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WEAKLY ISOTOPIC PLANAR TERNARY RINGS

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1. Introduction. This paper introduces two relations both weaker than isotopism which may hold between planar ternary rings. We will concentrate on the geometric consequences rather than the algebraic properties of these relations. It is well-known that every projective plane can be coordinatized by a planar ternary ring and every planar ternary ring coordinatizes a projective plane. If two planar ternary rings are isomorphic then their associated projective planes are isomorphic; however, the converse is not true. In fact, an algebraic bond which necessarily holds between the coordinatizing planar ternary rings of isomorphic projective planes has not been found. Such a bond must, of course, be weaker than isomorphism; furthermore, it must be weaker than isotopism. Here we show that it is even weaker than the two new relations introduced. This is significant because the weaker of our relations is, in a sense, the weakest possible algebraic relation which can hold between planar ternary rings which coordinatize isomorphic projective planes.

2. Definitions and theorems. Let $T = (R, t)$ and $T' = (R', t')$ be planar ternary rings.

Definition 2.1. (R, t) is isomorphic to (R', t') if and only if there exists a one-to-one function f from R onto R' such that if $d = t(a, b, c)$ then $f(d) = t'(f(a), f(b), f(c))$.

Definition 2.2. (R, t) is isotopic to (R', t') if and only if there exist one-to-one functions f, g, h from R onto R' such that $h(0) = 0$ (we use the symbol "0" for both T and T') and if $d = t(a, b, c)$ then $h(d) = t'(f(a), g(b), h(c))$.

Definition 2.3. (R, t) is w_1-isotopic to (R', t') if and only if there exist one-to-one functions $f, g, h_x, x \in R$ from R onto R' such that if $d = t(a, b, c)$ then $h_0(d) = t'(f(a), g(b), h_b(c))$.

Definition 2.4. (R, t) is w_2-isotopic to (R', t') if and only if there exist one-to-one functions $f, g, h_x, j_x, x \in R$ from R onto R' such that if $d = t(a, b, c)$ then $j_a(d) = t'(f(a), g(b), h_b(c))$.

While the weak forms of isotopism above may appear artificial on first glance, w_1-isotopism, in particular, arises naturally from a consideration of finite affine planes. An affine plane of order n generates $n - 1$ mutually orthogonal Latin squares of order n. Using the coordinization of a planar ternary ring we may generate these Latin squares by fixing m (the "slope")

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and letting k run over the x-y plane such that $y = t(x, m, k)$. This yields a Latin square for each $m \neq 0$. Now consider the following alterations on these Latin squares: for a given m relabel the k's (thus yielding the functions h_m); relabel the m's (yielding the function g); relabel the x's and y's (yielding f and h_0, respectively). The relabellings constitute a w_1-isotopism. Also the new set of Latin squares represents an affine plane isomorphic to the original affine plane. This discussion suggests a theorem (actually Corollary 2.9) which we now list with three others for comparison.

We use the notation of [**5**]; thus u, v, o, e is the reference quadrangle, ϕ associates points, Φ associates lines of an isomorphism (ϕ, Φ), π_T represents the projective plane coordinatized by the planar ternary ring T.

THEOREM 2.5. *T is isomorphic to T' if and only if there exists an isomorphism (ϕ, Φ) from π_T to $\pi_{T'}$ such that $\phi: u, v, o, e \to u', v', o', e'$.*

THEOREM 2.6. *T is isotopic to T' if and only if there exists an isomorphism (ϕ, Φ) from π_T to $\pi_{T'}$, such that $\phi: u, v, o \to u', v', o'$.*

THEOREM 2.7. *T is w_1-isotopic to T' if and only if there exists an isomorphism (ϕ, Φ) from π_T to $\pi_{T'}$ such that $\phi: u, v \to u', v'$.*

THEOREM 2.8. *T is w_2-isotopic to T' if and only if there exists an isomorphism (ϕ, Φ) from π_T to $\pi_{T'}$ such that $\Phi: v \to v'$ and $\Phi: uv \to u'v'$.*

The first two theorems are well-known; for example, see [**5**, Chapter 9]. We prove the latter two here. Non-ideal points are represented by ordered pairs from T, ideal points except v are represented by elements of T, v is represented by z. Lines not through z are represented by pairs, lines through z, except the ideal line are represented by elements; the ideal line is represented by Z. Also $[x, y] \in \langle m, k\rangle$ if and only if $y = t(x, m, k)$, $[x] \in \langle m, k\rangle$ if and only if $x = m$, and $[x, y] \in \langle k\rangle$ if and only if $x = k$.

Proof of Theorem 2.7. Suppose T and T' are w_1-isotopic. Define ϕ and Φ as follows:

$$\begin{aligned}\phi:\ & [x, y] \to [f(x), h_0(y)]\\ & [x] \to [g(x)]\\ & z \to z\\ \Phi:\ & \langle m, k\rangle \to \langle g(m), h_m(k)\rangle\\ & \langle k\rangle \to \langle f(k)\rangle\\ & Z \to Z.\end{aligned}$$

Thus

$$\begin{aligned}[x, y] \in \langle m, k\rangle &\Leftrightarrow y = t(x, m, k)\\ &\Leftrightarrow h_0(y) = t'(f(x), g(m), h_m(k))\\ &\Leftrightarrow [f(x), h_0(y)] \in \langle g(m), h_m(k)\rangle\\ &\Leftrightarrow \phi([x, y]) \in \Phi(\langle m, k\rangle).\end{aligned}$$

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Clearly if $p = [x]$ or z or $L = \langle k \rangle$ or Z then $p \in L$ if and only if $\phi(p) \in \Phi(L)$. Thus (ϕ, Φ) is an isomorphism from π_T to $\pi_{T'}$. Now $\phi: u \to u'$ because $u = [0]$ and $\phi([0]) = [g(0)] = [0] = u'$. We obtained $g(0) = 0$ as follows: $[x, k] \in \langle 0, k \rangle$ so $[f(x), h_0(k)] \in \langle g(0), h_0(k) \rangle$. Thus $h_0(k) = t'(f(x), g(0), h_0(k))$ and so $g(0) = 0$. Finally $\phi: v \to v'$ because clearly $\phi: z \to z'$.

Suppose (ϕ, Φ) is an isomorphism from π_T to $\pi_{T'}$ and $\phi: u, v \to u', v'$. Define f on T by $f(x) = x'$ where x' is such that $\Phi: \langle x \rangle \to \langle x' \rangle$. This definition is valid because $\phi: z \to z'$. Define g by $g(m) = m'$ where m' is such that $\phi: [m] \to [m']$. This definition is valid because $\Phi: Z \to Z'$. Define h_m by $h_m(k) = k'$ where $\Phi: \langle m, k \rangle \to \langle g(m), k' \rangle$. Since (ϕ, Φ) is an isomorphism, f, g, h_m are all one-to-one functions from T onto T'. Now

$$\begin{aligned} y = t(x, m, k) &\Leftrightarrow [x, y] \in \langle m, k \rangle \\ &\Leftrightarrow \phi([x, y]) \in \Phi(\langle m, k \rangle) \\ &\Leftrightarrow \phi(\langle x \rangle \cap \langle 0, y \rangle) \in \langle g(m), h_m(k) \rangle \\ &\Leftrightarrow \langle f(x) \rangle \cap \langle g(0), h_0(y) \rangle \in \langle g(m), h_m(k) \rangle \\ &\quad \text{(because } g(0) = 0 \text{ since } \phi: u \to u') \\ &\Leftrightarrow [f(x), h_0(y)] \in \langle g(m), h_m(k) \rangle \\ &\Leftrightarrow h_0(y) = t'(f(x), g(m), h_m(k)). \end{aligned}$$

Thus f, g, h_m is a w_1-isotopism from T to T'.

The proof of Theorem 2.8 is similar to that of 2.7 with the following exceptions: If T and T' are w_2-isotopic let $\phi: [x, y] \to [f(x), j_x(y)]$. If (ϕ, Φ) is an isomorphism from π_T to $\pi_{T'}$ define $j_x(y) = y'$ where y' is such that $\phi: [x, y] \to [f(x), y']$.

COROLLARY 2.9. *T is w_1-isotopic to T' if and only if there exists an isomorphism* (ϕ, Φ) from α_T to $\alpha_{T'}$ (affine planes) *such that $\Phi(X)$ is parallel to X' and $\Phi(Y)$ is parallel to Y' where X and X' represent the respective x-axes of α_T and $\alpha_{T'}$ and Y and Y' represent the y-axes.*

COROLLARY 2.10. *T is w_2-isotopic to T' if and only if there exists an isomorphism (ϕ, Φ) from α_T to $\alpha_{T'}$ such that $\Phi(Y)$ is parallel to Y'.*

Clearly each successive definition 2.1 to 2.4 is weaker than the preceding in the following sense: if T and T' are isomorphic then they are isotopic; if they are isotopic then they are w_1-isotopic; if they are w_1-isotopic then they are w_2-isotopic. The following example shows that each successive definition is strictly weaker.

Example 2.11. Let π be the plane coordinatized by the right nearfield of order nine. That the following statements are valid follow directly from Theorems 2.5-2.8 and the nature of the group of collineations of π. (See Andre [**2**].)

(1) $T(u, v, o, e)$ and $T(u, v, o, [2, 1])$ are isomorphic but not identical.
(2) $T([2], v, o, e)$ and $T([2], v, o, [1, 0])$ are isotopic but not isomorphic.
(3) $T(o, v, u, e)$ and $T(o, v, [2], e)$ are w_1-isotopic but not isotopic.

(4) $T(u, v, o, e)$ and $T([2], v, o, e)$ are w_2-isotopic but not w_1-isotopic.
(5) $T(u, v, o, e)$ and $T(u, [2], o, e)$ are not w_2-isotopic.

3. Some consequences. The concepts of w_1 and w_2 isotopism easily transfer to algebras with two binary operations such as a field, division ring, nearfield, semifield, and quasifield. As with isotopism this is simply done by adapting it to the addition and multiplication that are definable on a linear ternary ring. It is known that as the algebraic structure of these linear ternary rings which coordinatize isomorphic planes is strengthened the bond between these rings is strengthened. In this regard we know the following two theorems.

THEOREM 3.1. *If N and N' are planar nearfields which coordinatize isomorphic planes π_N and $\pi_{N'}$ then N and N' are isomorphic.*

For a proof of this, see [**5**, Chapter 12].

THEOREM 3.2. (Albert [**1**]). *If S and S' are semifields which coordinatize isomorphic planes π_S and $\pi_{S'}$ then S and S' are isotopic.*

It is an open problem to find the algebraic bond which is appropriate for the coordinatizing quasifields of isomorphic planes. Neither of the weak isotopisms provides the complete answer but we may offer the following:

THEOREM 3.3. *If there exists an isomorphism from π_T to $\pi_{T'}$ mapping the ideal line of π_T to the ideal line of $\pi_{T'}$ and if the collineation groups on the ideal points of one of these planes is transitive (doubly transitive) then T and T' are w_2-isotopic (w_1-isotopic).*

This follows from a direct application of Theorem 2.8. (Theorem 2.7.).

COROLLARY 3.4. *If the collineation group on the ideal points of π is transitive (doubly transitive) and Q and Q' are coordinatizing right quasifields of π then Q and Q' are w_2-isotopic (w_1-isotopic).*

This follows from Theorem 3.3. using the identity map as the isomorphism.

COROLLARY 3.5. *If α is a translation plane and the collineation group of α is flag transitive then all coordinatizing quasifields are w_2-isotopic.*

This follows from Corollary 3.4. using the appropriate definitions.

While these theorems are not completely satisfactory they are satisfying because those projective planes coordinatized by right quasifields (hence the translation planes also) whose collineation groups are doubly transitive or transitive on ideal points are completely known for the known finite projective planes. See Dembrowski [**3**, pp. 235, 236]. A more satisfactory answer will not be forthcoming within the current format. Equivalence relations in the form of one-to-one mappings of T onto T' must necessarily map $z \to z$ and $Z \to Z'$ under the coordinatization of π_T and $\pi_{T'}$. Thus, by Theorem 2.8. w_2-isotopism is the weakest algebraic bond of this type. Hence we must await a slightly different kind of algebraic bond in order to completely answer questions such as: what is the bond which necessarily relates

(1) T to T' if π_T is isomorphic to $\pi_{T'}$?

(2) T to T' if α_T is isomorphic to $\alpha_{T'}$?

(3) Q to Q' (Q and Q' are right quasifields) if π_Q is isomorphic to $\pi_{Q'}$?

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