\,\mathrm{d}\xi \geqslant \int_0^\infty \frac{g(\xi)}{\sqrt{s(\xi)}}\,\mathrm{d}\xi = \infty.$$

□

Proof of assertion (C) of Theorem 3.1. Equality (3.8) is an immediate consequence of equalities (3.5). □

Proof of assertion (B) of Theorem 3.1. The proof of equality (3.6) is based on the study of integrals (4.8) and (4.9) for $|x| \to \infty$:

$$J_1(x,\alpha) = \int_x^\infty g(t)^\alpha \exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t, \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R}, \tag{4.8}$$

$$J_2(x,\alpha) = \int_{-\infty}^x g(t)^\alpha \exp\biggl(-\int_t^x g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t, \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \tag{4.9}$$

We need some auxiliary assertions.

Lemma 4.4. *There exists an $m \gg 1$ such that, for $|x| \geqslant m$, the following inequalities hold:*

$$J_1(x,\alpha) \leqslant 3g(x)^{\alpha-1}, \qquad J_2(x,\alpha) \leqslant 3g(x)^{\alpha-1}. \tag{4.10}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Proof. Both inequalities are checked in a similar way. Let us prove the first estimate. Condition (a) of the theorem implies that there exists an $x_0 \gg 1$ such that the following inequality holds:

$$\frac{|\alpha-1|}{\tilde{s}(x)} \leqslant \tfrac{1}{2} \quad \text{for } |x| \geqslant x_0,\ \tilde{s}(x) = \inf_{|t|\geqslant|x|} s(t). \tag{4.11}$$

Denote

$$W(x,b) \stackrel{\text{def}}{=} \int_x^b g(t)^\alpha \exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t, \quad b \geqslant x,\ x \in \mathbb{R}. \tag{4.12}$$

Let $b \geqslant x \geqslant x_0$. Integrating by parts, we get

$$\begin{aligned} W(x,b) = g(x)^{\alpha-1} - g(b)^{\alpha-1}\exp\biggl(-\int_x^b g(\xi)\,\mathrm{d}\xi\biggr) \\ + (\alpha-1)\int_x^b g'(t)g(t)^{\alpha-2}\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t. \end{aligned} \tag{4.13}$$

From (4.13), (3.3) and (4.11) it follows that

$$\begin{aligned} W(x,b) &\leqslant g(x)^{\alpha-1} + |\alpha-1|\int_x^b \frac{|g'(t)|}{g(t)^2} g(t)^\alpha \exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t &(4.14)\\ &\leqslant g(x)^{\alpha-1} + |\alpha-1|\int_x^b \frac{g(t)^\alpha}{s(t)} \exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t \\ &\leqslant g(x)^{\alpha-1} + \frac{|\alpha-1|}{\tilde{s}(x)} W(x,b) &(4.15)\\ &\leqslant g(x)^{\alpha-1} + \tfrac{1}{2}W(x,b). &(4.16) \end{aligned}$$

Then from (4.16), we get

$$W(x,b) \leqslant 2g(x)^{\alpha-1} \quad \text{for } b \geqslant x \geqslant x_0. \tag{4.17}$$

Thus, the integral $J_1(x,\alpha)$ converges, and we have the following estimate:

$$J_1(x,\alpha) = \lim_{b\to\infty} W(x,b) \leqslant 2g(x)^{\alpha-1}, \quad x \geqslant x_0. \tag{4.18}$$

Furthermore, repeating the proof of (4.17) for $x \leqslant -x_0$, we get

$$\begin{aligned} &W(x,-x_0) \leqslant g(x)^{\alpha-1} + \frac{|\alpha-1|}{\tilde{s}(x_0)} W(x,-x_0) \leqslant g(x)^{\alpha-1} + \tfrac{1}{2}W(x,-x_0) \quad \Longrightarrow \\ &W(x,-x_0) \leqslant 2g(x)^{\alpha-1} \quad \text{for } x \leqslant -x_0. \end{aligned} \tag{4.19}$$

Below, we will need estimates for $g(x)^{\alpha-1}$ for $x \leqslant -x_0$. To obtain them, we use (3.3). Let $\xi \in [x,-x_0]$. Then

$$\begin{aligned} &-\frac{g(\xi)}{s(\xi)} \leqslant \frac{g'(\xi)}{g(\xi)} \leqslant \frac{g(\xi)}{s(\xi)} \quad \Longrightarrow \\ &g(-x_0)\exp\biggl(-\int_x^{-x_0} \frac{g(\xi)}{s(\xi)}\,\mathrm{d}\xi\biggr) \leqslant g(x) \leqslant g(-x_0)\exp\biggl(\int_x^{-x_0} \frac{g(\xi)}{s(\xi)}\,\mathrm{d}\xi\biggr), \quad x \leqslant -x_0. \end{aligned} \tag{4.20}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Denote

$$c_1(x_0)=\max\{g(-x_0)^{-|\alpha-1|},g(-x_0)^{|\alpha-1|}\}. \tag{4.21}$$

From (4.20) and (4.21), it follows that

$$c_1(x_0)^{-1}\exp\biggl(-|\alpha-1|\int_x^{-x_0}\frac{g(\xi)}{s(\xi)}\,\mathrm{d}\xi\biggr)\leqslant g(x)^{\alpha-1}\leqslant c_1(x_0)\exp\biggl(|\alpha-1|\int_x^{-x_0}\frac{g(\xi)}{s(\xi)}\,\mathrm{d}\xi\biggr),\quad x\leqslant -x_0. \tag{4.22}$$

The required inequalities for $g(x)^{\alpha-1}$ now follow from (4.22) and (4.23):

$$c_1(x_0)^{-1}\exp\biggl(-\frac{1}{2}\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\leqslant g(x)^{\alpha-1}\leqslant c_1(x_0)\exp\biggl(\frac{1}{2}\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr),\quad x\leqslant -x_0. \tag{4.23}$$

In the next estimate of $J_1(x,d)$ for $x\leqslant -x_0$, we use (4.18) and (4.19):

$$\begin{aligned}J_1(x,\alpha)&=\int_x^{-x_0}g(t)^\alpha\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t+\int_{-x_0}^{x_0}g(t)^\alpha\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&\quad+\int_{x_0}^{\infty}g(t)^\alpha\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&=W(x,-x_0)+\exp\biggl(-\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\int_{-x_0}^{x_0}g(t)^\alpha\exp\biggl(-\int_{x_0}^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&\quad+\exp\biggl(-\int_x^{x_0}g(\xi)\,\mathrm{d}\xi\biggr)J_1(x_0,\alpha)\\&\leqslant 2g(x)^{\alpha-1}+\exp\biggl(-\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\\&\quad\times\biggl[\int_{-x_0}^{x_0}g(t)^\alpha\exp\biggl(-\int_{x_0}^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t+2g(x_0)^{\alpha-1}\exp\biggl(-\int_{-x_0}^{x_0}g(\xi)\,\mathrm{d}\xi\biggr)\biggr].\end{aligned} \tag{4.24}$$

Denote by $c_2(x_0)$ the constant in brackets from (4.24) and set $c(x_0)=c_1(x_0)c_2(x_0)$ (see (4.21)). From (4.24) and (4.23), for $x\leqslant -x_0$, we now get

$$J_1(x,\alpha)\leqslant g(x)^{\alpha-1}\biggl\{2+c(x_0)\exp\biggl(-\frac{1}{2}\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\biggr\},\quad x\leqslant -x_0. \tag{4.25}$$

According to (3.5), there exists $m\gg x_0$ such that

$$c(x_0)\exp\biggl(-\frac{1}{2}\int_x^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\leqslant 1. \tag{4.26}$$

Therefore, for $x\leqslant -m$, using (4.25) and (4.26), we get

$$J_1(x,\alpha)\leqslant g(x)^{\alpha-1}\biggl\{2+c(x_0)\exp\biggl(-\frac{1}{2}\int_{-m}^{-x_0}g(\xi)\,\mathrm{d}\xi\biggr)\biggr\}\leqslant 3g(x)^{\alpha-1}.$$

Since (4.18) obviously holds for $x\geqslant m\gg x_0$, the lemma is proved. □

Corollary 4.5. *For $x \in \mathbb{R}$, the following estimate holds:*

$$J_1(x,\alpha) \leqslant c(\alpha) g(x)^{\alpha-1}, \quad \alpha \in \mathbb{R}. \tag{4.27}$$

Proof. Let m be chosen as in Lemma 4.4. The functions $J_1(x,\alpha)$ and $g(x)^{\alpha-1}$ are continuous and positive for $x \in \mathbb{R}$, and therefore the function $J_1(x,\alpha)g(x)^{1-\alpha}$ attains its maximum M on the segment $[-m, m]$. Then by (4.10) we conclude that (4.27) holds for $c(\alpha) = \max\{3, M\}$. □

Lemma 4.6. *Let $|x| \gg 1$, $t \in [x, \Delta^+(x)]$. Then (see (3.4))*

$$\exp\biggl(-\frac{\nu}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi\biggr) \leqslant \frac{g(t)}{g(x)} \leqslant \exp\biggl(\frac{\nu}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi\biggr). \tag{4.28}$$

Proof. From (3.2), we get

$$\Delta^+(x) = x + \frac{s(x)}{g(x)} = x\biggl[1 + \frac{s(x)}{xg(x)}\biggr] \longrightarrow \begin{cases} \infty, & \text{if } x \to \infty, \\ -\infty, & \text{if } x \to -\infty. \end{cases} \tag{4.29}$$

Therefore, for all $|x| \gg 1$ for $\xi \in [x, \Delta^+(x)]$, we can apply (3.3) and (3.4):

$$-\frac{\nu g(\xi)}{s(x)} \leqslant -\frac{g(\xi)}{s(\xi)} \leqslant \frac{g'(\xi)}{g(\xi)} \leqslant \frac{g(\xi)}{s(\xi)} \leqslant \nu\frac{g(\xi)}{s(x)} \quad \Longrightarrow$$

$$-\frac{\nu}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi \leqslant \ln\frac{g(t)}{g(x)} \leqslant \frac{\nu}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi \quad \Longrightarrow \quad (4.28).$$

□

Lemma 4.7. *For $|x| \gg 1$, the following inequality holds (see (3.4)):*

$$\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi \geqslant \frac{s(x)}{\nu}\ln(1+\nu). \tag{4.30}$$

Proof. For $t \in [x, \Delta^+(x)]$, using (4.28) we get

$$\nu\frac{g(x)}{s(x)} \leqslant \frac{\nu g(t)}{s(x)}\exp\biggl(\frac{\nu}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi\biggr) \quad \Longrightarrow$$

$$\nu = \nu\frac{g(x)(\Delta^+(x)-x)}{s(x)} \leqslant \int_x^{\Delta^+(x)} \frac{\nu g(t)}{s(x)}\exp\biggl(\int_x^t \frac{\nu g(\xi)}{s(x)}\,\mathrm{d}\xi\biggr)\,\mathrm{d}t$$

$$= \exp\biggl(\frac{\nu}{s(x)}\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\biggr) - 1 \quad \Longrightarrow$$

$$\exp\biggl(\frac{\nu}{s(x)}\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\biggr) \geqslant \nu + 1 \quad \Longrightarrow \quad (4.30).$$

□

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

In the next lemma we consider the integral

$$\tilde J_1(x,\alpha)\stackrel{\text{def}}{=}\int_x^{\Delta^+(x)} g(t)^\alpha\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t,\quad x\in\mathbb{R}.$$

Lemma 4.8. *For all $|x|\gg 1$, the following relations hold:*

$$\tilde J_1(x,\alpha)=g(x)^{\alpha-1}(1+\delta(x)),\tag{4.31}$$

$$|\delta(x)|\leqslant\frac{c(\nu,\alpha)}{s(x)},\quad c(\nu,\alpha)\leqslant 1+2\nu|\alpha-1|.\tag{4.32}$$

Proof. According to (4.28), for $|x|\gg 1$ and $t\in[x,\Delta^+(x)]$ we have

$$\exp\biggl(-\frac{\nu|\alpha-1|}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\leqslant\biggl(\frac{g(t)}{g(x)}\biggr)^{\alpha-1}\leqslant\exp\biggl(\frac{\nu|\alpha-1|}{s(x)}\int_x^t g(\xi)\,\mathrm{d}\xi\biggr).\tag{4.33}$$

Furthermore, for all $|x|\gg 1$, obviously the following inequalities hold:

$$\tfrac12\geqslant\frac{1+\nu|\alpha-1|}{s(x)},\quad s(x)\geqslant 4.\tag{4.34}$$

Therefore, from (4.33) and (4.34) it follows that

$$\begin{aligned}\tilde J_1(x,\alpha)&=g(x)^{\alpha-1}\int_x^{\Delta^+(x)}\biggl(\frac{g(t)}{g(x)}\biggr)^{\alpha-1}g(t)\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&\leqslant g(x)^{\alpha-1}\int_x^{\Delta^+(x)}g(t)\exp\biggl(-\biggl(1-\frac{\nu|\alpha-1|}{s(x)}\biggr)\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&=g(x)^{\alpha-1}\biggl(1-\frac{\nu|\alpha-1|}{s(x)}\biggr)^{-1}\biggl[1-\exp\biggl(-\biggl(1-\frac{\nu|\alpha-1}{s(x)}\biggr)\int_x^{\Delta^+(x)}g(\xi)\,\mathrm{d}\xi\biggr)\biggr]\\&\leqslant g(x)^{\alpha-1}\biggl(1-\frac{\nu|\alpha-1|}{s(x)}\biggr)\\&\leqslant g(x)^{\alpha-1}\biggl[1+\frac{2\nu|\alpha-1|}{s(x)}\biggr].\end{aligned}\tag{4.35}$$

Similarly, using (4.33), (4.34), (4.30) and well-known elementary inequalities (see [**1**, Chapter 4, § 14], we get

$$\begin{aligned}\tilde J_1(x,\alpha)&=g(x)^{\alpha-1}\int_x^{\Delta^+(x)}\biggl(\frac{g(t)}{g(x)}\biggr)^{\alpha-1}g(t)\exp\biggl(-\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\&\geqslant g(x)^{\alpha-1}\int_x^{\Delta^+(x)}g(t)\exp\biggl(-\biggl(1+\frac{\nu|\alpha-1|}{s(x)}\biggr)\int_x^t g(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= g(x)^{\alpha-1}\left(1+\frac{\nu|\alpha-1|}{s(x)}\right)^{-1}\left[1-\exp\left(-\left(1+\frac{\nu|\alpha-1|}{s(x)}\right)\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)\right]\\
&\geqslant g(x)^{\alpha-1}\left(1+\frac{\nu|\alpha-1|}{s(x)}\right)^{-1}\left[1-\frac{1}{(1+\nu)^{s(x)/\nu}}\right]\\
&\geqslant g(x)^{\alpha-1}\left(1+\frac{\nu|\alpha-1|}{s(x)}\right)^{-1}\left(1-\frac{1}{s(x)}\right)\\
&\geqslant \left(1-\frac{1+\nu|\alpha-1|}{s(x)}\right)g(x)^{\alpha-1}. \qquad (4.36)
\end{aligned}
$$

Relations (4.31) and (4.32) follow from estimates (4.35) and (4.36). □

Lemma 4.9. *For all $|x| \gg 1$, the following relations hold:*

$$J_1(x,\alpha) = g(x)^{\alpha-1}(1+\varepsilon_1(x)), \qquad (4.37)$$

$$|\varepsilon_1(x)| \leqslant \frac{c(\nu,\alpha)}{s(x)}, \quad c(\nu,\alpha) \leqslant 5+2\nu|\alpha-1|. \qquad (4.38)$$

Proof. The following inequalities, where $|x| \gg 1$, are based on Lemma 4.9:

$$
\begin{aligned}
J_1(x,\alpha) &= \int_x^{\infty} g(x)^{\alpha}\exp\left(-\int_x^t g(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\\
&\geqslant \int_x^{\Delta^+(x)} g(t)^{\alpha}\exp\left(-\int_x^t g(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\\
&= \tilde{J}_1(x,\alpha)\\
&= g(x)^{\alpha-1}(1+\delta(x)). \qquad (4.39)
\end{aligned}
$$

Furthermore, to prove the upper estimate of $J_1(x,\alpha)$, we use relations (4.31), (4.32), (4.27), (4.33) and (4.34):

$$
\begin{aligned}
J_1(x,\alpha) &= \tilde{J}_1(x,\alpha) + \exp\left(-\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)J_1(\Delta^+(x),\alpha)\\
&\leqslant g(x)^{\alpha-1}(1+\delta(x)) + c(\alpha)g(\Delta^+(x))^{\alpha-1}\exp\left(-\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)\\
&= g(x)^{\alpha-1}\left[1+\delta(x)+c(\alpha)\left(\frac{g(\Delta^+(x))}{g(x)}\right)^{\alpha-1}\exp\left(-\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)\right]\\
&\leqslant g(x)^{\alpha-1}\left[1+\delta(x)+c(\alpha)\exp\left(-\left(1-\frac{\nu|\alpha-1|}{s(x)}\right)\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)\right]\\
&\leqslant g(x)^{\alpha-1}\left[1+\delta(x)+c(\alpha)\exp\left(-\frac{1}{2}\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right)\right]. \qquad (4.40)
\end{aligned}
$$

From (4.30), it follows that, for $|x| \gg 1$, the following inequality holds:

$$c(\alpha)\exp\left(-\frac{1}{4}\int_x^{\Delta^+(x)} g(\xi)\,\mathrm{d}\xi\right) \leqslant 1. \qquad (4.41)$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Therefore, using (4.41) and (4.30) for $|x| \gg 1$, one can continue estimate (4.40):

$$\begin{aligned} J_1(x,\alpha) &\leqslant g(x)^{\alpha-1}\left[1+\delta(x)+\exp\left(-\frac{1}{4}\int_x^{\Delta^{(x)}} g(\xi)\,\mathrm{d}\xi\right)\right] \\ &\leqslant g(x)^{\alpha-1}\left[1+\delta(x)+\frac{1}{(1+\nu)^{s(x)/4\nu}}\right] \\ &\leqslant g(x)^{\alpha-1}\left[1+\delta(x)+\frac{4}{s(x)}\right] \\ &\leqslant g(x)^{\alpha-1}\left(1+\frac{5+2\nu|\alpha-1|}{s(x)}\right). \end{aligned} \tag{4.42}$$

The assertion of the lemma follows from (4.39) and (4.42). □

Remark 4.10. For $x \to \infty$, Lemma 4.9 follows from the assertions given in [**8**, Chapter II, § 2, no. 4]. The proof of (4.37) given above works both for $x \to \infty$ and for $x \to -\infty$.

To prove assertion (B) of Theorem 3.1, note that one can prove the following lemma in a similar way to Lemma 4.9,

Lemma 4.11. *For all $|x| \gg 1$, the following relations hold (see (4.9)):*

$$J_2(x,\alpha) = g(x)^{\alpha-1}(1+\varepsilon_2(x)), \tag{4.43}$$

$$|\varepsilon_2(x)| \leqslant \frac{c(\nu,\alpha)}{s(x)}, \quad c(\nu,\alpha) \leqslant 5+2\nu|\alpha-1|. \tag{4.44}$$

From Lemmas 4.9 and 4.11 and the obvious equality (see (3.11)),

$$F(x,\alpha) = J_1(x,\alpha) + J_2(x,\alpha), \quad x \in \mathbb{R},$$

we get assertion (B). The theorem is proved. □

5. Proof of the theorem on an asymptotic majorant

In this section we prove Theorem 3.3. From now on we assume that its hypotheses are fulfilled and we do not include them in the statements of the auxiliary assertions. Without additional comments, we denote by $k(x)$ a function satisfying the requirements of Definition 3.2. To prove equality (1.5), we need the following lemma.

Lemma 5.1. *For any $\alpha > 0$, for all $|x| \gg 1$, the following relations hold:*

$$G_\alpha(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t)^\alpha\,\mathrm{d}t = \frac{d(x)^{\alpha+1}}{\alpha 2^{\alpha-1}}(1+\varepsilon(x)), \tag{5.1}$$

$$\sqrt{k(x)}|\varepsilon(x)| \leqslant c(\alpha). \tag{5.2}$$

Proof. From (2.1), for $G_\alpha(x)$ we get

$$G_\alpha(x) = \rho(x)^{\alpha/2}\int_{-\infty}^{\infty} \rho(t)^{\alpha/2}\exp\left(-\frac{\alpha}{2}\left|\int_x^t \frac{\mathrm{d}\xi}{\rho(\xi)}\right|\right)\mathrm{d}t, \quad x \in \mathbb{R}. \tag{5.3}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

By (2.18), in order to prove (5.1), (5.2), it suffices to obtain an asymptotic formula for computing the integral $F(x,\alpha)$ (see (3.1)) with the following value of $g(x)$:

$$g(x)=\frac{\alpha}{2}\frac{1}{\rho(x)},\quad x\in\mathbb{R}. \tag{5.4}$$

Let us verify that in this case one can apply Theorem 3.1. From (5.4) and (2.19), we get, for all $|x|\gg 1$,

$$\frac{|g'(x)|}{g(x)^2}=\frac{2}{\alpha}|\rho'(x)|\leqslant\frac{2c}{\alpha}\frac{1}{\sqrt{k(x)}}. \tag{5.5}$$

Therefore, in connection with Theorem 3.1, we set

$$s(x)\overset{\text{def}}{=}\frac{\alpha\sqrt{k(x)}}{2c},\quad x\in\mathbb{R}. \tag{5.6}$$

Clearly, according to Definitions 3.2 and 2.13, the function $s(x)$ satisfies condition (a) of Theorem 3.1, and by (5.5) and (5.6) condition (c) also holds. Let us prove that conditions (b) and (d) also hold. Condition (b) is checked with the help of (5.4), (5.6), (2.13) and (3.9):

$$0<\frac{s(x)}{|x|g(x)}=\frac{\sqrt{k(x)}\rho(x)}{c|x|}\leqslant\frac{3\alpha}{2c}\frac{\sqrt{k(x)}d(x)}{|x|}\to 0\quad\text{as } |x|\to\infty\quad\Longrightarrow\quad\text{(b)}.$$

To check condition (d), note that from (2.13) and Definition 2.13 (1) it follows that, for all $|x|\gg 1$, we have the inclusions

$$\begin{aligned}\Delta(x)&=\left[x-\frac{s(x)}{g(x)},x+\frac{s(x)}{g(x)}\right]\\&=\left[x-\frac{\sqrt{k(x)}\rho(x)}{c},x+\frac{\sqrt{k(x)}\rho(x)}{c}\right]\\&\subseteq\left[x-\frac{3}{2c}\sqrt{k(x)}d(x),x+\frac{3}{2c}\sqrt{k(x)}d(x)\right]\\&\subseteq[x-k(x)d(x),x+k(x)d(x)].\end{aligned}$$

Therefore, using (2.16) we conclude that, for all $|x|\gg 1$ and $t\in\Delta(x)$, the following inequalities hold:

$$\frac{1}{\sqrt{a}}\leqslant\sqrt{\frac{k(t)}{k(x)}}=\sqrt{\frac{s(t)}{s(x)}}\leqslant\sqrt{a}\quad\Longrightarrow\quad\text{(d)}.$$

Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence, by (5.3), (5.4), (3.6) and (2.18), we have (see (5.4))

$$\begin{aligned}G_\alpha(x)&=\rho(x)^{\alpha/2}\int_{-\infty}^{\infty}\rho(t)^{\alpha/2}\exp\biggl(-\frac{\alpha}{2}\biggl|\int_x^t\frac{\mathrm{d}\xi}{\rho(\xi)}\biggr|\biggr)\,\mathrm{d}t\\&=\rho(x)^{\alpha/2}\int_{-\infty}^{\infty}\biggl(\frac{\alpha}{2}\frac{1}{g(t)}\biggr)^{\alpha/2}\exp\biggl(-\biggl|\int_x^t g(\xi)\,\mathrm{d}\xi\biggr|\biggr)\,\mathrm{d}t\end{aligned}$$

$$= 2\left(\frac{\alpha}{2}\right)^{\alpha/2} \rho(x)^{\alpha/2} g(x)^{-(\alpha/2)-1}\left(1 + O\left(\frac{1}{s(x)}\right)\right)$$
$$= \frac{4}{\alpha}\rho(x)^{\alpha+1}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right)$$
$$= \frac{4}{\alpha}\left(\frac{d(x)}{2}\right)^{\alpha+1}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right)$$
$$= \frac{d(x)^{\alpha+1}}{\alpha 2^{\alpha-1}}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right).$$

□

Let us now check (1.5). By Lemma 2.8 for $p \in [1, \infty]$ and $f \in L_p(\mathbb{R})$, the solution $y \in L_p(\mathbb{R})$ of (1.1) is of the form (2.11). Fix $x \in \mathbb{R}$. Then $y(x) = (Tf)(x)$, where T is the linear functional defined on $L_p(\mathbb{R})$ according to (2.11):

$$y(x) = (Tf)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t) f(t)\,\mathrm{d}t, \quad f \in L_p(\mathbb{R}). \tag{5.7}$$

From Lemma 2.12, It follows that the functional T is continuous. Therefore, according to general statements on the properties of linear continuous functionals defined on $L_p(\mathbb{R})$ (see [**9**, Chapter V, §§ 2.2, 2.3, Chapter VI, § 2]), we get

$$\sup_{y \in \mathcal{D}_p} |y(x)| = \|T\| = \begin{cases} \operatorname*{ess\,sup}_{t\in\mathbb{R}} G(x,t), & \text{for } p = 1, \\ \left(\int_{-\infty}^{\infty} G(x,t)^{p'}\,\mathrm{d}t\right)^{1/p'}, \quad p' = \dfrac{p}{p-1}, & \text{for } p \in (1,\infty), \\ \int_{-\infty}^{\infty} G(x,t)\,\mathrm{d}t, & \text{for } p = \infty. \end{cases} \tag{5.8}$$

From Lemma 5.1 and (5.8) for $p \in (1, \infty)$ and $|x| \gg 1$, it follows that

$$\sup_{y\in\mathcal{D}_p} |y| = \|T\| = \left(\int_{-\infty}^{\infty} G(x,t)^{p'}\,\mathrm{d}t\right)^{1/p'}$$
$$= \left[\frac{d(x)^{p'+1}}{p' 2^{p'-1}}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right)\right]^{1/p'}$$
$$= \frac{d(x)^{2-1/p}}{(p')^{1/p'} 2^{1/p}}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right)$$
$$= \frac{\ell(p)}{q^*(x)^{1-1/2p}}\left(1 + O\left(\frac{1}{\sqrt{k(x)}}\right)\right) \implies (1.5). \tag{5.9}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Let now $p = 1$. First note that according to (2.10) and Lemma 2.5, the following relations hold:

$$\begin{aligned} G(x,t) &= \begin{cases} u(x)v(t), & x \geqslant t, \\ u(t)v(x), & x \leqslant t \end{cases} \\ &= \rho(x) \begin{cases} \dfrac{v(t)}{v(x)}, & x \geqslant t, \\ \dfrac{u(t)}{u(x)}, & x \leqslant t \end{cases} \\ &\leqslant \rho(x), \quad x \in \mathbb{R} \quad \Longrightarrow \quad \operatorname*{ess\,sup}_{t\in\mathbb{R}} G(x,t) = \rho(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5.10}$$

Then (1.5) follows from (5.7), (5.8), (5.10) and (2.18). To prove (1.5) for $p = \infty$, we have (according to (5.8)) to repeat the chain of computation (5.9) with $p' = 1$. Equation (1.5) is thus proved. The assertion of the theorem on the representation of the asymptotic majorant $\varkappa_\rho(x)$ of solutions of (1.1) in the form (1.7) is a standard consequence of equality (1.5) and the definitions of limit and supremum. □

6. Proof of the asymptotic formula for computing Otelbaev's function for $|x| \to \infty$

Below we prove Theorem 3.4. We need the following lemma.

Lemma 6.1. *Under the conditions of Theorem 3.4, we have (see (2.21) and (3.12))*

$$h_1(x) \leqslant 2\hat{h}_1(x), \quad x \in \mathbb{R}. \tag{6.1}$$

Proof. The following relations are based only on the definition of $h_1(x)$ (see (2.21)),

$$\begin{aligned} h_1(x) &= \frac{1}{\sqrt{q_1(x)}} \sup_{t\in A(x)} \left| \int_0^t (q_1(x+s) - 2q_1(x) + q_1(x-s))\,\mathrm{d}s \right| \\ &= \frac{1}{\sqrt{q_1(x)}} \sup_{t\in A(x)} \left| \int_0^t \int_0^s \int_{x-\xi}^{x+\xi} q_1''(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}s \right| \\ &\leqslant \frac{1}{\sqrt{q_1(x)}} \sup_{\xi\in A(x)} \left| \int_{x-\xi}^{x+\xi} q_1''(\tau)\,\mathrm{d}\tau \right| \cdot \sup_{t\in A(x)} \int_0^t \int_0^s \mathrm{d}\xi\,\mathrm{d}s \\ &= 2\hat{h}_1(x). \end{aligned}$$

Equality (3.15) is implied now from (6.1) and Theorem 2.17. To prove (3.16), we note that these inequalities hold when $x \notin [-c, c]$, $c \gg 1$ (see (3.15)). The function

$$f(x) \overset{\text{def}}{=} d(x)q_1(x), \quad x \in [-c, c],$$

is continuous and positive for $x \in [-c, c]$ (see Lemma 2.16). Hence, its minimum m and maximum M on the segment $[-c, c]$ are finite positive numbers. Let $c_1 =$

$\max\{c, m^{-1}, M\}$. Then $c_1^{-1} \leqslant d(x)q_1(x) \leqslant c_1$ for $x \in \mathbb{R}$. Finally, criterion (3.17) for correct solvability of equation (1.1) in the spaces $L_p(\mathbb{R})$, $p \in [1, \infty]$, follows from (3.6) and Theorem 2.2. □

7. Examples

In this section, we give examples of applications of Theorems 3.1, 3.3 and 3.4.

Example 7.1. Using Theorem 3.1, we find an estimate (for $|x| \to \infty$) of the integral $F(x, \alpha)$ with $g(x) = \exp(x^2)$, $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$ (see (3.1)). In this case the function $g(x)$ is positive and differentiable for $x \in \mathbb{R}$ and, in addition,

$$\frac{|g'(x)|}{g(x)^2} = \frac{2|x|}{\mathrm{e}^{x^2}}, \quad x \in \mathbb{R}. \tag{7.1}$$

Let us verify that the function

$$s(x) = \frac{\mathrm{e}^{x^2}}{8\sqrt{1+x^2}}, \quad x \in \mathbb{R},$$

satisfies conditions (a)–(d) of Theorem 3.1. Indeed, $s(x)$ is continuous and positive for $x \in \mathbb{R}$, $s(x) \to \infty$ as $|s| \to \infty$, and

$$\lim_{|x|\to\infty} \frac{s(x)}{xg(x)} = \lim_{|x|\to\infty} \frac{1}{8x\sqrt{1+x^2}} = 0,$$

$$\frac{1}{s(x)} = \frac{8\sqrt{1+x^2}}{\mathrm{e}^{x^2}} \geqslant \frac{2|x|}{\mathrm{e}^{x^2}} = \frac{|g'(x)|}{g(x)^2}, \quad x \in \mathbb{R}.$$

Hence, conditions (a)–(c) of the theorem hold and it remains to check that condition (d) holds. Let us find $\Delta(x)$ (see (3.4)):

$$\Delta(x) = \left[x - \frac{s(x)}{g(x)}, x + \frac{s(x)}{g(x)}\right] = \left[x - \frac{1}{8\sqrt{1+x^2}}, x + \frac{1}{8\sqrt{1+x^2}}\right]. \tag{7.2}$$

Let $t \in \Delta(x)$. Then, by Lagrange's formula, we have

$$s(t) = s(x) + s'(\xi)(t - x), \quad x \in \mathbb{R}, \tag{7.3}$$

where ξ lies between t and x. Let

$$M(x) = \max_{t\in\Delta(x)} \frac{s(t)}{s(x)}, \qquad m(x) = \min_{t\in\Delta(x)} \frac{s(t)}{s(x)}. \tag{7.4}$$

Furthermore, with our choice of $s(x)$ we get

$$\frac{|s'(x)|}{s(x)} = \left|2x - \frac{x}{1+x^2}\right| = |x|\left|2 - \frac{1}{1+x^2}\right| \leqslant 2|x|, \quad x \in \mathbb{R}. \tag{7.5}$$

Therefore, from (7.3)–(7.5) it follows that

$$
\begin{aligned}
M(x) &= \max_{t\in\Delta(x)} |1 + \frac{s'(\xi)}{s(x)}(t-x)| \\
&\leqslant 1 + \max_{t\in\Delta(x)} \frac{|s'(\xi)|}{s(\xi)} \cdot \frac{s(\xi)}{s(x)} \cdot \frac{1}{8\sqrt{1+x^2}} \\
&\leqslant 1 + \frac{M(x)}{8\sqrt{1+x^2}} \max_{\xi\in\Delta(x)} 2|\xi| \\
&\leqslant 1 + \frac{M(x)}{4\sqrt{1+x^2}} \left[|x| + \frac{1}{8\sqrt{1+x^2}}\right] \\
&= 1 + \frac{M(x)}{4}\left[\frac{|x|}{\sqrt{1+x^2}} + \frac{1}{8(1+x^2)}\right] \\
&\leqslant 1 + \frac{M(x)}{4}(1+\tfrac{1}{8}) = 1 + \tfrac{9}{32}M(x) \quad \Longrightarrow \quad M(x) \leqslant \tfrac{32}{23}, \quad x\in\mathbb{R}. \qquad (7.6)
\end{aligned}
$$

Let us now use (7.6) to estimate $m(x)$:

$$
\begin{aligned}
m(x) &= \min_{t\in\Delta(x)} \left|1 + \frac{s'(x)}{s(x)}(t-x)\right| \\
&\geqslant \min_{t\in\Delta(x)} \left[1 - \frac{|s'(\xi)|}{s(\xi)} \cdot \frac{s(\xi)}{s(x)}|t-x|\right] \\
&\geqslant 1 - \frac{M(x)}{8\sqrt{1+x^2}} \max_{\xi\in\Delta(x)} 2|\xi| \\
&\geqslant 1 - \frac{8}{23}\left[\frac{|x|}{\sqrt{1+x^2}} + \frac{1}{8(1+x^2)}\right] \\
&\geqslant 1 - \tfrac{8}{23}(1+\tfrac{1}{8}) = \tfrac{14}{23} > \tfrac{1}{2}, \quad x\in\mathbb{R}. \qquad (7.7)
\end{aligned}
$$

Thus, using (7.6) and (7.7), we get

$$
\tfrac{1}{2} \leqslant m(x) \leqslant \frac{s(t)}{s(x)} \leqslant M(x) \leqslant 2, \quad t\in\Delta(x),\ x\in\mathbb{R}. \qquad (7.8)
$$

Since all the hypotheses of Theorem 3.1 are satisfied, we conclude that, for $\alpha\in\mathbb{R}$ and all $|x|\gg 1$, the following equality holds:

$$
\begin{aligned}
F(x,\alpha) &= \int_{-\infty}^{\infty} \exp(\alpha t^2) \exp\left(-\left|\int_x^t \exp(\xi^2)\,\mathrm{d}\xi\right|\right) \mathrm{d}t \\
&= 2\exp((\alpha-1)x^2)(1 + O(x\exp(-x^2))). \qquad (7.9)
\end{aligned}
$$

We emphasize that the constant in the symbol $O(\cdot)$ depends only on α and is absolute.

Example 7.2. Using Theorems 3.3 and 3.4, we find an asymptotic majorant $\varkappa_p(x)$ for solutions of equation (1.1) with coefficient

$$
q(x) = \mathrm{e}^{x^2} + \mathrm{e}^{x^2}\cos\mathrm{e}^{x^2}, \quad x\in\mathbb{R}. \qquad (7.10)
$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

Let us first establish correct solvability of this equation in $L_p(\mathbb{R})$, $p \in [1,\infty]$. To do this, let us find asymptotic estimates of $d(x)$ for $|x| \to \infty$. According to (3.10), set

$$q_1(x) = \mathrm{e}^{x^2}, \quad q_2(x) = \mathrm{e}^{x^2} \cos \mathrm{e}^{x^2}, \quad x \in \mathbb{R}. \tag{7.11}$$

Below we repeatedly use the following inequalities:

$$c^{-1} \leqslant \frac{q_1(t)}{q_1(x)} = \frac{\mathrm{e}^{t^2}}{\mathrm{e}^{x^2}} \leqslant c \qquad \text{for } t \in \tilde{\Delta}(x),\ x \in \mathbb{R}, \tag{7.12}$$

$$c^{-1} \leqslant \frac{q_1''(t)}{q_1''(x)} = \frac{1+2t^2}{1+2x^2}\frac{\mathrm{e}^{t^2}}{\mathrm{e}^{x^2}} \leqslant c \quad \text{for } t \in \tilde{\Delta}(x),\ x \in \mathbb{R}, \tag{7.13}$$

$$c^{-1} \leqslant \frac{|t|}{|x|} \leqslant c \qquad \text{for } t \in \tilde{\Delta}(x),\ |x| \gg 1, \tag{7.14}$$

$$c^{-1} \leqslant \frac{1+t^2}{1+x^2} \leqslant c \qquad \text{for } t \in \tilde{\Delta}(x),\ x \in \mathbb{R}. \tag{7.15}$$

Here

$$\tilde{\Delta}(x) = \left[x - \frac{4(1+x^2)}{\mathrm{e}^{x^2/2}}, x + \frac{4(1+x^2)}{\mathrm{e}^{x^2/2}}\right], \quad x \in \mathbb{R}.$$

Elementary inequalities (7.12)–(7.15) are checked for $|x| \gg 1$ in the same way as (7.8), and, on every infinite interval, estimates (7.12), (7.13) and (7.15) follow from continuity and positivity for $x \in \mathbb{R}$ of the functions under consideration. According to Theorem 3.4, we estimate $\hat{h}_1(x)$ and $\hat{h}_2(x)$ for $|x| \to \infty$. We have (see (7.13))

$$\begin{aligned}
\hat{h}_1(x) &= \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi)\,\mathrm{d}\xi \right| \\
&= \frac{2}{\mathrm{e}^{3x^2/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{(1+2\xi^2)\mathrm{e}^{\xi^2}}{(1+2x^2)\mathrm{e}^{x^2}}(1+2x^2)\mathrm{e}^{x^2}\,\mathrm{d}\xi \right| \\
&\leqslant c\frac{1+x^2}{\mathrm{e}^{x^2}}, \quad x \in \mathbb{R} \quad \Longrightarrow \quad \hat{h}_1(x) \leqslant c\frac{1+x^2}{\mathrm{e}^{x^2}}, \quad x \in \mathbb{R}.
\end{aligned}$$

Below, in order to estimate $h_2(x)$ for $|x| \gg 1$, we use the second main theorem (see [**13**, Chapter 12, § 12, no. 3] and (7.14):

$$\begin{aligned}
\hat{h}_2(x) &= \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi)\,\mathrm{d}\xi \right| \\
&= \frac{1}{\mathrm{e}^{x^2/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{2\xi\mathrm{e}^{\xi^2} \cos \mathrm{e}^{\xi^2}\,\mathrm{d}\xi}{2\xi} \right| \\
&\leqslant \frac{c}{|x|\mathrm{e}^{x^2/2}} \sup_{[\alpha,\beta] \subseteq \tilde{\Delta}(x)} \left| \int_{\alpha}^{\beta} 2\xi\mathrm{e}^{\xi^2} \cos \mathrm{e}^{\xi^2}\,\mathrm{d}\xi \right|
\end{aligned}$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

$$= \frac{c}{|x|e^{x^2/2}} \sup_{[\alpha,\beta]\subseteq\tilde{\Delta}(x)} \left| \int_\alpha^\beta d(\sin e^{\xi^2}) \right|$$
$$\leqslant \frac{c}{|x|e^{x^2/2}} \quad \Longrightarrow \quad \hat{h}_2(x) \leqslant \frac{c}{|x|e^{x^2/2}}, \quad |x| \gg 1.$$

Thus, $\hat{h}_1(x) \to 0$, $\hat{h}_2(x) \to 0$ as $|x| \to \infty$, and therefore by Theorem 3.4, we get

$$d(x) = \frac{1+\varepsilon(x)}{e^{x^2/2}}, \quad |\varepsilon(x)| \leqslant \frac{c}{|x|e^{x^2/2}} \quad \text{for } |x| \gg 1, \tag{7.16}$$

$$c^{-1}\frac{1}{e^{x^2/2}} \leqslant d(x) \leqslant \frac{c}{e^{x^2/2}}, \quad x \in \mathbb{R}. \tag{7.17}$$

In addition, since $q_1(x) = e^{x^2} \geqslant 1$ for $x \in \mathbb{R}$, in this case equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ for $p \in [1,\infty]$ (see (3.17)).

In order to apply Theorem 3.3, we now establish that $q \in \tilde{H}$. Set (see Definitions 2.13 and 3.2)

$$k(x) \overset{\text{def}}{=} 2(1+x^2), \quad x \in \mathbb{R}. \tag{7.18}$$

Let us check that with such a choice of $k(x)$ all requirements of Definition 2.13 are satisfied. Clearly, condition (1) of this definition is satisfied by the definition of $k(x)$, and condition (2) is satisfied because of (7.15)–(7.17). It remains to check inequalities (2.17). Denote

$$\Phi_1(x) = k(x)d(x) \sup_{t\in\omega(x)} \left| \int_0^t [q_1(x+\xi) - q_1(x-\xi)]\, d\xi \right|, \qquad x \in \mathbb{R}, \tag{7.19}$$

$$\Phi_2(x) = k(x)d(x)\left[\sup_{t\in\omega(x)} \left| \int_{x-t}^x q_2(\xi)\, d\xi \right| + \sup_{t\in\omega(x)} \left| \int_x^{x+t} q_2(\xi)\, d\xi \right| \right], \quad x \in \mathbb{R}, \tag{7.20}$$

where, as in (2.17), $\omega(x) = [0, k(x)d(x)]$, $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$, the following obvious inequality holds:

$$\Phi(x) = k(x)d(x) \sup_{t\in\omega(x)} \left| \int_0^t [q(x+\xi) - q(x-\xi)]\, d\xi \right| \leqslant \Phi_1(x) + \Phi_2(x), \tag{7.21}$$

and to prove (2.17) it is sufficient to show that the functions $\Phi_1(x)$ and $\Phi_2(x)$ are uniformly bounded for $x \in \mathbb{R}$. The following relations hold for $|x| \gg 1$ because of inequalities (7.12), (7.14) and (7.16):

$$\Phi_1(x) \leqslant \frac{c(1+x^2)}{e^{x^2/2}} \sup_{t\in\omega(x)} \left| \int_0^t [q_1(x+\xi) - q_1(x-\xi)]\, d\xi \right|$$
$$= \frac{c(1+x^2)}{e^{x^2/2}} \sup_{t\in\omega(x)} \left| \int_0^t \int_{x-\xi}^{x+\xi} q_1'(s)\, ds\, d\xi \right|$$
$$= \frac{c(1+x^2)}{e^{x^2/2}} \sup_{t\in\omega(x)} \left| \int_0^t \int_{x-\xi}^{x+\xi} 2se^{s^2}\, ds\, d\xi \right|$$

$$\begin{aligned}
&\leqslant c(1+x^2)|x|\mathrm{e}^{x^2/2}\sup_{t\in\omega(x)}\left|\int_0^t\int_{x-\xi}^{x+\xi}\mathrm{d}s\,\mathrm{d}\xi\right| \\
&= \frac{c|x|(1+x^2)^3}{\mathrm{e}^{x^2/2}} \\
&\leqslant c<\infty, \quad |x|\gg 1. \qquad (7.22)
\end{aligned}$$

Let us now estimate the first summand in (7.19). (The second one is estimated in a similar way.) Now, for $|x|\gg 1$, we use the second mean theorem and (7.16):

$$\begin{aligned}
k(x)d(x)\sup_{t\in\omega(x)}&\left|\int_{x-t}^{x} q_2(\xi)\,\mathrm{d}\xi\right| \\
&\leqslant \frac{c(1+x^2)}{\mathrm{e}^{x^2/2}}\sup_{t\in\omega(x)}\left|\int_{x-t}^{x}\frac{2\xi\mathrm{e}^{\xi^2}\cos\mathrm{e}^{\xi^2}\,\mathrm{d}\xi}{2\xi}\right| \\
&\leqslant \frac{c(1+x^2)}{|x|\mathrm{e}^{x^2/2}}\sup_{[\alpha,\beta]\subseteq\tilde{\Delta}(x)}\left|\int_{\alpha}^{\beta}2\xi\mathrm{e}^{\xi^2}\cos\mathrm{e}^{\xi^2}\,\mathrm{d}\xi\right| \\
&= \frac{c(1+x^2)}{|x|\mathrm{e}^{x^2/2}}\sup_{[\alpha,\beta]\subseteq\tilde{\Delta}(x)}\left|\int_{\alpha}^{\beta}\mathrm{d}(\sin\mathrm{e}^{\xi^2})\right| \\
&\leqslant \frac{c(1+x^2)}{|x|\mathrm{e}^{x^2/2}} \implies \Phi_2(x)\leqslant\frac{c(1+x^2)}{|x|\mathrm{e}^{x^2/2}}\leqslant c<\infty, \quad |x|\gg 1. \qquad (7.23)
\end{aligned}$$

Thus, according to (7.21)–(7.23), there exists $c\gg 1$ such that $\Phi(x)\leqslant c<\infty$ for $|x|\geqslant c$. Since the function $\Phi(x)$ is continuous for $x\in\mathbb{R}$ (see Lemma 2.16), $\Phi(x)$ is bounded on the segment $[-c,c]$ and is hence uniformly bounded for $x\in\mathbb{R}$. Hence, $q\in H$. From (7.16) we conclude that (3.9) holds,

$$\lim_{|x|\to\infty}\frac{\sqrt{k(x)}d(x)}{x}=\lim_{|x|\to\infty}\frac{\sqrt{2(1+x^2)}(1+\varepsilon(x))}{x\mathrm{e}^{x^2/2}}=0,$$

and therefore $q\in\tilde{H}$. According to Theorem 3.3, the asymptotic majorant $\varkappa_p(x)$ of solutions of (1.1) is given by (1.7) for all $p\in[1,\infty]$.

We now need the following simple assertion.

Lemma 7.3. *Suppose that for $p\in[1,\infty]$ equation (1.1) is correctly solvable in $L_p(\mathbb{R})$ and the function $\varkappa_p(x)$ is an asymptotic majorant of its solutions. Then, if a positive and continuous function $\tilde{\varkappa}_p(x)$ for $x\in\mathbb{R}$ satisfies condition (1.8), it is also an asymptotic majorant of solutions of (1.1).*

Proof. Let $\gamma>1$. Choose $\varepsilon>0$ so small that

$$\gamma_1=\gamma(1+\varepsilon)^{-1}>1.$$

From (1.8) it follows that there exists $c_1(\varepsilon)$ such that

$$\varkappa_p(x)\leqslant(1+\varepsilon)\tilde{\varkappa}_p(x) \quad \text{for } |x|\geqslant c_1(\varepsilon).$$

https://doi.org/10.1017/S001309150500074X Published online by Cambridge University Press

From Definition 1.1, we find that there exists $c_2(\gamma_1)$ such that, for all $|x| \geqslant c_2(\gamma_1)$, regardless of $y \in \mathcal{D}_p$, the following inequality holds:

$$|y(x)| \leqslant \gamma_1 \varkappa_p(x).$$

Let $c(\gamma) = \max\{c_1(\varepsilon), c_2(\gamma_1)\}$. Then, for $|x| \geqslant c(\gamma)$, we get

$$|y(x)| \leqslant \gamma_1 \varkappa_p(x) \leqslant \gamma_1(1+\varepsilon)\tilde{\varkappa}_p(x) = \gamma\tilde{\varkappa}_p(x), \quad y \in \mathcal{D}_p,$$

i.e. condition (1) of Definition 1.1 is satisfied.

Consider condition (2) of this definition for $\tilde{\varkappa}_p(x)$. Suppose it is not satisfied. Then there exists $\gamma_0 \in (0,1)$ such that, for all $|x| \geqslant c_1(\gamma_0) \gg 1$, the following inequality holds:

$$|y(x)| \leqslant \gamma_0 \tilde{\varkappa}_p(x) \quad \text{for } y \in \mathcal{D}_p. \tag{7.24}$$

Choose $\varepsilon > 0$ so small that $\gamma_1 = (1+\varepsilon)\gamma_0 < 1$. From (1.8) it follows that there exists a $c_2(\varepsilon)$ such that

$$\tilde{\varkappa}_p(x) \leqslant (1+\varepsilon)\varkappa_p(x) \quad \text{for all } |x| \geqslant c_1(\varepsilon). \tag{7.25}$$

Let $c(\gamma_0) = \max\{c_1(\gamma_0), c_2(\varepsilon)\}$. Then, from (7.23) and (7.24), for all $|x| \geqslant c(\gamma_0)$, regardless of $y \in \mathcal{D}_p$, we get

$$|y(x)| \leqslant \gamma_0 \tilde{\varkappa}_p(x) \leqslant \gamma_0(1+\varepsilon)\varkappa_p(x) = \gamma_1\varkappa_p(x) \quad \Longrightarrow \quad \text{a contradiction.}$$

Condition (2) of Definition 1.1 is also satisfied, and therefore $\tilde{\varkappa}_p(x)$ is an asymptotic majorant of the solutions of (1.1). □

From Lemma 7.3, (7.16) and (1.7), we conclude that the function (see (1.6))

$$\varkappa_p(x) = \frac{\ell(p)}{\mathrm{e}^{(1-(1/2p))x^2}}, \quad x \in \mathbb{R},\ p \in [1, \infty],$$

is an asymptotic majorant of the solutions of equations of (1.1) in the case (7.10).

Acknowledgements. The authors thank Professors Ja. M. Goltser, Z. S. Grinshpun and I. R. Liflyand for useful discussions.

References

1. E. F. Beckenbach and R. Bellman, *Inequalities* (Springer, 1961).
2. N. Chernyavskaya and L. Shuster, Estimates for the Green function of a general Sturm–Liouville operator and their applications, *Proc. Am. Math. Soc.* **127** (1999), 1413–1426.
3. N. Chernyavskaya and L. Shuster, Asymptotics on the diagonal of the Green function of a Sturm–Liouville operator and its applications, *J. Lond. Math. Soc.* **61** (2000), 506–530.
4. N. Chernyavskaya and L. Shuster, Regularity of the inversion problem for the Sturm–Liouville equation in the spaces L_p, *Meth. Applic. Analysis* **7** (2000), 65–84.
5. N. Chernyavskaya and L. Shuster, A criterion for correct solvability of the Sturm–Liouville equation in $L_p(\mathbb{R})$, *Proc. Am. Math. Soc.* **130** (2002), 1043–1054.

6. R. Courant, *Partial differential equations* (Wiley-Interscience, New York, 1962).
7. E. B. Davies and E. M. Harrell, Conformally flat Riemannian metrics, Schrödinger operators and semiclassical approximation, *J. Diff. Eqns* **66** (1987), 165–188.
8. M. V. Fedoryuk, *Asymptotics: integrals and series* (Nauka, Moscow, 1987).
9. L. V. Kantorovich and G. P. Akilov, *Functional analysis* (Nauka, Moscow, 1977).
10. K. Mynbaev and M. Otelbaev, *Weighted functional spaces and the spectrum of differential operators* (Nauka, Moscow, 1988) (in Russian).
11. M. Otelbaev, A criterion for the resolvent of a Sturm–Liouville operator to be kernel, *Math. Notes* **25** (1979), 296–297 (English translation).
12. W. A. Steklov, Sur une méthode nouvelle pour résoudre plusieurs problèmes sur le développement d'une fonction arbitraire en séries infinies, *C. R. Acad. Sci. Paris* **144** (1907), 1329–1332.
13. C. C. Titchmarsh, *The theory of functions* (Oxford University Press, 1932).