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COUNTEREXAMPLE TO A CONJECTURE OF GREENLEAF

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Greenleaf states the following conjecture in [1, p. 69]. Let G be a (connected, separable) amenable locally compact group with left Haar measure, μ, and let U be a compact symmetric neighbourhood of the unit. Then the sets, $\{U^m\}$, have the following property: given $\varepsilon > 0$ and compact $K \subset G$, $\exists m_0 = m_0(\varepsilon, K)$ such that

$$|\mu(xU^m \cap U^m)/\mu(U^m) - 1| < \varepsilon \quad \forall m \geq m_0 \quad \text{and} \quad \forall x \in K.$$

In this paper we exhibit a counterexample to this conjecture, the group G of pairs $\{(x, y) \mid x, y \in R, x > 0\}$ with multiplication, $(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + y_1)$, and the polygon, U, whose sides connect, in order, the points, $(1/n, n)$, $(1, n)$, (n, n^2), $(n, -n^2)$, $(1, -n)$, $(1/n, -n)$ and back to $(1/n, n)$, where $n \geq 2$. U us a compact symmetric neighbourhood of the unit $(1, 0)$. We prove that

$$\mu((a, 0)U^m \cap U^m)/\mu(U^m) \leq 1 - (1 - 1/a)/8 \quad \forall m = 1, 2, 3, \ldots, \text{if } n \geq 2 \text{ and } a > 1.$$

The group G can be regarded as a subset of the plane. It is easy to check by induction that:

(i) each U^m is symmetric about the x-axis.

(ii) $\max\{y \mid (x, y) \in U^m\} = \sum_{i=2}^{m+1} n^i \leq 2n^{m+1}$ ($n \geq 2$, always).

(iii) U^m is contained between the lines $x = 1/n^m$ and $x = n^m$.

(iv) each U^m is convex in the y-variable; that is, if $(x, y_1), (x, y_2) \in U^m$ and $0 \leq b \leq 1$, then $(x, by_1 + (1-b)y_2) \in U^m$.

Left Haar measure, μ, on G is given by $d\mu = x^{-2}\, dy\, dx$, so $\mu(U^m \cap \{(x, y) \mid x \geq 1\}) \leq \int_1^{n^m} 4n^{m+1}x^{-2}\, dx \leq 4n^{m+1}$. From (iv) we deduce that the upper part of the boundary of U^m is given by $\{(x, y) \in U^m \mid y = \max\{y_1 \mid (x, y_1) \in U^m\}\}$.

LEMMA. *$U^m \cap \{(x, y) \mid x \leq 1\}$, $m \geq 2$, is bounded above by the lines*

$$y = x\left(\sum_2^m n^i\right) + n \qquad \textit{from } x = 1/n^m \quad \textit{to} \quad x = 1/n^{m-1}$$

$$y = x\left(\sum_2^m n^i + n^m\right) \qquad x = 1/n^{m-1} \quad x = 1/n^{m-2}$$

$$y = x\left(\sum_2^{m-1} n^i\right) + n^2 + n^2 \qquad x = 1/n^{m-2} \quad x = 1/n^{m-3}$$

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$$\begin{array}{lll}
y = x\left(\sum_2^{m-1} n^i+n^{m-1}\right)+n^2 & x = 1/n^{m-3} & x = 1/n^{m-4} \\
y = x\left(\sum_2^{m-2} n^i\right)+n^3+\sum_2^3 n^i & x = 1/n^{m-4} & x = 1/n^{m-5} \\
y = x\left(\sum_2^{m-2} n^i+n^{m-2}\right)+\sum_2^3 n^i & x = 1/n^{m-5} & x = 1/n^{m-6} \\
\vdots & \vdots & \vdots \\
y = x\left(\sum_2^{m-k} n^i\right)+n^{k+1}+\sum_2^{k+1} n^i & x = 1/n^{m-2k} & x = 1/n^{m-2k-1} \\
y = x\left(\sum_2^{m-k} n^i+n^{m-k}\right)+\sum_2^{k+1} n^i & x = 1/n^{m-2k-1} & x = 1/n^{m-2k-2} \\
\vdots & \vdots & \vdots
\end{array}$$

Proof. We give an indication of the proof of the induction step. Suppose the formulae are true for m. Since U^{m+1} is convex in the y-variable, we calculate

$$\max\{y \mid (x_0, y) \in U^{m+1}\} = y_0 \quad \text{for each } x_0 \in [1/n^{m+1}, 1].$$

Now, if $(x_0, y) \in U^{m+1}$,

$$(x_0, y) = (x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2+y_1),$$

where $(x_1, y_1) \in U$ and $(x_2, y_2) \in U^m$. The way to get the maximum value of $x_1y_2+y_1$ is to choose $x_1 \in [1/n, n]$ as large as is compatible with $x_1x_2=x_0$ and $x_2 \in [1/n^m, n^m]$, then choose $y_1(y_2)$ as large as possible keeping $(x_1, y_1) \in U$ $((x_0/x_1, y_2) \in U^m)$.

The result of these instructions is that the upper boundary of U^{m+1} between $x=1/n^j$ and $x=1/n^{j-1}$ is:

(i) the left translate by (n, n^2) of the upper boundary of U^m between $x=1/n^{j+1}$ and $x=1/n^j$ if $j \le m-1$;

(ii) the right translate by $(1/n^m, \sum_{-m+2}^{1} n^i)$ of the upper boundary of U between $x=1$ and $x=n$ if $j=m$;

(iii) the right translate by $(1/n^m, \sum_{-m+2}^{1} n^i)$ of the upper boundary of U between $x=1/n$ and $x=1$ if $j=m+1$.

Using (ii) and (iii), the induction formulae are easily verified for $m=2$ (only two lines apply). We verify the formula for the boundary of U^{m+1} between $x=1/n^{m+1-2k}$ and $x=1/n^{m-2k}$, where $m+1-2k \le m-1$, namely, $k>0$.

Since

$$(n, n^2)\left(x, x\sum_2^{m-k+1} n^i+n^k+\sum_2^k n^i\right)=\left(nx, n\left(x\sum_2^{m-k+1} n^i+n^k+\sum_2^k n^i\right)+n^2\right)$$

which lies on the line, $y=x\sum_2^{(m+1)-k} n^i+n^{k+1}+\sum_2^{k+1} n^i$, we are finished. The other formulae can be verified similarly.

If $a>1$,

$$\mu(U^m\backslash(a, 0)U^m) \ge \int_{1/n^m}^{a/n^m} 2nx^{-2}\,dx \ge 2n^{m+1}(1-1/a),$$

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since all the line segments bounding U^m above lie above the line, $y=n$. It remains to show $\mu(U^m)\le 16n^{m+1}$. We have $\mu(U^m \cap \{(x,y) \mid x\ge 1\})\le 4n^{m+1}$ already, and use frequently the fact that $\sum_0^j n^i \le 2n^j$.

Computing an upper bound for $\mu(U^m \cap \{(x,y) \mid x \le 1\})$, we must sum some series of terms having powers of n ranging from m (or $m+1$) down to $m/2$ or $(m-1)/2$, depending whether m is even or odd. We add in all the powers down to zero to facilitate computation.

$$\int_{1/n^m}^{1/n^{m-2}} 4n^m x x^{-2}\,dx + \int_{1/n^{m-2}}^{1/n^{m-4}} 4n^{m-2} x x^{-2}\,dx + \cdots = 8n^m \log n + 8n^{m-2}\log n + \cdots \le 16 n^m \log n.$$

$$\int_{1/n^{m-1}}^{1/n^{m-2}} 2n^m x x^{-2}\,dx + \int_{1/n^{m-3}}^{1/n^{m-4}} 2n^{m-1} x x^{-2}\,dx + \cdots = 2n^m \log n + 2n^{m-1}\log n + \cdots \le 4n^m \log n.$$

$$\int_{1/n^m}^{1/n^{m-1}} 2n x^{-2}\,dx + \int_{1/n^{m-2}}^{1/n^{m-3}} 2n^2 x^{-2}\,dx + \cdots = 2(n^{m+1}-n^m) + 2(n^m - n^{m-1}) + \cdots \le 2n^{m+1}.$$

$$\int_{1/n^{m-2}}^{1} 2n^2 x^{-2}\,dx + \int_{1/n^{m-4}}^{1} 2n^3 x^{-2}\,dx + \cdots \le 2n^m + 2n^{m-1} + \cdots \le 4n^m.$$

Adding up, we have

$$\mu(U^m) \le 4n^{m+1} + 16n^m \log n + 4n^m \log n + 2n^{m+1} + 4n^m \le 16 n^{m+1},$$

since $(\log n)/n \le 2/5$ when $n \ge 2$.

Among the terms calculated when evaluating an upper bound for $\mu(U^m)$, the only one that has not been grossly overestimated and becomes dominant, as $n\to\infty$, is the second last one calculated, $2n^{m+1}$. Thus, by choosing n and a large, one could have $\mu((a,0)U^m \cap U^m)/\mu(U^m) < \varepsilon$, $\forall m = 1, 2, 3, \ldots$, for any given $\varepsilon > 0$

Reference

1. F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies No. 16, Van Nostrand, Princeton, N.J., 1969.

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