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UNIFORM FINITE GENERATION OF SU(2) AND SL(2, R)

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1. Introduction. A connected Lie group H is generated by a pair of one-parameter subgroups if every element of H can be written as a finite product of elements chosen alternately from the two one-parameter subgroups, i.e., if and only if the subalgebra generated by the corresponding pair of infinitesimal transformations is equal to the whole Lie algebra h of H (observe that the subgroup of all finite products is arcwise connected and hence, by Yamabe's theorem [**5**], is a sub-Lie group). If, moreover, there exists a positive integer n such that every element of H possesses such a representation of length at most n, then H is said to be uniformly finitely generated by the pair of one-parameter subgroups. In this case, define the order of generation of H as the least such n; otherwise define it as infinity. Since the order of generation of H will, in general, depend upon the pair of one-parameter subgroups, H may have many different orders of generation. However, it is a simple consequence of Sard's theorem [**4**, pp. 45–55] that the order of generation of H must always be $\geqq$ dimension of H.

The order of generation of $\mathrm{SU}(2)/\{I, -I\}$, i.e., of the isometry group of the spherical geometry may be any (finite) integer $\geqq 3$; the order of generation is determined by the cross-ratio of the fixed points of the pair of elliptic one-parameter subgroups [**1**]. In this paper, it is shown that precisely the same orders of generation are obtained for SU(2) and that the order of generation is determined by the Killing form of the pair of infinitesimal transformations. The order of generation of $\mathrm{SL}(2, \mathbf{R})/\{I, -I\}$, i.e., of the isometry group of the hyperbolic geometry, is infinite if both one-parameter subgroups are elliptic, 3 if exactly one is elliptic and 4 in all other cases, except that it is 6 if both are hyperbolic with interlacing fixed points [**2**]. In this paper, it is shown that this result is also true for $\mathrm{SL}(2, \mathbf{R})$ except that if both one-parameter subgroups are hyperbolic with interlacing eigenvectors, then the order of generation is 8 instead of 6.

2. Preliminaries. SU(2) consists of all 2×2 matrices

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha, \beta \text{ complex},$$

with $|\alpha|^2 + |\beta|^2 = 1$. The Lie algebra of SU(2) consists of all 2×2 skew-

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Hermitian matrices with trace 0, i.e., the infinitesimal transformations have the form

$$\epsilon = \begin{bmatrix} ib & \gamma \\ -\bar{\gamma} & -ib \end{bmatrix}, \quad b \text{ real}, \gamma \text{ complex}.$$

The one-parameter subgroups of SU(2) are the solutions of the differential system

$$\text{(1)} \qquad \frac{dA}{dt} = \begin{bmatrix} ib & \gamma \\ -\bar{\gamma} & -ib \end{bmatrix}, \quad A(0) = I$$

and hence have the form

$$A(t) = \exp\left\{t\begin{bmatrix} ib & \gamma \\ -\bar{\gamma} & -ib \end{bmatrix}\right\}, \quad -\infty < t < +\infty.$$

SL(2, **R**) consists of all 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \text{ real, with } ad - bc = 1.$$

The Lie algebra of SL(2, **R**) consists of all 2×2 (real) matrices with trace 0, i.e., the infinitesimal transformations have the form

$$\epsilon = \begin{bmatrix} u & v \\ w & -u \end{bmatrix}, \quad u, v, w \text{ real}.$$

The one-parameter subgroups of SL(2, **R**) are the solutions of the differential system

$$\text{(2)} \qquad \frac{dA}{dt} = \begin{bmatrix} u & v \\ w & -u \end{bmatrix}, \quad A(0) = I$$

and hence have the form

$$A(t) = \exp\left\{t\begin{bmatrix} u & v \\ w & -u \end{bmatrix}\right\}, \quad -\infty < t < +\infty.$$

Under the transformation on the Lie algebra of SU(2) or SL(2, **R**) induced by any inner automorphism of the respective group, the determinant of the infinitesimal transformation is an absolute invariant (this is true even if the "inner" automorphism is induced by an element of GL(2, **C**)). An infinitesimal transformation (only non-zero infinitesimal transformations will be considered henceforth) and the one-parameter subgroup that it generates are classified as elliptic, parabolic or hyperbolic depending upon whether its determinant is positive, zero, or negative respectively. Thus all one-parameter subgroups of SU(2) are elliptic while the one-parameter subgroup of SL(2, **R**) generated by an infinitesimal transformation

$$\begin{bmatrix} u & v \\ w & -u \end{bmatrix}$$

is elliptic if $u^2 + vw < 0$, parabolic if $u^2 + vw = 0$ and hyperbolic if $u^2 + vw > 0$.

Since a skew-Hermitian matrix can always be unitarily diagonalized, every one-parameter subgroup of SU(2) can be expressed in the form

$$(3) \qquad A(t) = U\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}U^{-1}, \quad U \in \mathrm{SU}(2).$$

Similarly it follows that the one-parameter subgroups of SL(2, **R**) can be expressed in the form

$$(4a) \qquad A(t) = C\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}C^{-1}, \quad C \in \mathrm{SL}(2, \mathbf{R}),$$

$$(4b) \qquad A(t) = C\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}C^{-1}, \quad C \in \mathrm{SL}(2, \mathbf{R}),$$

$$(4c) \qquad A(t) = C\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}C^{-1}, \quad C \in \mathrm{SL}(2, \mathbf{R}),$$

depending upon whether it is elliptic, parabolic, or hyperbolic, respectively. Note that a one-parameter subgroup is compact if and only if it is elliptic. Further, observe that in the elliptic case $A(t + \pi) = -A(t)$, while in both the parabolic and hyperbolic cases trace $A(t) \geqq 2$ and hence $A(t) \neq -A(s)$ for all t and s.

Since SU(2) has no two-dimensional connected Lie subgroups, any pair of distinct one-parameter subgroups of SU(2) generate SU(2). A pair of distinct infinitesimal transformations ϵ and η of SU(2) can be simultaneously transformed into exactly one of the normal forms

$$(5) \qquad \epsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}; \qquad \eta = \begin{bmatrix} di & i \\ i & -di \end{bmatrix}, \quad d \geqq 0$$

by means of a suitably chosen inner automorphism of SU(2).

The only two dimensional connected Lie subgroups of SL(2, **R**) are those that leave a one-dimensional subspace of $\mathbf{R}^2$ invariant. Hence a pair of distinct one-parameter subgroups of SL(2, **R**) generate SL(2, **R**) if and only if their infinitesimal transformations do not have a common eigenvector. A pair of distinct infinitesimal transformations ϵ and η of SL(2, **R**) with no common eigenvector can be simultaneously transformed into one and only one of the normal forms [**3**]

$$(6) \quad (a) \quad \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad \eta = \begin{bmatrix} 0 & 1 \\ -c & 0 \end{bmatrix}, \quad c > 1, \text{ both elliptic}$$

$$(b) \quad \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad \eta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \epsilon \text{ elliptic, } \eta \text{ parabolic}$$

$$(c) \quad \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad \eta = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, \quad c \geqq 1, \ \epsilon \text{ elliptic, } \eta \text{ hyperbolic}$$

$$(d) \quad \epsilon = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \qquad \eta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{both parabolic}$$

(e) $\epsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \eta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, ϵ hyperbolic, η parabolic

(f) $\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \eta = C\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}C^{-1}$ where

$$C = \begin{bmatrix} r/(r-1)^{\frac{1}{2}} & 1/(r-1)^{\frac{1}{2}} \\ 1/(r-1)^{\frac{1}{2}} & 1/(r-1)^{\frac{1}{2}} \end{bmatrix}, \quad r > 1,$$

both hyperbolic with eigenvectors separating

(g) $\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \eta = C\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}C^{-1}$ where

$$C = \begin{bmatrix} r/(r+1)^{\frac{1}{2}} & 1/(r+1)^{\frac{1}{2}} \\ -1/(r+1)^{\frac{1}{2}} & 1/(r+1)^{\frac{1}{2}} \end{bmatrix}, \quad r \geqq 1,$$

both hyperbolic with eigenvectors interlacing

by means of a suitably chosen inner automorphism of SL(2, **R**) ("inner" automorphisms induced by matrices with determinant equal to -1 may be needed).

If $U \in SU(2)$, let $w = \tilde{U}(z)$ be the unique transformation of the isometry group of the spherical geometry whose matrix is U; this is the natural map from SU(2) to $SU(2)/\{I, -I\}$. Under this map the one-parameter subgroup $A(t)$ of SU(2) given by (3) corresponds to the one-parameter subgroup $w = T_t(z) = \tilde{U}(e^{2it}\tilde{U}^{-1}(z))$ of the isometry group of the spherical geometry. If

$$U = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix},$$

then the fixed points of $T_t(z)$ are just $\tilde{U}(0) = \beta/\bar{\alpha}$ and $\tilde{U}(\infty) = \alpha/-\bar{\beta}$. Similarly, to each one-parameter subgroup of SL(2, **R**) there corresponds a one-parameter subgroup of the isometry group of the hyperbolic geometry (choose the Poincare' half-plane as the model for the hyperbolic geometry). Note that a pair of one-parameter subgroups $A(t)$ and $B(s)$ generate SU(2) (SL(2, **R**)) if and only if the corresponding pair $T_t(z)$ and $S_s(z)$ generate

$$(SU(2)/\{I, -I\})(SL(2, \mathbf{R})/\{I, -I\});$$

further, the order of generation of SU(2) (SL(2, **R**)) by $A(t)$ and $B(s)$ is always $\geqq$ the order of generation of $(SU(2)/\{I, -I\})$ $((SL(2, \mathbf{R})/\{I, -I\}))$ by $T_t(z)$ and $S_s(z)$. A sufficient condition for these orders of generation to be equal is that at least one of the one-parameter subgroups be elliptic; this follows from the fact that if $A(t)$ is elliptic, then $A(t + \pi) = -A(t)$ and hence a matrix W is representable as a product of length k ($k \geqq 2$) if and only if $-W$ is representable as a product of length k.

3. The orders of generation of SU(2). Since SU(2) is a compact semi-simple Lie group, the Killing form $\langle \epsilon, \eta \rangle$ is negative definite. If $A(t)$ and

$B(s)$ are the respective one-parameter subgroups generated by ϵ and η, define

$$G(A(t), B(s)) = \frac{|\langle \epsilon, \eta \rangle|}{\langle \epsilon, \epsilon \rangle^{\frac{1}{2}} \langle \eta, \eta \rangle^{\frac{1}{2}}}; \tag{7}$$

Note that $G(A(t), B(s))$ depends only on the pair of one-parameter subgroups $A(t)$ and $B(s)$ and not on the particular representatives ϵ and η that are chosen. It is well known that $G(A(t), B(s))$ is invariant under all inner automorphisms of the group, i.e.,

$$G(U^{-1}A(t)U, U^{-1}B(s)U) = G(A(t), B(s)), \qquad U \in \mathrm{SU}(2). \tag{8}$$

To explicitly compute $G(A(t), B(s))$, where $A(t)$ and $B(s)$ are distinct one-parameter subgroups, it is, in view of (8), permissible to assume that the corresponding pair of infinitesimal transformations ϵ and η is in normal form (5); a routine calculation yields

$$G(A(t), B(s)) = d/(d^2 + 1)^{\frac{1}{2}}. \tag{9}$$

Observe that $G(A(t), B(s))$ assumes all real values x satisfying $0 \leqq x < 1$.

THEOREM 1. SU(2) *is generated by any pair of distinct one-parameter subgroups* $A(t)$ *and* $B(s)$. *If* $G(A(t), B(s)) = 0$, *then the order of generation is* 3; *if*

$$\cos(\pi/k) < G(A(t), B(s)) \leqq \cos(\pi/(k + 1)), \tag{10}$$

then the order of generation is $k + 2$ $(k \geqq 2)$. *Thus the order of generation may be any integer* $\geqq 3$.

Proof. Since both (and hence at least one!) one-parameter subgroups are elliptic, the order of generation of SU(2) by $A(t)$ and $B(s)$ equals the order of generation of the isometry group of the spherical geometry by $T_t(z)$ and $S_s(z)$. Assume that the corresponding pair of infinitesimal transformations ϵ and η is in normal form (5); then the fixed points of $T_t(z)$ are 0 and ∞ and the fixed points of $S_s(z)$ are $1/((d^2 + 1)^{\frac{1}{2}} - d)$ and $-1/((d^2 + 1)^{\frac{1}{2}} + d)$. Since the fixed points interlace, their cross-ratio, determined to within reciprocity, must be negative; the unique value of the cross-ratio which is $\geqq -1$ is given by

$$-\frac{(d^2 + 1)^{\frac{1}{2}} - d}{(d^2 + 1)^{\frac{1}{2}} + d}. \tag{11}$$

Choose ψ as the unique angle, $0 < \psi \leqq \pi/2$, such that

$$\tan^2(\psi/2) = \frac{(d^2 + 1)^{\frac{1}{2}} - d}{(d^2 + 1)^{\frac{1}{2}} + d}; \tag{12}$$

ψ is, in fact, just the angle between the axes of the one-parameter rotation subgroups of SO(3) corresponding to $T_t(z)$ and $S_s(z)$ [**1**]. From (9) and (12) it follows that

$$G(A(t), B(s)) = \cos \psi. \tag{13}$$

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But the order of generation of the isometry group of the spherical geometry, i.e., of SO(3), by $T_t(z)$ and $S_s(z)$ is 3 if $\psi = \pi/2$ and it is $k + 2$ if $\pi/(k + 1) \leqq \psi < \pi/k$ $k \geqq 2$ [1].

4. The orders of generation of SL(2, **R**). The matrices $A(t)$ $(A \neq \pm I)$ of an elliptic one-parameter subgroup have no (real) eigenvalues. The matrices $A(t)$ $(A \neq I)$ of a parabolic one-parameter subgroup have eigenvalue 1 with geometric multiplicity one; if

$$A(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

then the set of eigenvectors of $A(t)$ $(A \neq I)$ consists of all vectors $(x, 0)$, $x \neq 0$. The line $x_2 = 0$ is both the source and the sink of this parabolic one-parameter subgroup: if $0 < \arg v < \pi$ (if $v = (v_1, v_2) \neq (0, 0)$, define $\arg v = \arg v_1 + iv_2$), then for all t, $0 < \arg A(t)v < \pi$ and $\lim_{t\to+\infty} \arg A(t)v = 0$, $\lim_{t\to-\infty} \arg A(t)v = \pi$; similarly, if $\pi < \arg v < 2\pi$, then for all t, $\pi < \arg A(t)v < 2\pi$ and $\lim_{t\to+\infty} \arg A(t)v = \pi$, $\lim_{t\to-\infty} \arg A(t)v = 2\pi$. The matrices $A(t)$ $(A \neq I)$ of a hyperbolic one-parameter subgroup have distinct eigenvalues e^t and e^{-t}; if

$$A(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix},$$

then the set of eigenvectors of $A(t)$ $(A \neq I)$ consists of all vectors $(x, 0)$ (belonging to eigenvalue e^t) and $(0, x)$ (belonging to eigenvalue e^{-t}), $x \neq 0$. The line $x_1 = 0$ is the source of this hyperbolic one-parameter subgroup: if $0 < \arg v < \pi$, then $\lim_{t\to-\infty} \arg A(t)v = \pi/2$ while if $\pi < \arg v < 2\pi$, then $\lim_{t\to-\infty} \arg A(t)v = 3\pi/2$. Similarly, the line $x_2 = 0$ is its sink. Note also that if for any integer n, $n\pi/2 \leqq \arg v \leqq (\pi + n\pi)/2$, then for all t, $n\pi/2 \leqq \arg A(t)v \leqq (\pi + n\pi)/2$.

Suppose that $C \in \mathrm{SL}(2, \mathbf{R})$ and that $\{v, w\}$ is a linearly independent set in $\mathbf{R}^2$; let $u_1 = Cv$, $u_2 = Cw$. Then for each $\lambda \neq 0$, there exists a matrix $D \in \mathrm{SL}(2, \mathbf{R})$ such that $Dv = \lambda u_1$ and $Dw = (1/\lambda)u_2$, and, conversely, if $D \in \mathrm{SL}(2, \mathbf{R})$ is such that $Dv = \lambda u_1$, $Dw = \tau u_2$, then $\tau = 1/\lambda$. It follows that if $A(t)$ is the hyperbolic one-parameter subgroup with u_1 and u_2 as eigenvectors, then every $D \in \mathrm{SL}(2, \mathbf{R})$ such that $Dv = \lambda u_1$, $Dw = \tau u_2$, $\lambda > 0$, $\tau > 0$ can be uniquely expressed in the form

$$D = A(t)C. \tag{14}$$

THEOREM 2. *If* SL(2, **R**) *is generated by a pair of one-parameter subgroups, then the order of generation is* 3, 4, 8 *or* ∞. *It is* ∞ *if both are elliptic,* 3 *if exactly one is elliptic, and* 4 *in all other cases except that it is* 8 *if both are hyperbolic with interlacing eigenvectors.*

Proof. Refer to the list of normal forms in (6). In cases (a), (b) and (c), since at least one infinitesimal transformation is elliptic, the order of generation of

$SL(2, \mathbf{R})$ by $A(t)$ and $B(s)$ equals the order of generation of the isometry group of the hyperbolic geometry by $T_t(z)$ and $S_s(z)$. But this order is ∞ if both are elliptic while it is 3 if exactly one is elliptic [**2**].

(d) $$A(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad B(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}:$$

Since

(15) $$A(t)B(s)A(v) = \begin{bmatrix} 1 + st & v + t + stv \\ s & 1 + sv \end{bmatrix},$$

a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $SL(2, \mathbf{R})$ can be represented by a product (15) if and only if $c \neq 0$: Choose $s = c$, $t = (a - 1)/c$, $v = (d - 1)/c$. Similarly, it can be represented by a product $B(v)A(t)B(s)$ if and only if $b \neq 0$. Since

(16) $$\begin{bmatrix} a & 0 \\ 1 & 1/a \end{bmatrix} B(-a) = \begin{bmatrix} a & 0 \\ 1 & 1/a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$$

where

$$\begin{bmatrix} a & 0 \\ 1 & 1/a \end{bmatrix}$$

can be generated by a product (15), the order of generation of $SL(2, \mathbf{R})$ is 4.

(e) $$A(t) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \qquad B(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}:$$

Note that the set of eigenvectors of $A(t)$ ($A \neq I$) consists of all vectors $(-x,x)$ and (x, x), $x \neq 0$.

Let $u_1 = (-1, 1)$, $u_2 = (1, 1)$ and let (v, w) be any ordered pair of linearly independent vectors that is oriented in the same way as (u_1, u_2), i.e., if $\theta = \arg v$, $\Phi = \arg w$, then $\theta - \pi < \Phi < \theta$; only pairs (v, w) so oriented will be considered. Define order (v, w) (with respect to (u_1, u_2)) as the least positive integer k for which there exists both a matrix $C \in SL(2, \mathbf{R})$ expressible as a product of $A(t)$ and $B(s)$ of length k and also $\lambda_0 > 0$, $\tau_0 > 0$ with $Cv = \lambda_0 u_1$, $Cw = \tau_0 u_2$. Since both u_1 and u_2 are eigenvectors of $A(t)$, the last (reading from right to left) element of any product of $A(t)$and$B(s)$ of length $k = \text{order}(v, w)$ that represents C must always be a $B(s)$. If order$(v, w) = k$, then it follows from (14) that every matrix $D \in SL(2, \mathbf{R})$ such that $Dv = \lambda u_1$, $Dw = \tau u_2$, $\lambda > 0$, $\tau > 0$ can be expressed as a product of $A(t)$ and $B(s)$ of length at most $k + 1$. It will be shown below that order$(v, w) \leqq 3$ for all (v, w); it follows that the order of generation of $SL(2, \mathbf{R})$ is $\leqq 4$. In fact, it is equal to 4, since, in this case, the order of generation of $SL(2, \mathbf{R})/\{I, -I\}$ is 4 [**2**].

Let $\theta = \arg v$, $\Phi = \arg w$; a simple calculation shows that order$(v, w) = 1$ if and only if $0 < \theta < \pi$ and $\cot \Phi = \cot \theta + 2$. Hence order$(v, w) \leqq 2$ if and only if there exists an $A(t)$ such that $0 < \arg A(t)v < \pi$ and

(17) $$\cot(\arg A(t)w) = \cot(\arg A(t)v) + 2.$$

In particular, if $-\pi/4 < \Phi < \theta < \pi/4$ or if $-\pi/4 < \Phi < \pi/4$ and $\pi/4 < \theta < 3\pi/4$, then $\operatorname{order}(v, w) \leqq 2$: in both cases, let t_0 be the unique t such that $\arg A(t)w = 0$; then the function

(18) $$f(t) = \cot(\arg A(t)w) - \cot(\arg A(t)v)$$

is continuous on $(t_0, +\infty)$ and satisfies $\lim_{t\to+\infty} f(t) = \cot(\pi/4) - \cot(\pi/4) = 0$, $\lim_{t\to t_0^+} f(t) = +\infty$. Hence there is a $t > t_0$ for which $f(t) = 2$; note that for this t, $0 < \arg A(t)v < 3\pi/4 < \pi$. A similar argument shows that if $3\pi/4 < \Phi < \theta < 5\pi/4$ or if $\pi/4 < \Phi < 3\pi/4$ and $3\pi/4 < \theta < 5\pi/4$, then again $\operatorname{order}(v, w) \leqq 2$.

To show that $\operatorname{order}(v, w) \leqq 3$ for all (v, w), consider the four cases: (a′) $\theta = 0$, (b′) $0 < \theta < \pi$, (c′) $\theta = \pi$, and (d′) $\pi < \theta < 2\pi$. In case (a′), $-\pi < \Phi < 0$ and hence it is possible to find a $B(s)$ such that $-\pi/4 < \arg B(s)w < \arg B(s)v \equiv 0 < \pi/4$. In case (b′), it is possible to find a $B(s)$ such that $-\pi/4 < \arg B(s)w < \pi/4$ and $\pi/4 < \arg B(s)v < 3\pi/4$. In case (c′), $0 < \Phi < \pi$ and hence it is possible to find a $B(s)$ such that $3\pi/4 < \arg B(s)w < \arg B(s)v \equiv \pi < 5\pi/4$. Finally, in case (d′), if $\pi \leqq \Phi < \theta$, then it is possible to find a $B(s)$ such that $3\pi/4 < \pi \leqq \arg B(s)w < \arg B(s)v < 5\pi/4$ while if $\theta - \pi < \Phi < \pi$, then it is possible to find a $B(s)$ such that $\pi/4 < \arg B(s)w < 3\pi/4$ and $3\pi/4 < \pi < \arg B(s)v < 5\pi/4$. Hence in all cases it is possible to find a $B(s)$ such that $\operatorname{order}(B(s)v, B(s)w) \leqq 2$, i.e., in all cases $\operatorname{order}(v, w) \leqq 3$.

In both cases (f) and (g), let $u_1 = (0, 1)$, $u_2 = (1, 0)$ (u_1 and u_2 are eigenvectors of $A(t)$) and then define $\operatorname{order}(v, w)$ exactly as in case (e). Observe that if $\theta = \arg v$, $\Phi = \arg w$, then $\operatorname{order}(v, w)$ is uniquely determined by θ and Φ since if $\lambda > 0$, $\tau > 0$

(19) $$\operatorname{order}(\lambda v, \tau w) = \operatorname{order}(v, w)$$

(f) $A(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$,

$$B(s) = \begin{bmatrix} r/(r-1)^{\frac12} & 1/(r-1)^{\frac12} \\ 1/(r-1)^{\frac12} & 1/(r-1)^{\frac12} \end{bmatrix} \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \begin{bmatrix} 1/(r-1)^{\frac12} & -1/(r-1)^{\frac12} \\ -1/(r-1)^{\frac12} & r/(r-1)^{\frac12} \end{bmatrix}$$

where $r > 1$: choose α, $0 < \alpha < \pi/4$ such that $\cot \alpha = r$. Note that the set of eigenvectors of $B(s)$ ($B \neq I$) consists of all vectors (rx, x) and (x, x), $x \neq 0$. The source of $B(s)$ is the line $x_1 = x_2$ and its sink is the line $x_1 = rx_2 = (\cot \alpha)x_2$.

If $\operatorname{order}(v, w) = 1$, then θ and Φ must satisfy $\pi/4 < \theta < \pi + \alpha$, $-3\pi/4 < \Phi < \alpha$; moreover, for each θ, $\pi/4 < \theta < \pi + \alpha$, there is a unique $\Phi = g(\theta)$, $-3\pi/4 < \Phi < \alpha$ such that if $\arg v = \theta$ and $\arg w = g(\theta)$, then $\operatorname{order}(v, w) = 1$. In fact, choose $s(\theta)$ as the unique solution of $\arg B(s)v = \pi/2$; then

(20) $$\Phi = g(\theta) = \arg B^{-1}(s(\theta))u_2 = \arg B(-s(\theta))u_2.$$

The continuous, decreasing function $s(\theta)$ satisfies

$$\lim_{\theta\to(\pi/4)^+} s(\theta) = +\infty, \qquad \lim_{\theta\to(\pi+\alpha)^-} s(\theta) = -\infty;$$

hence the continuous function $g(\theta)$ is increasing on $(\pi/4, \pi + \alpha)$ and satisfies

$$\lim_{\theta\to(\pi/4)^+} g(\theta) = -3\pi/4 \qquad \lim_{\theta\to(\pi+\alpha)^-} g(\theta) = \alpha.$$

Denote the unique solution of $g(\theta) = -\pi/2$ by γ, and let $\delta = g(\pi)$; since $g(\pi/2) = 0$, $\pi/4 < \gamma < \pi/2$ and $0 < \delta < \alpha$.

From (20) and a straightforward calculation it follows that

(21) $$\cot g(\theta) = r + 1 - (r/\cot\theta);$$

define $h(\theta) = \cot\theta/\cot g(\theta)$. A routine calculation yields

(22) $$\frac{dh}{d\theta} = \frac{-(\csc^2\theta)(r + 1 - (2r/\cot\theta))}{\cot^2 g(\theta)}$$

and hence $dh/d\theta > 0$ on $\pi/4 < \theta < \pi/2$, $\theta \neq \gamma$ while $dh/d\theta < 0$ on $\pi/2 < \theta < \pi + \alpha$, $\theta \neq \pi$. Thus the behavior of $h(\theta)$ is as follows: on $(\pi/4, \gamma)$, h is strictly increasing,

$$\lim_{\theta\to(\pi/4)^+} h(\theta) = 1, \qquad \lim_{\theta\to\gamma^-} h(\theta) = +\infty;$$

on $(\gamma, \pi/2]$, h is strictly increasing,

$$\lim_{\theta\to\gamma^+} h(\theta) = -\infty, \qquad \lim_{\theta\to(\pi/2)^-} h(\theta) = h(\pi/2) = 0;$$

on $[\pi/2, \pi)$, h is strictly decreasing,

$$\lim_{\theta\to(\pi/2)^+} h(\theta) = h(\pi/2) = 0, \qquad \lim_{\theta\to\pi^-} h(\theta) = -\infty;$$

on $(\pi, \pi + \alpha)$, h is strictly decreasing,

$$\lim_{\theta\to\pi^+} h(\theta) = +\infty, \qquad \lim_{\theta\to(\pi+\alpha)^-} h(\theta) = 1.$$

The graph of $h(\theta)$ is sketched in Figure 1 below.

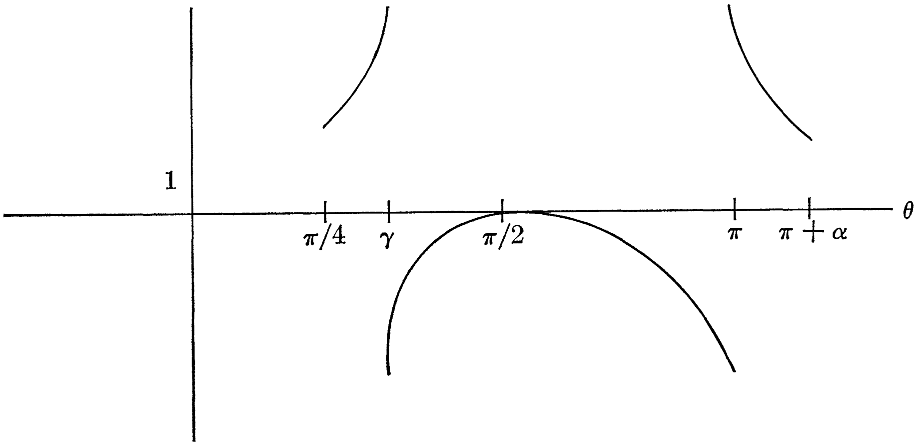

Figure 1. Graph of $h(\theta)$.

The function $h(\theta)$ will play a crucial role in the determination of order(v, w); this results from the fact that all the matrices $A(t)$ preserve the ratio $\cot \theta / \cot \Phi$, i.e.,

$$\frac{\cot(\arg A(t)v)}{\cot(\arg A(t)w)} = \frac{e^{2t} \cot \arg v}{e^{2t} \cot \arg w} = \frac{\cot \theta}{\cot \Phi}. \tag{23}$$

It follows immediately from the above characterization of pairs of vectors (v, w) of order 1 that order$(v, w) \leqq 2$ if and only if there exists an $A(t)$ such that $\pi/4 < \arg A(t)v < \pi + \alpha$ and $\arg A(t)w = g(\arg A(t)v)$. Consider the following cases: (a′) $0 < \theta < \pi/2$, (b′) $\theta = \pi/2$, (c′) $\pi/2 < \theta < \pi$, (d′) $\theta = \pi$, (e′) $\pi < \theta < 3\pi/2$, and (f′) $3\pi/2 \leqq \theta \leqq 2\pi$. In case (a′), order $(v, w) \leqq 2$ if and only if $\theta - \pi < \Phi < 0$. First, note that for all $A(t)$, $0 < \arg A(t)v < \pi/2$. If $0 \leqq \Phi < \theta$, then $0 \leqq \cot \theta / \cot \Phi < 1$. It follows from (23) and the fact that on $(\pi/4, \pi/2) h(\theta)$ never assumes a value $y \in [0, 1)$ that for all $A(t)$, order$(A(t)v, A(t)w) > 1$, i.e., order $(v, w) > 2$. If $\theta - \pi < \Phi < 0$, then either $\cot \theta / \cot \Phi > 1$ ($\theta - \pi < \Phi < -\pi/2$) or $\cot \theta / \cot \Phi < 0$ ($-\pi/2 < \Phi < 0$) or $\Phi = -\pi/2$; if $\Phi \neq \pi/2$, there exists a unique θ_0, $\pi/4 < \theta_0 < \pi/2$, $\theta_0 \neq \gamma$ such that $h(\theta_0) = \cot \theta / \cot \Phi$ (if $\Phi = -\pi/2$, let $\theta_0 = \gamma$). Choose $A(t)$ so that $\arg A(t)v = \theta_0$; from (23) it follows that $\arg A(t)w = g(\theta_0)$, and hence order$(v, w) \leqq 2$. In case (b′), order$(v, w) > 2$ unless $\Phi = 0$. In case (c′), order$(v, w) \leqq 2$ if and only if $0 < \Phi < \pi/2$: note that on $(\pi/2, \pi)$ the range of $h(\theta)$ consists of all real numbers <0. In case (d′), order$(v, w) \leqq 2$ if and only if $0 < \Phi < \pi/2$: note that there is an $A(t)$ such that $\arg A(t)w = g(\pi) = \delta$, $0 < \delta < \alpha$ if and only if $0 < \Phi < \pi/2$. In case (e′), order $(v, w) \leqq 2$ if and only if $\theta - \pi < \Phi < \pi/2$: note that on $(\pi, \pi + \alpha)$ the range of $h(\theta)$ consists of all real numbers >1. Finally, in case (f′), order(v, w) is always >2 since for all $A(t)$, $3\pi/2 \leqq \arg A(t)v \leqq 2\pi$.

If $\alpha \leqq \Phi < \theta \leqq \pi/4$, then for all $B(s)$, $\alpha \leqq \arg B(s)w < \arg B(s)v \leqq \pi/4$. Hence for all $B(s)$, order$(B(s)v, B(s)w) > 2$, i.e., order$(v, w) > 3$. A similar argument shows that if $\pi + \alpha \leqq \Phi < \theta \leqq 5\pi/4$, then again order$(v, w) > 3$. In all *other* cases, order(v, w) is at most 3. To establish this consider the three cases: (a′) $-3\pi/4 < \theta \leqq \pi/4$, (b′) $\pi/4 < \theta < \pi + \alpha$, and (c′) $\pi + \alpha \leqq \theta \leqq 5\pi/4$. In case (a′), $\theta - \pi < \Phi < \alpha$ (the case $\alpha \leqq \Phi < \theta \leqq \pi/4$ is excluded) and hence it is always possible to find a $B(s)$ such that $0 < \arg B(s)v \leqq \pi/4$ and $\pi < \arg B(s)w < 2\pi$. In case (b′), choose $B(s)$ such that $\pi/2 < \arg B(s)v < \pi$ and $0 < \arg B(s)w < \pi/2$. In case (c′), $\theta - \pi < \Phi < \pi + \alpha$ (the case $\pi + \alpha \leqq \Phi < \theta < 5\pi/4$ is excluded) and hence it is possible to chose $B(s)$ such that $\pi + \alpha \leqq \arg B(s)v \leqq 5\pi/4$ and $0 < \alpha < \arg B(s)w < \pi/2$. Thus in all three cases it is possible to find a $B(s)$ such that order$(B(s)v, B(s)w) \leqq 2$, i.e., order$(v, w) \leqq 3$, provided that neither $\alpha \leqq \Phi < \theta \leqq \pi/4$ nor $\pi + \alpha \leqq \Phi < \theta \leqq 5\pi/4$ hold. In both these exceptional cases order$(v, w) = 4$, in the first case, choose any $A(t)$ such that $0 < \arg A(t)v \leqq \alpha$; then order$(A(t)v, A(t)w) = 3$. In the second case, choose any $A(t)$ such that $\pi < \arg A(t)v \leqq \pi + \alpha$; again order$(A(t)v, A(t)w) = 3$.

Equation (14) implies that unless $\alpha \leqq \Phi < \theta \leqq \pi/4$ or $\pi + \alpha \leqq \Phi < \theta \leqq 5\pi/4$, every matrix $D \in SL(2, \mathbf{R})$ such that $Dv = \lambda u_1$, $Dw = \tau u_2$, $\lambda > 0$, $\tau > 0$

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can be expressed as a product of $A(t)$ and $B(s)$ of length at most 4; in the two exceptional cases, (14) provides the estimate 5. This last estimate is unsatisfactory since the direct analysis given below shows that even in both these exceptional cases D can still be expressed as a product of length 4.

First, consider the case $\alpha \leqq \Phi < \theta \leqq \pi/4$. Let t_0 be the unique solution of $\arg A(t)v = \alpha$; then $0 < \arg A(t_0)w < \arg A(t_0)v = \alpha$. Let s_0 be the unique solution of $\arg B(s)A(t_0)w = 0$; it follows that for all s, $-\infty < s < s_0$, $-3\pi/4 < \arg B(s)A(t_0)w < 0$. Moreover, since $A(t_0)v$ is an eigenvector of $B(s)$ belonging to eigenvalue e^s, it follows that

(24) $$B(s)A(t_0)v = e^s A(t_0)v.$$

Notice that for each s, $-\infty < s < s_0$,

(25) $$\text{order}(B(s)A(t_0)v, B(s)A(t_0)w) = 2;$$

in fact, there is for each such s, a unique $\theta(s)$ such that

(26) $$h(\theta(s)) = \cot\alpha/\cot(\arg B(s)A(t_0)w)$$

(if $\arg B(s)A(t_0)w = -\pi/2$, define $\theta(s) = \gamma$). The function $\theta(s)$ is continuous and increasing on $(-\infty, s_0)$, $\lim_{s\to-\infty}\theta(s) = \beta$ where β is the unique solution on $(\pi/4, \gamma)$ of $h(\theta) = \cot\alpha$, $\lim_{s\to s_0-}\theta(s) = \pi/2$. Further, for each s, $-\infty < s < s_0$, there is a unique $t(s)$ such that $\arg(A(t(s))B(s)A(t_0)v) = \theta(s)$, i.e., such that

(27) $$e^{2t(s)}\cot\alpha = \cot\theta(s)$$

and there is a unique $p(s)$ such that $\arg(B(p(s))A(t(s))B(s)A(t_0)v) = \pi/2$, i.e., such that

(28) $$\arg B(p(s))\tilde{v}(s) = \pi/2$$

where $\tilde{v}(s) = (\cot\theta(s), 1)$. The functions $t(s)$ and $p(s)$ are both continuous and decreasing on $(-\infty, s_0)$, $\lim_{s\to-\infty} t(s) = \frac{1}{2}\ln(\cot\beta/\cot\alpha)$, $\lim_{s\to s_0-} t(s) = -\infty$, $\lim_{s\to-\infty} p(s) = p_0$, where p_0 is the unique solution of (28) when $\tilde{v}(s) = (\cot\beta, 1)$, $\lim_{s\to s_0-} p(s) = 0$. Now for each s, $-\infty < s < s_0$, the product

$$B(p(s))A(t(s))B(s)A(t_0)$$

of length 4 satisfies

(29) $$\arg(B(p(s))A(t(s))B(s)A(t_0)v) = \pi/2,$$
$$\arg(B(p(s))A(t(s))B(s)A(t_0)w) = 0.$$

The function $\lambda(s)$ defined by

(30) $$\lambda(s) = ||B(p(s))A(t(s))B(s)A(t_0)v|| = e^s||B(p(s))A(t(s) + t_0)v||$$

is continuous on $(-\infty, s_0)$; clearly $\lambda(s) > 0$. In fact, on $(-\infty, s_0)$, $\lambda(s)$ assumes *all* positive real values. To establish this observe that

(31) $$\lim_{s\to-\infty}\lambda(s) = 0, \qquad \lim_{s\to s_0^-}\lambda(s) = +\infty.$$

This completes the analysis in the case $\alpha \leqq \Phi < \theta \leqq \pi/4$.

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If $\pi + \alpha \leqq \Phi < \theta \leqq 5\pi/4$, let t_0 be the unique solution of $\arg A(t)v = \pi + \alpha$. Let s_0 be the unique solution of $\arg B(s)A(t_0)w = \pi/2$; it follows that for all s, $-\infty < s < s_0$, $\pi/4 < \arg B(s)A(t_0)w < \pi/2$. It is easily verified that equations (24)–(31) inclusive remain valid for this case, with the sole exception that $\tilde{v}(s)$ now must equal $(-\cot\theta(s), -1)$. Note also that in this case, $\theta(s)$ is decreasing, $\lim_{s\to-\infty}\theta(s) = \tilde{\beta}$ where $\tilde{\beta}$ is the unique solution on $(\pi, \pi+\alpha)$ of $h(\theta) = \cot\alpha$, $\lim_{s\to s_0^-}\theta(s) = \pi$ while both the functions $t(s)$ and $p(s)$ are now increasing, $\lim_{s\to-\infty} t(s) = \frac{1}{2}\ln(\cot\tilde{\beta}/\cot\alpha)$, $\lim_{s\to s_0^-} t(s) = +\infty$, $\lim_{s\to-\infty} p(s) = \tilde{p}_0$, $\lim_{s\to s_0^-} p(s) = \tilde{q}_0$ where $\tilde{p}_0$ and $\tilde{q}_0$ are the unique solutions of (28) when $\tilde{v}(s)$ equals $(-\cot\tilde{\beta}, -1)$ and $(-1, 0)$, respectively. Since both (29) and (31) again hold, the analysis of the case $\pi + \alpha \leqq \Phi < \theta \leqq 5\pi/4$ is complete.

(g) $A(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$,

$$B(s) = \begin{bmatrix} r/(r+1)^{\frac{1}{2}} & 1/(r+1)^{\frac{1}{2}} \\ -1/(r+1)^{\frac{1}{2}} & 1/(r+1)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \begin{bmatrix} 1/(r+1)^{\frac{1}{2}} & -1/(r+1)^{\frac{1}{2}} \\ 1/(r+1)^{\frac{1}{2}} & r/(r+1)^{\frac{1}{2}} \end{bmatrix}$$

where $r \geqq 1$: choose α, $0 < \alpha \leqq \pi/4$ such that $\cot\alpha = r$. Note that the set of eigenvectors of $B(s)$ $(B \neq I)$ consists of all vectors $(rx, -x)$ and (x, x), $x \neq 0$. The source of $B(s)$ is the line $x_1 = x_2$ and its sink is the line $x_1 = -rx_2 = -(\cot\alpha)x_2$.

If $\text{order}(v, w) = 1$, then θ and Φ must satisfy $\pi/4 < \theta < \pi - \alpha$, $-\alpha < \Phi < \pi/4$; moreover, for each θ, $\pi/4 < \theta < \pi - \alpha$, there is a unique $\Phi = g(\theta)$, $-\alpha < \Phi < \pi/4$, such that if $\arg v = \theta$, $\arg w = \Phi$, then $\text{order}(v, w) = 1$. In fact, choose $s(\theta)$ as the unique solution of $\arg B(s)v = \pi/2$; then

$$\Phi = g(\theta) = \arg B^{-1}(s(\theta))u_2 = \arg B(-s(\theta))u_2. \tag{32}$$

The continuous, decreasing function $s(\theta)$ satisfies

$$\lim_{\theta\to(\pi/4)^+} s(\theta) = +\infty, \qquad s(\pi/2) = 0, \qquad \lim_{\theta\to(\pi-\alpha)^-} s(\theta) = -\infty;$$

hence the continuous function $g(\theta)$ is decreasing on $(\pi/4, \pi-\alpha)$ and satisfies

$$\lim_{\theta\to(\pi/4)^+} g(\theta) = \pi/4, \qquad g(\pi/2) = 0, \qquad \lim_{\theta\to(\pi-\alpha)^-} g(\theta) = -\alpha.$$

From (32) and a straightforward calculation it follows that

$$\cot g(\theta) = 1 - r + (r/\cot\theta); \tag{33}$$

define $h(\theta) = \cot\theta/\cot g(\theta)$. The continuous function $h(\theta)$ satisfies

$$\lim_{\theta\to(\pi/4)^+} h(\theta) = \lim_{\theta\to(\pi-\alpha)^-} h(\theta) = 1, \qquad h(\pi/2) = 0.$$

Moreover, a routine calculation shows that $dh/d\theta < 0$ on $(\pi/4, \pi/2)$, and $dh/d\theta > 0$ on $(\pi/2, \pi-\alpha)$ and hence $h(\theta)$ is decreasing on $(\pi/4, \pi/2]$ and increasing on $[\pi/2, \pi-\alpha)$. The graph of $h(\theta)$ is sketched in Figure 2 below.

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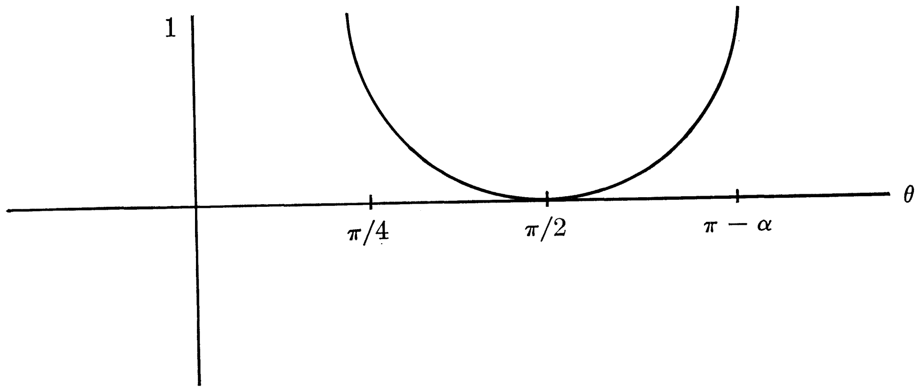

Figure 2. Graph of $h(\theta)$.

Clearly if $\theta = \pi/2$, order$(v, w) \leqq 2$ if and only if $\Phi = 0$. From an analysis completely similar to that used in case (f) it easily follows that if $\theta \neq \pi/2$, order$(v, w) \leqq 2$ if and only if $0 < \theta < \pi$, $\theta \neq \pi/2$ and $0 < \cot \theta/\cot \Phi < 1$. In other words, order$(v, w) \leqq 2$ if and only if either (a′) $0 < \Phi < \theta < \pi/2$ or (b′) $\theta = \pi/2$, $\Phi = 0$ or (c′) $\pi/2 < \theta < \pi$, $\theta - \pi < \Phi < 0$.

Next it will be shown that a sufficient condition for order(v, w) to be $\leqq 3$ is that $\pi/4 < \theta < \pi - \alpha$. Without loss of generality, assume that $\theta = \pi/2$; then $-\pi/2 < \Phi < \pi/2$. If $-\alpha < \Phi < \pi/2$, there exists a $B(s)$ such that $0 < \arg B(s)w < \arg B(s)v < \pi/2$ while if $-\pi/2 < \Phi \leqq -\alpha$, there exists a $B(s)$ such that $\pi/2 < \arg B(s)v < \pi - \alpha < \pi$ and $\arg B(s)v - \pi < \arg B(s)w \leqq -\alpha < 0$. Hence in both cases there exists a $B(s)$ such that order$(B(s)v, B(s)w) \leqq 2$, i.e., order$(v, w) \leqq 3$.

It is easily verified that for any vector $v \neq 0$ there exists a product $A(q)B(p)A(t)B(s)$ of length at most 4 such that

$$\pi/4 < \arg(A(q)B(p)A(t)B(s)v) < \pi - \alpha \tag{34}$$

(in fact, a product of length 3 suffices unless $\theta = -\pi/2$). Hence order$(v, w) \leqq 7$ for all (v, w) (in fact, if $\theta \neq -\pi/2$, order$(v, w) \leqq 6$). From (14) it follows that the order of generation is $\leqq 8$; surprisingly, equality holds.

Note that order$(-u_1, -u_2) =$ order$((0, -1), (-1, 0))$ must be odd since $-u_1$ and $-u_2$ are eigenvectors of $A(t)$. It follows that to show that order$(-u_1, -u_2) = 7$, it suffices to prove that for all real s, t and p

$$\text{order}(B(p)A(t)B(s)(-u_1), B(p)A(t)B(s)(-u_2)) > 2. \tag{35}$$

Assume first that $5\pi/4 < \arg B(s)(-u_1) \leqq 3\pi/2$; then $\pi \leqq \arg B(s)(-u_2) < \arg B(s)(-u_1)$. It follows that for all t

$$\pi \leqq \arg A(t)B(s)(-u_2) < \arg A(t)B(s)(-u_1) \leqq 3\pi/2 \tag{36}$$

and hence for all p

$$\begin{aligned} \pi - \alpha &< \arg B(p)A(t)B(s)(-u_2) \\ &< \arg B(p)A(t)B(s)(-u_1) < 2\pi - \alpha. \end{aligned} \tag{37}$$

Equation (35) follows immediately from (37) together with the characterization of pairs of vectors (v, w) of order $\geqq 2$. If $-\pi/2 < \arg B(s)(-u_1) < -\alpha$, then $\pi - \alpha < \arg B(s)(-u_2) < \pi$. It follows that

(38) $$-\pi/2 < \arg A(t)B(s)(-u_1) < 0,$$
$$\pi/2 < \arg A(t)B(s)(-u_2) < \pi$$

and hence for all p

(39) $$-3\pi/4 < \arg B(p)A(t)B(s)(-u_1) < \pi/4$$

and

$$\pi/4 < \arg B(p)A(t)B(s)(-u_2) < 5\pi/4.$$

Again (35) follows directly from (39). Thus the order of generation is $\geqq 7$; to prove that it equals 8 it suffices to show that $-I$ cannot be represented as a product of length $\leqq 7$.

Since $-I(-u_1) = u_1$, $-I(-u_2) = u_2$, $-I$ cannot be represented as a product of length <7, and if it could be represented as a product of length 7, such a product would have to begin an end with a $B(s)$. Hence assume

(40) $$B(z)A(y)B(x)A(q)B(p)A(t)B(s) = -I;$$

it follows that

(41) $$B(z+s)A(y)B(x)A(q)B(p)A(t) = B(s)(-I)B(-s) = -I,$$

i.e., $-I$ could then be represented as a product of length 6, a contradiction.

Remark 1. To understand why in case (g) $-I$ could not be expressed as a product of length 7, consider the "inner" automorphism of $SL(2, \mathbf{R})$ induced by

(42) $$C = \begin{bmatrix} 1 & r \\ 1 & -1 \end{bmatrix}, \qquad C \in GL(2, \mathbf{R}).$$

Note that although $C \notin SL(2, \mathbf{R})$, if $D \in SL(2, \mathbf{R})$, then $C^{-1}DC \in SL(2, \mathbf{R})$. Clearly C transforms the pair of infinitesimal transformations ϵ, η of case (g) into the pair η, ϵ since C takes the eigenvectors of ϵ into those of η and the eigenvectors of η into those of ϵ. It follows that order$((0, -1), (-1, 0))$ with respect to $((0, 1), (1, 0))$ must equal order$((-r, 1), (-1, -1))$ with respect to $((r, -1), (1, 1))$. Now if $-I$ were expressible as a product of length 7 whose first and last elements were a $B(s)$, then since both $(r, -1)$ and $(1, 1)$ are eigenvectors of $B(s)$ it would follow that order$((-r, 1), (-1, -1))$ with respect to $((r, -1), (1, 1))$ would be 5 and not 7.

Remark 2. Observe that in both cases (e) and (g), the order of generation of $SL(2, \mathbf{R})$ is equal to the maximum of order$(v, w) + 1$ while in (f) it equals the maximum of order(v, w).

Remark 3. Note that the order of generation of $SL(2, \mathbf{R})$ is uniquely determined by the type of the infinitesimal transformations, i.e., elliptic, parabolic or hyperbolic, except that if both are hyperbolic, then the order of generation depends upon whether or not the eigenvectors separate or interlace.

5. Let $n \neq \infty$ be the order of generation of a connected Lie group H by $A(t)$ and $B(s)$. It is of interest to determine whether every element of H can, in fact, be expressed as a product of length n whose last element is an $A(t)$; a dual question may be asked of $B(s)$. Note that any element that can be expressed as a product of length $<n$ can be expressed both as a product of length n whose last element is an $A(t)$ and one whose last element is a $B(s)$ by inserting $I = A(0) = B(0)$ an appropriate number of times.

If there is any automorphism of the group that interchanges the two one-parameter subgroups, then both questions must have the same answer. The same conclusion holds under the quite different assumption that n is even; if an element is not representable as a product of length n ending in an $A(t)$, then its inverse is not representable as a product of length n ending in a $B(s)$.

For SU(2) both questions must have the same answer since every pair of one-parameter subgroups of SU(2) can be interchanged by some inner automorphism. The answer is affirmative; this follows immediately from the fact that it is affirmative for $SU(2)/\{I, -I\}$ [**1**].

Every element of $SL(2, \mathbf{R})$ can be expressed as a product $A(t)B(s)A(u)$ but not all elements can be expressed as a product $B(s)A(t)B(v)$ in cases (b) and (c) ($A(t)$ is elliptic, $B(s)$ is parabolic or hyperbolic, respectively); again this follows directly from the corresponding assertion for $SL(2, \mathbf{R})/\{I, -I\}$ [**2**]. In cases (d)-(g) inclusive the order of generation of $SL(2, \mathbf{R})$ is even and hence both questions must have the same answer. It is affirmative except for case (f), i.e., except if both one-parameter subgroups are hyperbolic with eigenvectors separating. In case (d), this is clear from the proof of Theorem 2; in the remaining cases see Remark 2 above.

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References

1. F. Lowenthal, *Uniform finite generation of the rotation group*, Rocky Mountain J. Math. *1* (1971), 575–586.

2. ——— *Uniform finite generation of the isometry groups of Euclidean and non-Euclidean geometry*, Can. J. Math. *23* (1971), 364–373.

3. ——— *On subsemigroups of the projective group on the line*, Can. J. Math. *20* (1968), 1001–1011.

4. S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs, N.J., 1964).

5. Hidehiko Yamabe, *On an arcwise connected subgroup of a Lie group*, Osaka J. Math. *2* (1950), 13–14.

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