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ON ZERO-TRACE COMMUTATORS

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We present some results concerning the trace of certain trace class commutators of operators acting on a separable, complex Hilbert space. It is shown, among other things, that if X is a Hilbert-Schmidt operator and A is an operator such that $AX - XA$ is a trace class operator, then $tr(AX - XA) = 0$ provided one of the following conditions holds : (a) A is subnormal and $A^*A - AA^*$ is a trace class operator, (b) A is a hyponormal contraction and $1 - AA^*$ is a trace class operator, (c) A^2 is normal and $A^*A - AA^*$ is a trace class operator, (d) A^2 and A^3 are normal. It is also shown that if A is a self - adjoint operator, if f is a function that is analytic on some neighbourhood of the closed disc$\{z : |z| \leq ||A||\}$, and if X is a compact operator such that $f(A)X - Xf(A)$ is a trace class operator, then $tr(f(A)X - Xf(A))=0$.

An operator means a bounded linear operator on a separable, complex Hilbert space H . Let $B(H)$, $K(H)$, C_2 , and C_1 denote respectively, the algebra of all bounded linear operators acting on H , the class of compact operators, the Hilbert - Schmidt class, and the trace class operators in $B(H)$. It is known that $K(H)$, C_2 and C_1

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are two-sided ideals in $B(H)$ and that if X and Y are in C_2, then $XY \in C_1$.

If $T \in C_1$ and $\{e_i\}$ is an orthonormal basis of H, then the trace of T, denoted by $tr\ T = \sum_i (Te_i, e_i)$ is independent of the choice of $\{e_i\}$. If X and Y in $B(H)$ are such that both XY and YX lie in C_1, then $tr\ (XY) = tr\ (YX)$ [7, Corollary 3.8].

If H is finite dimensional, then every commutator, that is, operator of the form $AX - XA$, has zero trace. In fact by the Shoda - Albert and Muckenhoupt result [2], an operator on a finite dimensional Hilbert space is a commutator if and only if it has trace 0. If, however H is infinite dimensional, and $AX - XA$ is in C_1, then $tr\ (AX - XA)$ may not be zero even though A is a normal operator. For example, if U is the unilateral shift operator, if $A = aU + bU^*$ where $|a| = |b| \neq 0$, and if $X = U$, then $AX - XA = b(1 - UU^*)$ is a rank one operator, hence in C_1, but $tr\ (AX - XA) = b$. But if A is assumed to be diagonalizable, and X is in $B(H)$ such that $AX - XA \in C_1$, then $tr\ (AX - XA) = 0$ (just evaluate the trace using the eigenvectors of A). Also if X is required to be compact and A is a self - adjoint operator such that $AX - XA \in C_1$ then $tr\ (AX - XA) = 0$, a result which is due to Helton and Howe [3, Lemma 1.3].

In [8], G. Weiss proved that if N is a normal operator, and X is a Hilbert-Schmidt operator such that $NX - XN \in C_1$, then $tr(NX - XN) = 0$.

The purpose of this note is to extend the result of Weiss to non-normal cases, and the result of Helton and Howe to non self-adjoint cases. For other extensions the reader is refered to [4]. In [8] the question as to whether Weiss' theorem remains true under the weaker assumption that $X \in K(H)$ was raised. Namely, if N is a normal operator and X is a compact operator such that $NX - XN \in C_1$, must $tr(NX - XN) = 0$? In [9] it was observed that if C_1 possessed the generalized Fuglede property (that is for normal N and $X \in B(H)$, $NX - XN \in C_1$ implies $N^*X - XN^* \in C_1$), then the answer to this question would be yes.

Motivated by the work in [5] we now present the following generalizations of Weiss' result.

THEOREM 1. *Let* $A \in B(H)$ *be subnormal with* $A^*A - AA^* \in C_1$. *If* $X \in C_2$ *and* $AX - XA \in C_1$, *then* $tr(AX - XA) = 0$.

Proof. By assumption there exists a Hilbert space H_1 and there exists a normal operator N on $H \oplus H_1$ such that $N = \begin{bmatrix} A & R \\ 0 & A_1 \end{bmatrix}$.

Let $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Then $Y \in C_2$ as an operator acting on $H \oplus H_1$.

Now $NY - YN = \begin{bmatrix} AX - XA & -XR \\ 0 & 0 \end{bmatrix}$. N being normal implies that $A^*A - AA^* = RR^* \in C_1$. Thus $R \in C_2$ and so $XR \in C_1$. Hence $NY - YN \in C_1$. Weiss' result now implies that $tr(NY - YN) = 0$. But $tr(NY - YN) = tr(AX - XA)$. Therefore $tr(AX - XA) = 0$ as required.

COROLLARY. *Let* $A \in B(H)$ *be a subnormal and rationally cyclic operator. If* $X \in C_2$ *and* $AX - XA \in C_1$, *then* $tr(AX - XA) = 0$.

Proof. The conclusion follows from Theorem 1 and the fact that if A is a rationally cyclic hyponormal operator, then $A^*A - AA^* \in C_1$ [1].

If $A \in B(H)$ is a hyponormal contraction, and if $1 - AA^* \in C_1$, then $1 - A^*A \in C_1$. In fact it follows from the hypothesis that $1 - AA^* \geq 0$, $1 - A^*A \geq 0$, and $((1 - A^*A)f,f) \leq ((1 - AA^*)f,f)$ for any vector $f \in H$ [5, Lemma 1].

THEOREM 2. *Let* $A \in B(H)$ *be a hyponormal contraction with* $1 - AA^* \in C_1$. *If* $X \in C_2$ *and* $AX - XA \in C_1$, *then* $tr(AX - XA) = 0$.

Proof. Let $U = \begin{bmatrix} A & (1-AA^*)^{\frac{1}{2}} \\ (1-A^*A)^{\frac{1}{2}} & -A^* \end{bmatrix}$ on $H \oplus H$. Then U is

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unitary [2]. Let $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Then $UY - YU = \begin{bmatrix} AX - XA & -X(1-AA^*)^{\frac{1}{2}} \\ (1 - A^*A)^{\frac{1}{2}}X & 0 \end{bmatrix}$

Since $1 - AA^* \in C_1$, it follows that $1 - A^*A \in C_1$. Thus both $(1 - AA^*)^{\frac{1}{2}}$ and $(1 - A^*A)^{\frac{1}{2}}$ lie in C_2. But $X \in C_2$ implies that $(1 - AA^*)^{\frac{1}{2}}X \in C_1$ and $(1 - A^*A)^{\frac{1}{2}}X \in C_1$. Since $AX - XA \in C_1$, it follows that $UY - YU \in C_1$. Now Weiss' result implies that $tr(UY - YU) = 0$ since U is unitary and $Y \in C_2$. But $tr(UY - YU) = tr(AX - XA)$ and so the proof is complete.

THEOREM 3. *Let $T \in B(H)$ be such that T^2 is normal and $T^*T - TT^* \in C_1$. If $X \in C_2$ and $TX - XT \in C_1$, then $tr(TX - XT = 0$.*

Proof. Since T^2 is normal, it follows by Radjavi's and Rosenthal's structure theorem [6] that $T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix}$, where A,B are normal operators, $C \geq 0$ and one - to - one, $BC = CB$, and $\sigma(B)$ is contained in the closed upper half - plane. Now $T^*T - TT^* \in C_1$ implies that $C^2 \in C_1$. Hence $C \in C_2$. Therefore $T = N+K$, where $N = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -B \end{bmatrix}$ is normal and $K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{bmatrix} \in C_2$. Thus $TX - XT = NX - XN + KX - XK$. Since KX and XK both lie in C_1 and $TX - XT \in C_1$, it follows that $NX - XN \in C_1$, from which it follows by Weiss' result that $tr(NX - XN)=0$. Since $tr(KX - XK)= 0$, it follows that $tr(TX - XT) = 0$ as required.

THEOREM 4. *Let $T \in B(H)$ be such that T^2 and T^3 are normal. If $X \in C_2$ and $TX - XT \in C_1$, then $tr(TX - XT) = 0$.*

Proof. Since T^2 is normal, it follows that $T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix}$

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as in the proof of Theorem 3 above, where A, B are normal operators, $C \geq 0$ and one - to - one and $BC = CB$.

Now $T^3 = \begin{bmatrix} A^3 & 0 & 0 \\ 0 & B^3 & B^2C \\ 0 & 0 & -B^3 \end{bmatrix}$. But T^3 being normal implies that $B^{*3}B^3 = B^3B^{*3} + B^{*2}B^2C^2$. Hence $B^{*2}B^2C^2 = 0$. Since C is one - to - one, it follows that $B^{*2}B^2 = 0$. Thus $B^2 = 0$ and so $B = 0$ since it is normal. Therefore $T = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{bmatrix}$. Let $X = [X_{ij}](i, j = 1,2,3)$ be the corresponding matrix representation of X . Then

$$TX - XT = \begin{bmatrix} AX_{11} - X_{11}A & AX_{12} & AX_{13}-X_{12}C \\ CX_{31} - X_{21}A & CX_{32} & CX_{33}-X_{22}C \\ -X_{31}A & 0 & -X_{32}C \end{bmatrix}.$$

Since $TX - XT \in C_1$, it follows that every entry of this matrix is in C_1. Therefore $tr(TX - XT) = tr(AX_{11} - X_{11}A) + tr(CX_{32}) - tr(X_{32}C)$. Since A is normal and $X_{11} \in C_2$, it follows that $tr(AX_{11} - X_{11}A) = 0$. Now CX_{32} and $X_{32}C$ both lie in C_1 . Thus $tr(CX_{32}) = tr(X_{32}C)$ and so $tr(TX - XT) = 0$ as required.

Before focusing our attention on the Helton - Howe result, we give the following related result.

THEOREM 5. *Let $V \in B(H)$ be an isometry of finite multiplicity. If $X \in K(H)$ and $VX - XV \in C_1$, then $tr(VX - XV) = 0$.*

Proof. By the observation in [9]it is sufficient to show that $V^*X - XV^* \in C_1$. Since $1 - VV^*$ is of finite rank, it follows that $VV^* = 1 + C$ for some finite rank operator C .

Now $V^*(VX - XV)V^* \in C_1$. Thus $XV^* - V^*XVV^* \in C_1$ and so $XV^* - V^*X(1 + C) \in C_1$. Therefore $V^*X - XV^* \in C_1$ as required.

Remark. The unilateral shift and unitary operators are important special cases for which Theorem 5 holds.

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Our first generalization of the Helton - Howe result can be stated as follows.

THEOREM 6. *Let* $A \in B(H)$ *be self - adjoint. If* $f = p/q$ *is a rational function with poles off* $\sigma(A)$ *, and* $X \in K(H)$ *with* $S = f(A)X - Xf(A) \in C_1$ *, then* $tr(q(A)Sq(A)) = 0$ *.*

Proof. We consider the following cases.

Case (i) If f is constant, then the result holds trivially.

Case (ii) If $f(t) = t^n (n \geq 1)$, then

$$f(A)X - Xf(A) = A(\sum_{k=0}^{n-1} A^{n-1-k} XA^k) - (\sum_{k=0}^{n-1} A^{n-1-k} XA^k) A$$

$= AY - YA$ for some $Y \in K(H)$. Hence the result in this case follows from the Helton - Howe lemma.

Case (iii). If f is a polynomial, then the result follows from case (ii) by addition.

Case (iv). If $f(t) = p(t)/q(t)$, where p, q are polynomials and q has no zeros on $\sigma(A)$, then by the spectral mapping theorem it follows that $q(A)$ is invertible.

Now $P(A)Xq(A) - q(A)XP(A) = q(A)\ Sq(A) \in C_1$ and so

$q(A)Sq(A) = [P(A)(Xq(A)) - (Xq(A))P(A)] - [q(A)(XP(A)) - (XP(A))q(A)]$.

By case (iii) we obtain that $q(A)Sq(A) = (AY - YA) - (AZ - ZA)$, where Y and Z are compact operators. Hence $q(A)Sq(A) = A(Y - Z) - (Y - Z)A$, and so $tr(q(A)Sq(A)) = 0$ by the Helton - Howe lemma.

Notice that the Helton - Howe lemma becomes a special case of Theorem 6 upon taking $p(t) = t$ and $q(t) = 1$.

We conclude with the following result.

THEOREM 7. *Let* $A \in B(H)$ *be self - adjoint. If* f *is a function that is analytic on some neighbourhood of the closed disc* $\{z : |z| \leq ||A||\}$ *and* $X \in K(H)$ *with* $T = f(A)X - Xf(A) \in C_1$ *, then* $tr\ T = 0$ *.*

Proof. Without loss of generality, we may assume that $||A|| \leq 1$ and $||X|| \leq 1$. Thus f is analytic on the disc $D = \{z: |z| < 1 + r\}$

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for some $r > 0$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be the power series expansion of f . Let $f_n(z) = \sum_{k=0}^{n} a_k z^k$. Then $f_n(z) \longrightarrow f(z)$ uniformly on the closed unit disc and so $f_n(A) \longrightarrow f(A)$. Therefore

$$\begin{aligned} T &= \lim_{n \to \infty} f_n(A)X - X \lim_{n \to \infty} f_n(A) \\ &= \lim_{n \to \infty} (f_n(A)X - Xf_n(A)) \\ &= \lim_{n \to \infty} (AX_n - X_nA) \text{ , where } \{X_n\} \text{ is a sequence of compact} \end{aligned}$$

operators as shown in case (iii) of Theorem 6. In fact it is not hard to see that $X_n = a_1X + a_2(AX + XA) + \ldots + a_n(\sum_{k=0}^{n-1} A^{n-k-1}XA^k)$. For $n > m$ we have

$$X_n - X_m = a_{m+1}(\sum_{k=0}^{m} A^{m-k}XA^k) + a_{m+2}(\sum_{k=0}^{m+1} A^{m+1-k}XA^k) + \ldots + a_n(\sum_{k=0}^{n-1} A^{n-k-1}XA^k)$$

and so $||X_n - X_m|| \le (m+1)|a_{m+1}| + (m+2)|a_{m+2}| + \ldots + n|a_n|$. Since f is analytic on D , it follows that $\sum_{k=m+1}^{n} k|a_k| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Therefore $\{X_n\}$ is a Cauchy sequence of compact operators, hence it is convergent to some compact operator Y . Now $T = AY - YA$. Since A is self - adjoint and Y is compact and $T \in C_1$, it follows by the Helton - Howe lemma that $tr\ T = 0$ as required.

We would like to remark here that $f(A)$ as described in Theorem 7 is normal operator but it need not be self - adjoint.

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