proof, on the ground that in the same way the angles of a spherical triangle might be proved equal to two right angles. On the contrary, a similar mode of proof will show that the angles of a spherical triangle are greater than two right angles. For we must now consider the revolution of planes containing the great circles of which the sides of the spherical triangle are arcs. Suppose, then, a plane by revolving to coincide in turn with the three sides of a spherical triangle. This plane in its three positions has always one point common, that is the centre of the sphere. The result of the three revolutions through the three spherical angles, is that the plane coincides with its original position, but with ends reversed. Now a plane can thus reverse its position by turning through two right angles, only on condition that it remains, during the revolution, perpendicular to the same fixed plane, that is that its axis of revolution is not subjected to tilting. Now, this is a condition that cannot be satisfied by a plane which coincides in turn with the three sides of a spherical triangle (except in the case when one side vanishes). Hence the three angles of a spherical triangle are greater than two right angles.

Ninth Meeting, July 11th, 1884.

Dr R. M. Ferguson in the Chair.

Application of the Multiplication of Matrices to prove a Theorem in Spherical Geometry.

By Professor Chrystal, University of Edinburgh.

The theorem in question is that if two of the diagonals of a spherical quadrilateral be quadrantal arcs, the third diagonal is also a quadrantal arc. (Fig. 31.)

Denote the direction cosines of the radius to the point 1 by l_1, m_1, n_1, &c., and $l_1\ l_2 + m_1\ m_2 + n_1\ n_2$ by 12.

Then our conditions give $12 = 0$, $34 = 0$, and we have to prove $56 = 0$.

The equation to the plane 13 is

https://doi.org/10.1017/S0013091500037111 Published online by Cambridge University Press

$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$, say $A_{13}x + B_{13}y + C_{13}z = 0$; and similarly the equation to the plane 24 is $A_{24}x + B_{24}y + C_{24}z = 0$.

Hence the direction cosines of 6 are the *full* minors of $\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix}$ divided by the square root of the sum of the squares of these minors. Hence the cosine 56 is $\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix} \times \begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix}$ divided by the product of the square roots of the sums of the squares of the full minors of the two matrices.

Now by a double application of the multiplication of matrices

$$\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix} \times \begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix}$$

$$= \begin{vmatrix} A_{13}A_{14} + B_{13}B_{14} + C_{13}C_{14} & A_{13}A_{23} + B_{13}B_{23} + C_{13}C_{23} \\ A_{24}A_{14} + B_{24}B_{14} + C_{24}C_{14} & A_{24}A_{23} + B_{24}B_{23} + C_{24}C_{23} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_4 & m_4 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \\ \begin{vmatrix} l_2 & m_2 & n_2 \\ l_4 & m_4 & n_4 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_4 & m_4 & n_4 \end{vmatrix} & \begin{vmatrix} l_2 & m_2 & n_2 \\ l_4 & m_4 & n_4 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} 11 & 14 \\ 31 & 34 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 32 & 33 \end{vmatrix} \\ \begin{vmatrix} 21 & 24 \\ 41 & 44 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 42 & 43 \end{vmatrix} \end{vmatrix}.$$

Since $12 = 0$, $34 = 0$, $11 = 22 = 33 = 44 = 1$, we have for the value of the last determinant

$$\begin{vmatrix} -13\cdot14, & -13\cdot23 \\ -24\cdot14, & -24.23 \end{vmatrix} = 0;$$

which proves the proposition above stated.

The above process may be applied to the calculation of relations connecting the cosines of the arcs connected with the spherical quadrilateral in general. For example

$$57 = \frac{\begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix} \times \begin{vmatrix} A_{12} & B_{12} & C_{12} \\ A_{34} & B_{34} & C_{34} \end{vmatrix}}{\sqrt{\begin{vmatrix} B_{14} & C_{14} \\ B_{23} & C_{23} \end{vmatrix}^2 + \&c. + \&c.} \times \sqrt{\begin{vmatrix} B_{12} & C_{12} \\ B_{34} & C_{34} \end{vmatrix} + \&c. + \&c.}}$$

$$= \frac{\begin{vmatrix} \begin{vmatrix} 11 & 12 \\ 41 & 42 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 43 & 44 \end{vmatrix} \\ \begin{vmatrix} 21 & 22 \\ 31 & 32 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \end{vmatrix}}{\sqrt{(A_{14}^2+B_{14}^2+C_{14}^2)(A_{23}^2+B_{23}^2+C_{23}^2)-(A_{14}A_{23}+B_{14}B_{23}+C_{14}C_{23})^2}\ \sqrt{\&c.}}.$$

Since $A_{14}^2+B_{14}^2+C_{14}^2=(l_1^2+m_1^2+n_1^2)\,(l_4^2+m_4^2+n_4^2)$
$-(l_1l_4+m_1m_4+n_1n_4)^2=1-14^2$,

and $A_{14}A_{23}+B_{14}B_{23}+C_{14}C_{23}=\begin{vmatrix} 12 & 13 \\ 42 & 43 \end{vmatrix}$;

we get

$$57=\frac{\begin{vmatrix} 1 & 34 & 12 \\ 14 & 13 & 24 \\ 23 & 24 & 13 \end{vmatrix}}{\sqrt{\left\{(1-14^2)(1-23^2)-\begin{vmatrix} 12 & 13 \\ 42 & 43 \end{vmatrix}^2\right\}\left\{(1-12^2)(1-34^2)-\begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix}^2\right\}}};$$

In the particular case of the quadrantal quadrilateral this reduces to

$$57=\frac{13^2-24^2}{\sqrt{\left\{(1-14^2)(1-23^2)-13^2 24^2\right\}\left\{1-(13\cdot 24-14\cdot 23)^2\right\}}};$$

from which 56 is obtained by interchanging 1 and 2.

On the Discrimination of Conics enveloped by the rays joining the corresponding points of two projective ranges.

By Professor CHRYSTAL.

It is evident in the first place as is pointed out by Steiner that the conic will be a parabola if, and cannot be a parabola unless the point at infinity on one range correspond to the point at infinity on the other, that is, the two ranges must be similar. This is the converse of the well-known proposition that a movable tangent to a parabola divides two fixed tangents similarly.

Steiner however does not take up the other cases, nor does Reye, or any other writer on the projective geometry of conics so far as I am aware.

We may however proceed in general as follows :