Age at entry	Annual premium per cent.			
	$p=0$.	$p=\frac{1}{3}$.	$p=\frac{2}{3}$.	$p=1$.
30	4·881	4·865	4·820	4·769
40	5·856	5·843	5·782	5·697
50	6·738	6·748	6·752	6·745

We see from these figures that the variation in the amount of the yearly premium for different values of p is very trifling, and we see moreover, on comparing these figures with those before found that notwithstanding the considerable change made in the suppositions as to surrender, the annual premium is as nearly as possible the same. We may conclude from these results that it is at least highly probable that in such contracts as we have been considering, the speculative element under any circumstances is extremely small.

When $p_1=1$ and $p_2=0$ we get from (1) and (2) $\varpi=\dfrac{M_x-\frac{4}{5}M_{x+2}}{D_x+D_{x+1}}$, and when $p_1=p_2=p_3\ldots=p_9=1$ we find $\varpi=\dfrac{M_x}{N_{x-1}-N_{x+9}}$, the latter being the ordinary formula when a whole life assurance is paid for by ten equal annual premiums.

I am, Sir,

Your obedient servant,

SAMUEL YOUNGER.

316, *Regent Street,*
26th Sept., 1868.

*** We readily give insertion to the above letter on a subject which is not only of theoretical interest but may become of some practical importance. We should have preferred, however, to see the numerical examples worked out by the Experience Table, instead of the Carlisle, when probably some of the irregularities in the results would have disappeared. We should be glad now to see the question treated in another way, viz. by a comparison of the amount of the paid-up policy which the value of the policy would purchase, according to the office single premiums, with the amount of that granted under the regulations quoted above.—Ed. *J. I. A.*

ON A FORMULA IN THE CALCULUS OF FINITE DIFFERENCES.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—I am not about to enter upon the consideration of a theory proposed by some writers, that all mathematical evidence resolves into a perception of identity, and that mathematical propositions are only diversified expressions of the simple formula, $a=a$.* It must however be admitted

* The following is quoted by Dugald Stewart ("Philosophy of the human mind," part 2, cap. 1) from a writer on the subject referred to. *Omnes Mathematicorum propositiones sunt identicæ et repræsentantur hac formulâ, $a=a$.* He adds, "This sentence, which I quote from a dissertation published at Berlin about 50 years ago" (1813), "expresses in a few words what seems to be now the prevailing opinion (more particularly on the Continent) concerning the nature of Mathematical evidence."

that some results which are in the form of equations, approach more nearly to this character than others. In some cases, the equality of two quantities can only be arrived at by means of a long series of intermediate steps, while in others the connexion is more immediate, as in the case of the Binomial theorem $(x+a)^n=x^n+nx^{n-1}a+\&c.$; or the more general expression, viz. Stirling's theorem, $fx=fo+f'o.x+f''o.\frac{x^2}{1.2}+\&c.$ These are not however mere equivalent algebraical expressions, as, for instance, the second side of the latter equation shows the result of the operation on x signified by f on the first side of it. But if we write down such an equation as $a-d=a-b+b-c+c-d$, the character of it will not be altered by making the terms on the second side of it which cancel one another, more numerous or complicated, or even by supposing some law followed in their arrangement.

Now the two fundamental theorems of the Calculus of Finite Differences are

$$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{1.2}\Delta^2 u_0+\&c.=(1+\Delta)^x u_0 \quad . \quad . \quad (1)$$

and $$\Delta^n u_0=u_n-nu_{n-1}+\frac{n(n-1)}{1.2}u_{n-2}-\&c.=(u-1)^n \quad . \quad . \quad (2)$$

If in the first of these equations are substituted the values of $\Delta^n u_0$, from $n=1$ to $n=x$, as deduced from the second equation, the result is the identical equation, $u_x=u_x$.*

Again, according to the property of derivation—one of frequent application in analysis—whereby differences and differential coefficients are treated as the primitive functions of differences or differential coefficients of higher orders, the form of equation (1) will equally hold if for u_x is substituted one of its differences or some function of which u_x is a difference.

Referring now to the demonstration of a formula for interpolation given in vol. xiv., page 244, of this *Journal*, I think the result there arrived at is to be considered not as a fresh property, but rather as involving and illustrating the original properties of the Calculus of Finite Differences. Now when we have arrived at some conclusions, in elementary Geometry for instance, such conclusions are felt to have the same cogency as the axioms and definitions from which they proceed; and in other cases, we

* The distinction I wish to point out between different kinds of equations may perhaps be put in a clearer light as follows. If in the equation, $u_x=(1+\Delta)^x u_0$, have been substituted the values of all the differences previous to $\Delta^x u_0$, as found by means of equation (2) from x terms of the series u_0, u_1, u_2, &c., the former equation is reduced to $u_x=F(u_0, u_1, u_2, \ldots . u_{x-1}, \Delta^{x-1}u_0)$; and to find $\Delta^x u_0$, the next, namely the $x+1$th, term must be introduced. But this is u_x, the term on the first side of the equation. In other words, the previous terms of the series, and consequently the previous differences, are quite arbitrary in reference to u_x considered only as a *given* quantity, and which might be a term in an infinite number of series. But if $u_x=fx$, a known function of x, it is $=fo+f'o.x+f''o.\frac{x^2}{1.2}+\&c.$, (Stirling's theorem), and since fo and all the differential coefficients $f'o, f''o, f'''o$, &c., are determined if they are all finite, the value of fx will be found. We must draw a distinction between using a formula containing given differences as an instrument to find unknown terms of a series, and deducing the relation between given terms of a series and their differences as shown in equations (1) and (2); and if the terms of the latter series follow some law, that will furnish a particular case of the general relation alluded to.

should, I think, endeavour, so to speak, to 'account for' results at which we may have arrived. If that be not done, a demonstration may be, to adopt an Aristotelian phrase, one ὅτι, but not one διότι.* I have thought that the result arrived at or rather discovered, I believe, by Mr. Berridge, may be also produced in a different form as follows:

Let S_x denote the sum of x consecutive terms commencing with u_0 of a series u_0, u_1, u_2, &c.; and if we give to x successively the values 0, 1, 2, 3, &c., we have the following.

$$S_0=0$$
$$S_1=u_0$$
$$S_2=u_0+u_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_5=u_0+u_1+u_2+u_3+u_4$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_x=u_0+u_1+u_2+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad +u_{x-1}$$

The first differences of successive terms of the series S_0, S_1, $S_2 \ldots S_x$ thus formed are equal to successive terms of the series u_0, u_1, u_2, $\ldots u_x$; thus $S_1-S_0=u_0$, $S_2-S_1=u_1$, $\ldots \;. S_x-S_{x-1}=u_{x-1}$. Let these differences be denoted by δS_0, δS_1, &c., so that the differences of succeeding orders, of S_0, are denoted by δS_0, $\delta^2 S_0$, $\delta^3 S_0$, &c., and let the differences of successive orders, of u_0, in the series u_0, u_1, u_2, &c., be denoted by δu_0, $\delta^2 u_0$, $\delta^3 u_0$, &c., then

$$\delta S_0=u_0,\ \delta^2 S_0=\delta u_0,\ \delta^3 S_0=\delta^2 u_0,\ \delta^4 S_0=\delta^3 u_0 \ldots . \delta^x S_0=\delta^{x-1} u_0 \quad . \quad . \quad (a)$$

Suppose we wish to interpolate intermediate terms of the series S_0, S_1, S_2, &c., between values of S_x taken at successive intervals of p terms from the commencement, viz. between S_0, S_p, $S_{2p} \ldots . S_{mp}$;—for simplicity let $p=5$, and let the number of the orders of differences of the latter terms be 4, and the differences of S_0 of the 1st, 2nd, &c. orders as found from them, be denoted by ΔS_0, $\Delta^2 S_0$, &c. Then the first differences of the series, S_0, S_5, S_{10}, &c., are equal to sums of five consecutive terms of the series, u_0, u_1, u_2, &c., commencing respectively with u_0, u_5, u_{10}, &c., viz.,

$$\Delta S_0=u_0+u_1+u_2+u_3+u_4$$
$$\Delta S_5=u_5+u_6+u_7+u_8+u_9$$
$$\&c. \quad \&c. \quad \&c. \quad \&c.$$

But these differences are the same as the quantities denoted by Σ_1, Σ_2,

* To show that this idea of the different modes of proof adopted by mathematical writers is not merely chimerical, I append the following remarks in reference to Dr. Wallis. "Sa façon de démontrer, qui est fondée sur induction plutôt que sur un raisonnement à la mode d'Archimède, fera quelque peine aux novices, qui veulent des syllogismes demonstratifs depuis le commencement jusqu'à la fin. Ce n'est pas que je ne l'approuvre, mais toutes ses propositions pouvant être démontrées *viâ ordinariâ, legitimâ, et Archimedœâ* en beaucoup moins de paroles, que n'en contient son livre, je ne sçai pas pourquoi il a préféré cette manière à l'ancienne, qui est plus convainquante et plus élégante ainsi que j'espère lui faire voir à mon premier loisir." Lettre de M. de Fermat à M. le Chev. Kenelme Digby (Fermat's varia opera Mathematica, p. 191—as quoted by Dugald Stewart, "Philosophy of the human mind," part 2, cap. 9).

&c., in Mr. Berridge's letter, and the other differences Δ^2S_0, &c., are the same as the differences $\Delta\Sigma_1$, &c., therein, *i.e.*

$$\Delta S_0=\Sigma_1,\ \Delta^2S_0=\Delta\Sigma_1,\ \Delta^3S_0=\Delta^2\Sigma_1,\ \Delta^4S_0=\Delta^3\Sigma_1 \quad . \quad . \quad . \quad (b)$$

It has been shown in vol. xiv., page 23, of the *Journal*,* that

$$\left.\begin{aligned}
\delta u_0 &= \frac{1}{5}\Delta u_0-\frac{2}{5^2}\Delta^2u_0+\frac{6}{5^3}\Delta^3u_0-\frac{21}{5^4}\Delta^4u_0\\
\delta^2u_0 &= \frac{1}{5^2}\Delta^2u_0-\frac{4}{5^3}\Delta^3u_0+\frac{16}{5^4}\Delta^4u_0\\
\delta^3u_0 &= \frac{1}{5^3}\Delta^3u_0-\frac{6}{5^4}\Delta^4u_0\\
\delta^4u_0 &= \frac{1}{5^4}\Delta^4u_0
\end{aligned}\right\}\quad(3)$$

Substituting S_0 for u_0 in these formulæ they become

$$\left.\begin{aligned}
\delta S_0 &= \frac{1}{5}\Delta S_0-\frac{2}{5^2}\Delta^2S_0+\frac{6}{5^3}\Delta^3S_0-\frac{21}{5^4}\Delta^4S_0\\
\delta^2S_0 &= \frac{1}{5^2}\Delta^2S_0-\frac{4}{5^3}\Delta^3S_0+\frac{16}{5^4}\Delta^4S_0\\
\delta^3S_0 &= \frac{1}{5^3}\Delta^3S_0-\frac{6}{5^4}\Delta^4S_0\\
\delta^4S_0 &= \frac{1}{5^4}\Delta^4S_0
\end{aligned}\right\}\quad(4)$$

Again substituting for the quantities in the latter formulæ, their values as contained in the systems of equations (a) and (b), we have

$$\left.\begin{aligned}
u_0 &= \frac{1}{5}\Sigma_1-\frac{2}{5^2}\Delta\Sigma_1+\frac{6}{5^3}\Delta^2\Sigma_1-\frac{21}{5^4}\Delta^3\Sigma_1\\
\delta u_0 &= \frac{1}{5^2}\Delta\Sigma_1-\frac{4}{5^3}\Delta^2\Sigma_1+\frac{16}{5^4}\Delta^3\Sigma_1\\
\delta^2u_0 &= \frac{1}{5^3}\Delta^2\Sigma_1-\frac{6}{5^4}\Delta^3\Sigma_1\\
\delta^3u_0 &= \frac{1}{5^4}\Delta^3\Sigma_1
\end{aligned}\right\}\quad(5)$$

The coefficients in (3) and (5) are the same; but the two systems of equations cannot coexist, because in (4) $\delta^5S_0=\delta^4u_0=0$, and this is an equation of condition for (5).

I am, Sir,
Yours obediently,

7, Royal Exchange,
6th November, 1868.

THOMAS CARR.

* The same results are also given by Dr. Farr, vol. ix., page 136, but in a different form.

https://doi.org/10.1017/S2046167400003943 Published online by Cambridge University Press