Can. J. Math., Vol. XXXIV, No. 5, 1982, pp. 1047–1058

PROPERTY $k_{\alpha,n}$ ON SPACES WITH STRICTLY POSITIVE MEASURE

S. ARGYROS AND N. KALAMIDAS

In this paper we study intersection properties of measurable sets with positive measure in a probability measure space, or equivalently, intersection properties of open subsets on a compact space with a strictly positive measure.

The first result in this direction is due to Erdös and it is a negative solution to the problem of calibers on such spaces. In particular, under C.H., Erdös proved that Stone's space of Lebesque measurable sets of [0, 1] modulo null sets, does not have $\aleph_1$-caliber.

A statement in measure theoretic language equivalent to Erdös' example is the following. Under C.H. there is an uncountable family $\{K_\xi, \xi < \omega^+\}$ of closed subsets of [0, 1] with positive measure, such that for every uncountable set $I \subset \omega^+$ there are $\xi_1, \ldots, \xi_{m(I)} \in I$ with

$$\lambda(K_{\xi_1} \cap \ldots \cap K_{\xi_{m(I)}}) = 0$$

Some extensions of this result under G.C.H. for cardinals of the form β^+ with cf $(\beta) = \omega$ are contained in [**2**]. In the same paper, also, there are some positive results about calibers for certain cardinals. (A topological space X has α-caliber if every family $\{U_\xi : \xi < \alpha\}$ of open non-empty subsets of X contains a subfamily with cardinality α and with non-empty intersection.)

In the present paper we prove that families of measurable sets satisfy a property weaker than the α-caliber property, namely property $k_{\alpha,n}$ for all $n < \omega$ and cf $\alpha > \omega$.

In particular, in the first section we prove that if (X, Σ, μ) is a probability measure space and α a cardinal with uncountable cofinality, then for every family $\{A_\xi : \xi < \alpha\}$ of elements of Σ with positive measure and for every $n < \omega$ there is $I_n \subset \alpha$ with $|I_n| = \alpha$ and for every $\xi_1, \ldots, \xi_n \in I_n$ we have that

$$\mu(A_{\xi_1} \cap \ldots \cap A_{\xi_n}) > 0.$$

In the last part of this section we present relations between decomposition properties of $\mathcal{T}^*(X)$ (open non-empty subsets of X) and intersection properties.

Received September 30, 1980 and in revised form September 30, 1981.

The second section mainly contains an application of the above result for compact topological spaces. So, starting from purely functional analytic assumptions on $C(X)$ we can get intersection properties for the space X.

0. Preliminaries.

0.1. *Definition.* Let X be a topological space, α an infinite cardinal and n a positive integer. We say that X *satisfies the property* $k_{\alpha,n}$ if for every family $\{U_\xi, \xi < \alpha\}$ of open subsets of X, there is a subfamily $\{U_\xi, \xi \in A\}$ with $|A| = \alpha$ that satisfies the n-intersection property (i.e., for every $\xi_1, \ldots, \xi_n$ points of A, it follows that $U_{\xi_1} \cap U_{\xi_2} \cap \ldots \cap U_{\xi_n} \neq \emptyset$).

0.2. *Definition.* Let X be a topological space. We say that X *satisfies the property* $(*)$ if the family of non-empty open subsets of X can be written in the following way:

$$\mathcal{T}^*(X) = \bigcup_{n<\omega} \mathcal{T}_n$$

such that the subfamily $\mathcal{T}_n$ contains at most n pairwise disjoint open subsets for every $n < \omega$.

0.3. *Definition.* Let X be a topological space and $\mathcal{S} \subset \mathcal{T}^*(X)$. For $F \subset \mathcal{S}$, F finite we set

$$\text{cal } (F) = \max \{k : \exists\ I \subset F, |I| = k \text{ such that } \cap I \neq \emptyset\}.$$

Now we correspond to the family $\mathcal{S}$ the number

$$k(\mathcal{S}) = \inf \left\{ \frac{\text{cal } (F)}{|F|}, F \subset \mathcal{S}, F : \text{finite} \right\},$$

and we say that the space X *satisfies the property* $(**)$ if the non-empty open subsets of X can be written in the following way:

$$\mathcal{T}^*(X) = \bigcup_{n<\omega} \mathcal{T}_n$$

such that $k(\mathcal{T}_n) > 0$ for every $n < \omega$.

0.4. THEOREM ([**10**]). *Let X be a compact totally disconnected topological space and $\mathcal{S}$ a family of open-and-closed subsets of X with $k(\mathcal{S}) \geqq \delta > 0$. Then there is a regular Borel probability measure μ on X with $\mu(U) \geqq \delta$ for every $U \in \mathcal{S}$.*

0.5. *Definition.* Let X be a compact topological space. We say that X has a *strictly positive measure* if there is a regular Borel measure μ on X with $\mu(U) > 0$ for every $U \in \mathcal{T}^*(X)$.

0.6. *Remark.* It is easy to prove that the existence of a strictly positive measure for a space X, implies the property $(**)$. Kelley's Theorem 0.4 now gives the equivalence of these properties about compact spaces.

For non-compact spaces this is not true. For example if we consider the space

$$X = \{p \in \{0, 1\}^{\omega^+} : |\{i : p(i) = 1\}| < \omega\}$$

then X satisfies the property (**) but there is no Borel measure that is positive on all non-empty open subsets of X.

0.7. THEOREM (Erdös-Rado, regular case). *Let α be an infinite regular cardinal and $\{F_\xi, \xi < \alpha\}$ a family of finite subsets of α. Then there are a subset $A \subset \alpha$, $|A| = \alpha$ and a finite set F such that $F_{\xi_1} \cap F_{\xi_2} = F$ for every $\xi_1, \xi_2 \in A$ with $\xi_1 \neq \xi_2$.*

0.8. THEOREM ([**1**], singular case). *Let α be an infinite singular cardinal and $\{F_\xi, \xi < \alpha\}$ a family of finite subsets of α. Then there are A_σ, $\sigma <$ cf α disjoint subsets of α with $|A_\sigma| = \alpha_\sigma$,*

$$\sigma < \operatorname{cf}\alpha \quad \textit{and} \quad \sum_{\sigma<\operatorname{cf}\alpha} \alpha_\sigma = \alpha \quad \textit{and} \quad E_\sigma, \sigma < \alpha,$$

E finite subsets of α such that

(i) *For $\xi_1, \xi_2 \in A_\sigma$ $\xi_1 \neq \xi_2$ then*

$$F_{\xi_1} \cap F_{\xi_2} = E_\sigma, \textit{for } \sigma < \operatorname{cf}\alpha \quad \textit{and}$$

(ii) *For $\xi_1 \in A_{\sigma_1}, \xi_2 \in A_{\sigma_2}, \sigma_1 \neq \sigma_2$ then*

$$F_{\xi_1} \cap F_{\xi_2} = E \quad \textit{for } \sigma_1, \sigma_2 < \operatorname{cf}\alpha.$$

For a detailed study of known results about intersection and decomposition properties on topological spaces we refer the reader to [**4**].

1. The main theorem.

1.1. LEMMA. *Let $1 \leqq p < \omega$, $1 \leqq n < \omega$, $1 \leqq N < \omega$ with $p \leqq 2^N$ and let $\theta_1, \theta_2, \ldots, \theta_p$ be positive real numbers. Then*

(a) $$\frac{\theta_1^n + \ldots + \theta_p^n}{2p} \geqq \left(\frac{\theta_1 + \ldots + \theta_p}{2p}\right)^n \quad \textit{and}$$

(*b*) $$\sum_{i=1}^{p} \frac{1}{2^N} \theta_i^n \geqq \frac{1}{2^{n-1}} \left(\sum_{i=1}^{p} \frac{1}{2^N} \theta_i\right)^n.$$

Proof. Part (a) follows from Holder's inequality and (b) follows from (a).

1.2. LEMMA. *Let I be a non-empty set, α an uncountable cardinal with* cf $\alpha > \omega$, $\delta > 0$, $\{U_\xi, \xi < \alpha\}$ *a family of non-empty open and closed subsets of $\{0, 1\}^I$ such that $\mu(U_\xi) \geqq \delta$ for $\xi < \alpha$. Then there is $A \subset \alpha$, $|A| = \alpha$*

such that

$$\mu(U_{\xi_1} \cap \ldots \cap U_{\xi_n}) \geqq \left(\frac{\delta}{4}\right)^n \quad \textit{for} \quad 1 \leqq n < \omega, \xi_1, \ldots \xi_n \in A$$

(*where* μ *is the usual product measure on* $\{0, 1\}^I$).

Proof. Let F_ξ be a finite subset of I such that U_ξ depends on F_ξ for $\xi < \alpha$. We distinguish two cases.

Case 1. α is a regular cardinal.

By the Erdös-Rado theorem [**5**] there are $B \subset \alpha$ with $|B| = \alpha$ and a finite set $F \subset I$ such that

$$F_\xi \cap F_{\xi'} = F \quad \text{for } \xi, \xi' \in B, \xi \neq \xi'.$$

We set

$$H_\xi^x = \{y \in \{0, 1\}^{F_\xi - F} : (x, y) \in \pi_{F_\xi}(U_\xi)\}$$

for $x \in \{0, 1\}^F$, $\xi \in B$ and

$$T_\xi = \{x \in \{0, 1\}^F : H_\xi^x \neq \emptyset\}$$

for $\xi \in B$, where π_{F_ξ} is the usual projection. Then

$$\pi_{F_\xi}(U_\xi) = \bigcup_{x \in \{0,1\}^F} (\{x\} \times H_\xi^x) = \bigcup_{x \in T_\xi} (\{x\} \times H_\xi^x) \quad \text{for } \xi \in B.$$

Since α is regular and uncountable, there are $T = \{x_1, \ldots, x_p\} \subset \{0, 1\}^F$ and $C \subset B$, with $|C| = \alpha$ such that

$$T_\xi = T \quad \text{for } \xi \in C.$$

We note that $|T| = p \leqq 2^{|F|}$. Furthermore, there are $A \subset C$, with $|A| = \alpha$ and positive rational numbers $\theta_1, \theta_2, \ldots, \theta_p$ such that

$$\mu(H_\xi^{x_i}) = \theta_i \quad \text{for } 1 \leqq i \leqq p, \xi \in A.$$

Let now $1 \leqq n < \omega$ and $\xi_1, \xi_2, \ldots, \xi_n \in A$. Then

$$U_{\xi_1} \cap U_{\xi_2} \cap \ldots \cap U_{\xi_n} = \left(\bigcup_{i=1}^{p} \{x_i\} \times H_{\xi_1}^{x_i} \times \ldots \times H_{\xi_n}^{x_i}\right) \times \{0, 1\}^{I - (F_{\xi_1} \cup \ldots \cup F_{\xi_n})}$$

and hence using Lemma 1.1

$$(*) \quad \mu(U_{\xi_1} \cap \ldots \cap U_{\xi_n}) = \sum_{i=1}^{p} \frac{1}{2^{|F|}} \mu(H_{\xi_1}^{x_i}) \ldots \mu(H_{\xi_n}^{x_i}) = \sum_{i=1}^{p} \frac{1}{2^{|F|}} \theta_i^n \geqq \frac{1}{2^{n-1}} \left(\sum_{i=1}^{p} \frac{1}{2^{|F|}} \theta_i\right)^n \geqq \left(\frac{\delta^n}{2^{n-1}}\right).$$

From this it follows that

$$\mu(U_{\xi_1} \cap \ldots \cap U_{\xi_n}) \geqq \left(\frac{\delta}{4}\right)^n.$$

https://doi.org/10.4153/CJM-1982-076-3 Published online by Cambridge University Press

Case 2. α is a singular cardinal.

Let $\{\alpha_\sigma, \sigma < \operatorname{cf} \alpha\}$ be a family of uncountable regular cardinals with

$$\alpha = \sum_{\sigma < \operatorname{cf}\alpha} \alpha_\sigma.$$

By Theorem 0.8 there is a set $B \subset \alpha$,

$$B = \bigcup_{\sigma < \operatorname{cf}\alpha} B_\sigma,$$
$$|B_\sigma| = \alpha_\sigma \text{ for } \sigma < \operatorname{cf}\alpha,$$
$$B_\sigma \cap B_{\sigma'} = \emptyset \text{ for } \sigma < \sigma' < \operatorname{cf}\alpha$$

and there are finite sets F, $\{E^\sigma, \sigma < \operatorname{cf}\alpha\}$ such that

(i) $F_\xi \cap F_{\xi'} = E^\sigma$ for $\xi, \xi' \in B_\sigma$, $\xi \neq \xi'$, $\sigma < \operatorname{cf}\alpha$,
(ii) $F_\xi \cap F_{\xi'} = F$ for $\xi \in B_\sigma$, $\xi' \in B_{\sigma'}$, $\sigma < \sigma' < \operatorname{cf}\alpha$.

We set

$$H_\xi^x = \{y \in \{0, 1\}^{F_\xi - F} : (x, y) \in \pi_{F_\xi}(U_\xi)\} \quad \text{for } x \in \{0, 1\}^F, \xi \in B$$

and

$$T_\xi = \{x \in \{0, 1\}^F : H_\xi^x \neq \emptyset\} \quad \text{for } \xi \in B.$$

Then

$$\pi_{F_\xi}(U_\xi) = \bigcup_{x \in T_\xi} (\{x\} \times H_\xi^x) \quad \text{for } \xi \in B.$$

Since $\operatorname{cf}\alpha > \omega$, there are $T = \{x_1, \ldots, x_k\} \subset \{0, 1\}^F$, $I \subset \operatorname{cf}\alpha$ with $|I| = \operatorname{cf}\alpha$ and $C_\sigma \subset B_\sigma$, with $|C_\sigma| = \alpha_\sigma$ for $\sigma \in I$, such that

$$T_\xi = T \text{ for } \xi \in C_\sigma, \sigma \in I.$$

We note that $|T| = p \leqq 2^{|F|}$. Furthermore, there are $J \subset I$ with $|J| = \operatorname{cf}\alpha$, $D_\sigma \subset C_\sigma$ with $|D_\sigma| = \alpha_\sigma$ for $\sigma \in J$ and positive rational numbers $\theta_1, \theta_2, \ldots, \theta_p$ such that

$$\mu(H_\xi^{x_i}) = \theta_i \quad \text{for } 1 \leqq i \leqq p, \xi \in D_\sigma, \sigma \in J.$$

It follows that

$$\pi_{F_\xi}(U_\xi) = \bigcup_{i=1}^{p} (\{x_i\} \times H_\xi^{x_i}) \quad \text{and}$$

$$\mu(U_\xi) = \sum_{i=1}^{p} \frac{1}{2^{|F|}} \theta_i \geqq \delta \quad \text{for } \xi \in D_\sigma, \sigma \in J.$$

Finally there is, by case 1, $E_\sigma \subset D_\sigma$ for $\sigma \in J$, with $|E_\sigma| = \alpha_\sigma$ such that

$$\mu\left(H_{\xi_1}{}^{x_i} \times \{0, 1\}^{I - F_{\xi_1}} \cap \ldots \cap H_{\xi_k}{}^{x_i} \times \{0, 1\}^{I - F_{\xi_k}}\right) \geqq \frac{\theta_i^{\,k}}{2^{k-1}}$$

for $1 \leqq i \leqq p$, $1 \leqq k < \omega$, $\xi_1, \ldots, \xi_k \in E_\sigma$ for $\sigma \in J$. We set

$$A = \bigcup_{\sigma \in J} E_\sigma;$$

it is clear that $|A| = \alpha$. Let $1 \leqq n < \omega$ and $\xi_1, \ldots, \xi_n \in A$. Then there are $\sigma_1 < \ldots < \sigma_m < \text{cf } \alpha$, $\sigma_1, \ldots, \sigma_m \in J$ such that, setting

$$\Phi_j = \{\xi_1, \ldots, \xi_n\} \cap E_{\sigma_j} \quad \text{for } 1 \leqq j \leqq m,$$

we have

$$\{\xi_1, \ldots, \xi_n\} = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_m.$$

Then

$$U_{\xi_1} \cap U_{\xi_2} \cap \ldots \cap U_{\xi_n} = \bigcap_{j=1}^{m} \left(\bigcap_{\xi \in \Phi_j} U_\xi \right)$$
$$= \bigcup_{i=1}^{p} \left(\{x_i\} \times \prod_{j=1}^{m} \left(\bigcap_{\xi \in \Phi_j} (H_\xi^{x_i} \times \{0, 1\}^{I - F_\xi}) \right) \right)$$

and

$$\mu(U_{\xi_1} \cap U_{\xi_2} \cap \ldots \cap U_{\xi_n}) = \sum_{i=1}^{p} \frac{1}{2^{|F|}} \prod_{j=1}^{m} \mu\left(\bigcap_{\xi \in \Phi_j} (H_\xi^{x_i} \{0, 1\}^{I - F_\xi}) \right)$$
$$\geqq \sum_{i=1}^{p} \frac{1}{2^{|F|}} \prod_{j=i}^{m} \frac{\theta_i^{|\Phi_j|}}{2^{|\Phi_j| - 1}} = \sum_{i=1}^{p} \frac{1}{2^{|F|}} \frac{1}{2^{n-m}} \theta_i^n$$
$$= \frac{1}{2^{|F|}} \frac{1}{2^{n-m}} \sum_{i=1}^{p} \theta_i^n \geqq \frac{1}{2^{n-1}} \frac{1}{2^{|F|}} \sum_{i=1}^{p} \theta_i^n$$
$$\geqq \frac{1}{2^{n-1}} \cdot \frac{1}{2^{n-1}} \cdot \delta^n \geqq \left(\frac{\delta}{4} \right)^n \quad \text{(from (*))} .$$

The proof of the lemma is complete.

1.3. THEOREM. *Let* $(X, \mathscr{S}, \mu)$ *be a probability measure space,* α *a cardinal number with* cf $\alpha > \omega$ *and* $\{E_\xi, \xi < \alpha\} \subset \mathscr{S}$ *with* $\mu(E_\xi) > 0$ *for* $\xi < \alpha$. *Then for every* $1 \leqq n < \omega$, *there are* $A_n \subset \alpha$, $|A_n| = \alpha$, *and* $\delta_n > 0$ *such that*

$$\mu(E_{\xi_1} \cap E_{\xi_2} \cap \ldots \cap E_{\xi_n}) \geqq \delta_n \quad \textit{for } \xi_1, \xi_2, \ldots, \xi_n \in A_n.$$

Proof. Let $(\mathscr{B}, \lambda)$ be the measure algebra of $(X, \mathscr{S}, \mu)$ and $\Omega = S(\mathscr{B})$. It is enough to prove that if $\{V_\xi, \xi < \alpha\} \subset \mathscr{B}$ with $\lambda(V_\xi) > 0$ for $\xi < \alpha$, then for every $1 \leqq n < \omega$, there are $A_n \subset \alpha$, $|A_n| = \alpha$, $\delta_n > 0$ such that

$$\lambda(V_{\xi_1} \cap \ldots \cap V_{\xi_n}) \geqq \delta_n \quad \text{for } \xi_1, \ldots, \xi_n \in A_n.$$

Let $\{p_i, i < \gamma\}$ be the isolated elements (if any) of Ω, and let $\{\Omega_m, m < \delta\}$ be the partition of Ω given by Maharam's classification theorem [**12**]. Then since cf $\alpha > \omega$ and $\gamma + \delta \leqq \omega$, either there is $i < \gamma$ with $|A| = \alpha$, such that $p_i \in V_\xi$, for $\xi \in A$, in which case we are reduced to a trivial situation, or there is $m < \delta$ and $A \subset \alpha$ with $|A| = \alpha$ such that

$$V_\xi \cap \Omega_m \neq \emptyset \quad \text{for } \xi \in A.$$

Then it is enough to prove that if $\{B_\xi, \xi \in A\}$ is a family of Borel sets of $\{0, 1\}^{\alpha_m}$ with $\mu_m(B_\xi) > 0$ for $\xi \in A$, then for every $1 \leqq n < \omega$, there are $A_n \subset A$, $|A_n| = \alpha$, $\delta_n > 0$ such that

$$\mu(B_{\xi_1} \cap \ldots \cap B_{\xi_n}) \geqq \delta_n \quad \text{for } \xi_1, \ldots, \xi_n \in A_n.$$

Now there is $\delta > 0$ and $B \subset A$ with $|B| = \alpha$, such that

$$\mu(B_\xi) \geqq \delta \quad \text{for } \xi \in B.$$

Let $1 \leqq n < \omega$. Since μ is a regular measure there is an open and closed subset U_ξ of $\{0, 1\}^{\alpha_m}$ such that

$$\mu(B_\xi \Delta U_\xi) < \frac{1}{2^n}\left(\frac{\delta}{8}\right)^n \quad \text{for } \xi \in B.$$

Then

$$\mu(U_\xi) \geqq \mu(B_\xi \cap U_\xi) \geqq \delta - \frac{1}{2^n}\left(\frac{\delta}{8}\right)^n \geqq \frac{\delta}{2} \quad \text{for } \xi \in B.$$

By the previous lemma there is $A_n \subset B$ with $|A_n| = \alpha$, such that

$$\mu(U_{\xi_1} \cap \ldots \cap U_{\xi_n}) \geqq \left(\frac{\delta}{8}\right)^n \quad \text{for } \xi_1, \ldots, \xi_n \in A_n.$$

We note that

$$U_{\xi_1} \cap \ldots \cap U_{\xi_n} \subset B_{\xi_1} \cap \ldots \cap B_{\xi_n} \cup B_{\xi_1} \Delta U_{\xi_1} \cup \ldots \cup B_{\xi_n} \Delta U_{\xi_n}$$

hence

$$\left(\frac{\delta}{8}\right)^n \leqq \mu(U_{\xi_1} \cap \ldots \cap U_{\xi_n}) \leqq \mu(B_{\xi_1} \cap \ldots \cap B_{\xi_n}) + \sum_{k=1}^{n} \frac{1}{2^n}\left(\frac{\delta}{8}\right)^n$$

hence

$$\mu(B_{\xi_1} \cap \ldots \cap B_{\xi_n}) \geqq \frac{1}{2}\left(\frac{\delta}{8}\right)^n = \delta_n \quad \text{for } \xi_1, \ldots \xi_n \in A_n.$$

The proof of the theorem is complete.

1.4. *Remark.* The proof of the above theorem does not need Maharam's classification theorem in case α is a regular (uncountable) cardinal. Indeed, it follows from the following proposition.

1.5. PROPOSITION. *Let α be a regular uncountable cardinal $(X, \mathcal{S}, \mu)$ a probability measure space, $\{E_\xi, \xi < \alpha\} \subset \mathcal{S}$, $\delta > 0$ such that*

$$\mu(E_{\xi_1} \cap \ldots \cap E_{\xi_{n-1}}) \geqq \delta \quad \textit{for } \xi_1, \ldots, \xi_{n-1} < \alpha.$$

Then there is $A \subset \alpha$, $|A| = \alpha$ *such that*

(1) $\mu(E_{\xi_1} \cap \ldots \cap E_{\xi_n}) \geqq \delta/4k$ *for* $\xi_1, \ldots, \xi_n \in A$

(*where* k *is a positive number*, $k > 2/\delta$).

Proof. Suppose that this is not the case. We set $\Gamma_1 = \alpha$ and let

$$\mathscr{C}_1 = \{C : C \subset \Gamma_1, \{E_\xi, \xi \in C\} \text{ satisfies (1)}\}.$$

Then $\mathscr{C}_1 \neq \emptyset$ and it is inductive under set inclusion. Let $C_1 \in \mathscr{C}_1$, C_1 maximal. Then $|C_1| < \alpha$.

For every $\xi \in \Gamma_1 - C_1$ there are $\xi_2, \ldots, \xi_n \in C_1$ such that

$$\mu(E_\xi \cap E_{\xi_2} \cap \ldots \cap E_{\xi_n}) < \delta/4k.$$

Since α is regular, there are $\xi_2^{(1)}, \xi_3^{(1)}, \ldots, \xi_n^{(1)} \in C_1$ and $\Gamma_2 \subset \Gamma_1$ with $|\Gamma_2| = \alpha$ such that

$$\mu(E_\xi \cap E_{\xi_2^{(1)}} \cap \ldots \cap E_{\xi_n^{(1)}}) < \delta/4k \quad \text{for } \xi \in \Gamma_2.$$

We now repeat the same argument with Γ_2 in place of Γ_1, and we find $C_2 \subset \Gamma_2$, $\xi_2^{(2)}, \xi_3^{(2)}, \ldots, \xi_n^{(2)} \in C_2$, and $\Gamma_3 \subset \Gamma_2$, with $|\Gamma_3| = \alpha$ such that

$$\mu(E_\xi \cap E_{\xi_2^{(2)}} \cap \ldots \cap E_{\xi_n^{(2)}}) < \delta/4k \quad \text{for } \xi \in \Gamma_3.$$

We repeat the same argument k times, and thus find elements

$$\begin{array}{l} \xi_2^{(1)}, \xi_3^{(1)}, \ldots, \xi_n^{(1)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \xi_2^{(k)}, \xi_3^{(k)}, \ldots, \xi_n^{(k)} \end{array}$$

such that, if $1 \leqq l < m \leqq k$, then

$$\mu(E_{\xi_2^{(l)}} \cap \ldots \cap E_{\xi_n^{(l)}} \cap E_{\xi_2^{(m)}}) < \delta/4k.$$

We now set

$$A_m = E_{\xi_2^{(m)}} \cap \ldots \cap E_{\xi_n^{(m)}} - \bigcup_{l=1}^{m-1} E_{\xi_2^{(l)}} \cap \ldots \cap E_{\xi_n^{(l)}} \qquad \text{for } 1 \leqq m \leqq k$$

and we note that

$$\mu(A_m) \geqq \delta - (m-1)\frac{\delta}{4k} \geqq \frac{\delta}{2} \quad \text{for } 1 \leqq m \leqq k,$$

$$A_m \cap A_l = \emptyset \quad \text{for } 1 \leqq l < m \leqq k$$

a contradiction since $k \cdot \delta/2 > 1$, and $\mu(X) = 1$.

1.6. PROPOSITION. *Let X be a topological space with property* $(**)$. *Then X has property $k_{\alpha,n}$ for all cardinals α with* $\mathrm{cf}\,\alpha > \omega$ *and* $1 \leqq n < \omega$.

https://doi.org/10.4153/CJM-1982-076-3 Published online by Cambridge University Press

Proof. From [**8**], for the space X there is an extremally disconnected compact space GX and two maps:

$$p_1 : \mathscr{T}^*(X) \to \mathscr{T}^*(GX), \quad p_2 : \mathscr{T}^*(GX) \to \mathscr{T}^*(X)$$

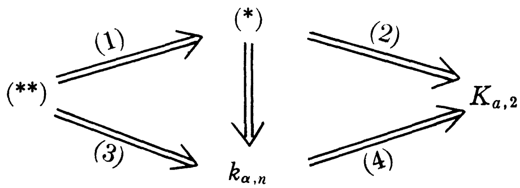
such that for $U_1, \ldots, U_n \in \mathscr{T}^*(X)$,

$$U_1 \cap \ldots \cap U_n = \emptyset \text{ implies } p_1(U_1) \cap \ldots \cap p_1(U_n) = \emptyset$$

and an analogous statement for p_2.

Since X satisfies property (**) from the existence of the map p_2 it follows that GX satisfies (**) and then from Kelley's theorem GX has a strictly positive measure. Now from Theorem 1.3 the space GX satisfies property $k_{\alpha,n}$ and finally from the existence of the map p_1 the space X satisfies property $k_{\alpha,n}$.

1.7. *Remark*. For an arbitrary topological space X and a cardinal α with cf $\alpha > \omega$ the following diagram holds.



(1) is due to Gaifman [**6**].
(2) is a consequence of the arrow-relation $\alpha \to (\alpha, \omega)^2$ of Erdös.
(3) is Proposition 1.6 (4) and is trivial.

Each of converses of the implications above is false.

(1) (*) $\Rightarrow$ (**). This is due to Gaifman [**8**].
(2) $k \to$ (*). This is due to Galvin-Hajnal [**7**].
(3) k_n, $n < \omega \to$ (**). This follows from Gaifman's example [**8**].
(4) $k \to k_n$, $n < \omega$. This follows from Argyros example [**3**].
(5) (*) $\to k_{\alpha,n}$ for $n > 2$. This follows from [**3**].

1.8. *Remark*. The case of $k_{\alpha,2}$ in Theorem 1.3 which follows from the arrow-relation $\alpha \to (\alpha, \omega)_2{}^2$ was well known. In particular the case $k_{\omega^+,2}$ has been established by Marczewski. Also W. Comfort informed us that K. Kunen also has given a proof of Proposition 1.5.

2.

2.1. *Definitions and notations*. For a compact space X we denote by $C(X)$ the space of all continuous real-valued functions defined on X.

By $M(X)$ we denote the space of all regular, Borel measures on X. We coincide this space with the conjugate of $C(X)$, via the Riesz representation theorem. Let α be a cardinal. We symbolize by l_α^1 the Banach space of all real functions f on α that are absolutely summable with

$$\|f\| = \sum_{\xi<\alpha} |f(\xi)|.$$

By l_α^∞ we denote the Banach space of all bounded real-valued functions defined on α with

$$\|f\| = \sup \{|f(\xi)| : \xi < \alpha\}.$$

Let E be a Banach space. We say that E contains isomorphically a copy of l_α^1 if there is an isomorphism $T : l_\alpha^1 \to E$. Equivalently l_α^1 is isomorphic with a subspace of E if there are $c > 0$ and a uniformly bounded sequence $\{x_\xi, \xi < \alpha\} \subset E$ such that

$$\left\|\sum_{i=1}^{\mu} \lambda_i x_{\xi_i}\right\| \geqq c \sum_{i=1}^{\mu} |\lambda_i| \quad \text{for every } \lambda_1, \lambda_2, \ldots, \lambda_\mu \in \mathbf{R}.$$

2.2. *Definition.* Let X be a compact, totally disconnected topological space, and $\mathscr{B}$ a base of clopen sets for its topology. For an infinite cardinal α we say that X *satisfies the property* P_α if for every family $\{U_\xi, \xi < \alpha\} \subset \mathscr{B}$ of pairwise different sets, l_α^1 embeds isomorphically into the closed linear span of the set

$$\{\chi_{U_\xi}, \xi < \alpha\}.$$

2.3. THEOREM. *Let X be a compact, totally disconnected space and $\mathscr{B}$ a base of clopen sets for its topology. We suppose that X satisfies property P_α for some infinite cardinal α with* $\operatorname{cf} \alpha > \omega$. *Then, the space X satisfies property $k_{\alpha,n}$ for every $n < \omega$.*

Proof. Let $\{U_\xi, \xi < \alpha\}$ be a family of elements of $\mathscr{B}$ and Z the closed linear span of the set $\{\chi_{U_\xi}, \xi < \alpha\}$. Because of P_α there is a uniformly bounded family $\{f_\xi, \xi < \alpha\} \subset Z$ and a constant $c > 0$ such that

$$\left\|\sum_{i=1}^{k} \lambda_i f_{\xi_i}\right\| \geqq c \sum_{i=1}^{k} |\lambda_i| \quad \text{for every } \lambda_1, \ldots, \lambda_k \in \mathbf{R}.$$

We approximate the elements $\{f_\xi, \xi < \alpha\}$ by finite linear combinations with rational coefficients $\{g_\xi, \xi < \alpha\}$ of the set $\{\chi_{U_\xi}, \xi < \alpha\}$ such that

$$\|f_\xi - g_\xi\| < c/2, \ \xi < \alpha.$$

It is easily verified that the family $\{g_\xi, \xi < \alpha\}$ is also equivalent with the usual basis of l_α^1.

Now passing to a subfamily, we find $A \subset \alpha$, $|A| = \alpha$ and rational numbers $r_1, r_2, \ldots, r_q$ such that

$$g_\xi = r_1\chi_{U_\xi^1} + \ldots + r_q\chi_{U_\xi^q}, \ \xi \in A.$$

We assume that α is singular. The regular case follows from similar (and easier) arguments. We consider the sets

$$F_\xi = \{U_\xi^1, U_\xi^2, \ldots, U_\xi^q\}, \quad \xi \in A$$

https://doi.org/10.4153/CJM-1982-076-3 Published online by Cambridge University Press

and apply Theorem 0.8. So there are sets $A_j, j < \text{cf}\,\alpha$, $E_j, j < \text{cf}\,\alpha$ and E such that

$$\left| \bigcup_{j<\text{cf}\,\alpha} A_j \right| = A$$

and if $\xi_1, \xi_2 \in A_j, \xi_1 \neq \xi_2$ then

$$F_{\xi_1} \cap F_{\xi_2} = E_j$$

and if $\xi_1 \in A_{j_1}, \xi_2 \in A_{j_2}, j_1 \neq j_2$ then

$$F_{\xi_1} \cap F_{\xi_2} = E.$$

Now for $j < \text{cf}\,\alpha$, let $\{\xi_1^p, \xi_2^p\}, p \in A_j'$ where $\xi_1^p, \xi_2^p \in A_j$, and for $p \neq p'$,

$$\{\xi_1^p, \xi_2^p\} \cap \{\xi_1^{p'}, \xi_2^{p'}\} = \emptyset \quad \text{and} \quad |A_j'| = |A_j|.$$

We set

$$\bar{g}_p = g_{\xi_1^p} - g_{\xi_2^p}, \quad p \in A_j', \quad j < \text{cf}\,\alpha.$$

It is easy to see that the family $\{\bar{g}_p, p \in A_j', j < \text{cf}\,\alpha\}$ is equivalent with the usual basis of l_α^1. So there is an isomorphism

$$T : l_\alpha^1 \to \langle \{\bar{g}_p, p \in A_j', j < \text{cf}\,\alpha\} \rangle \subset C(X).$$

Hence the conjugate operator

$$T^* : M(X) \to l_\alpha^\infty$$

is onto. Consequently there is a regular Borel measure μ on X such that

$$T^*(\mu) = (1, 1, \ldots).$$

So $\mu(\bar{g}_p) = 1, p \in A_j', j < \text{cf}\,\alpha$ and hence for p there exists a set $U_p^{i(p)}$ such that $\mu(U_p^{i(p)}) > 0$. Now from the construction of $\bar{g}_p$ it follows that the family

$$\{U_p^{i(p)}, p \in A_j', j < \text{cf}\,\alpha\}$$

has cardinality α.

The desired result is now a simple consequence of 1.3.

2.4. *Remark*. Let X be an arbitrary compact space. If for every closed subspace Z of $C(X)$ with $\dim Z = \alpha$, $\text{cf}\,\alpha > \omega$, l_α^1 is isomorphic to a subspace of Z, then with the same method it can be proved that the space X satisfies property $k_{\alpha,n}$ for $n < \omega$.

2.5. *Remark*. From results of [**1**] and [**9**] there follows the existence of compact spaces X with property P_α. (For example for an arbitrary I the space $\{0, 1\}^I$ satisfies property P_α when $\text{cf}\,(\alpha) > \omega$.) The cardinals α for which P_α holds in spaces of [**1**] and [**9**] satisfy a property stronger than $k_{\alpha,n}$; namely, these spaces have α-caliber. Recently, however, the authors have constructed an example of a space with P_{ω^+} property and without caliber ω^+. This example shows that property $k_{\alpha,n}$ is the best intersection property that can be obtained from P_α.

Acknowledgment. The authors are grateful to Professors Comfort and Negrepontis for the careful checking of proofs of results and their suggestions.

REFERENCES

1. S. Argyros and S. Negrepontis, *Universal embeddings of l_α^1 into $C(X)$ and $L^\infty(\mu)$*, Proceedings of Colloquium on Topology, Budapest (1978).
2. S. Argyros and A. Tsarpalias, *Calibers of compact spaces*, to appear in T.A.M.S.
3. S. Argyros, *On compact spaces without strictly positive measure* (to appear).
4. W. Comfort and S. Negrepontis, *Chain conditions in topology*, Cambridge University Press (to appear).
5. Erdös and Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. *44* (1969), 467–479.
6. H. Gaifman, *Concerning measures on Boolean algebras*, Pacific J. Math. *14* (1964), 61–73.
7. F. Galvin and A. Hajnal (manuscript).
8. F. Galvin, *Chain conditions and products* (to appear in Fund. Math.).
9. J. Hagler, *On the structure of S and C(S) for S dyadic*, T.A.M.S. *214* (1975), 415–427.
10. J. Kelley, *Measures in Boolean algebras*, Pacific J. Math. *9* (1959), 1165–1177.
11. E. Lacey, *The isometric structure of classical Banach spaces*, *208* (Springer-Verlag, Berlin, Heidelberg, New York).
12. D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. *28* (1942), 108–11.

Athens University,
Athens, Greece