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ON THE CENTRE OF THE CYCLOTOMIC HECKE ALGEBRA OF $G(m, 1, 2)$

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Abstract We compute the centre of the cyclotomic Hecke algebra attached to $G(m, 1, 2)$ and show that if $q \neq 1$, it is equal to the image of the centre of the affine Hecke algebra H_2^{aff}. We also briefly discuss what is known about the relation between the centre of an arbitrary cyclotomic Hecke algebra and the centre of the affine Hecke algebra of type A.

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1. On the centre of $\mathcal{K}_2^v$

1.1. The affine Hecke algebra associated to $\mathfrak{gl}_n$ has an interesting family of finite-dimensional quotients known as cyclotomic Hecke algebras. These may be viewed as deformations of the complex reflection groups $G(m, 1, n)$ (see [**10**] for a nice review of the subject). The study of their representation theory has proved to be a rich subject, and a natural first question in such a study is to understand the centre. Oddly, although the decomposition of the category of representations into blocks is now known [**9**], the centre itself is not well understood. In this note we describe the centre for the cases of $G(m, 1, 2)$ and discuss what is known in the general case.

1.2. We consider quotients of the affine Hecke algebra H_2^{aff} of type A_2: this is the algebra over $\mathcal{A} = \mathbb{Z}[q]$ generated by $T, X_1^{\pm 1}, X_2^{\pm 1}$, such that

(i) there is an injective algebra map $\mathcal{A}[X_1^{\pm 1}, X_2^{\pm 1}] \to H_2^{\mathrm{aff}}$,

(ii) $(T - q)(T + 1) = 0$,

(iii) $TX_1T = qX_2$.

Let S denote the image of the ring of Laurent polynomials $\mathcal{A}[X_1^{\pm}, X_2^{\pm}]$ in H_2^{aff}, and let W be the two-element group with non-trivial element s which acts on S by interchanging

X_1 and X_2. We write the action as $f \mapsto {}^s f$. For convenience of notation we set $Q = q - 1$. Relation (iii) above is then equivalent to

$$Tf = {}^s f T + Q \frac{f - {}^s f}{1 - X_1 X_2^{-1}}, \quad f \in S. \tag{1.1}$$

It is easy to check from this that the centre of H_2^{aff} is S^W, the algebra of symmetric functions in the $X_i^{\pm 1}$.

1.3. Now let $A = \mathbb{Z}[q^{\pm 1}, v_1^{\pm 1}, v_2^{\pm 1}, \dots, v_m^{\pm 1}]$ be a Laurent polynomial ring, and extend the scalars of H_2^{aff} to $A \otimes_{\mathcal{A}} H_2^{\mathrm{aff}}$. By abuse of notation we will again denote this algebra by H_2^{aff}, and similarly for the subalgebra S.

Definition 1.1. The cyclotomic Hecke algebra $\mathcal{K}_2^{\boldsymbol{v}}$ of type $G(m, 1, 2)$ is a quotient of H_2^{aff}: let

$$\begin{aligned} f_{\boldsymbol{v}} &= (x - v_1)(x - v_2) \cdots (x - v_m) \\ &= \sum_{j=0}^{m} (-1)^{m-j} e_{m-j} x^j, \end{aligned} \tag{1.2}$$

where the e_j are the elementary symmetric functions in the v_i. Let $J_{\boldsymbol{v}}$ be the two-sided ideal in H_2^{aff} generated by $f_1 = f_{\boldsymbol{v}}(X_1)$ and set $\mathcal{K}_2^{\boldsymbol{v}} = H_2^{\mathrm{aff}}/J_{\boldsymbol{v}}$. (Note that our definition coincides with that in [**2**] up to rescaling, after $v_1, v_2, \dots, v_m$ have been inverted, except that the 'q' therein is a square root of ours.)

We say that a polynomial p in S is *m-restricted* (or simply *restricted*, when the integer m is understood) if the monomials $X_1^i X_2^j$ occurring with non-zero coefficient in p all satisfy $0 \leqslant i, j \leqslant m - 1$. Let S_m denote the space of m-restricted polynomials. It is known [**3**, Proposition 3.4] that the image R_m of S in $\mathcal{K}_2^{\boldsymbol{v}}$ is isomorphic as an A-module to S_m, and, moreover, $\mathcal{K}_2^{\boldsymbol{v}} = R_m \oplus R_m T$ as an A-module, i.e. every element of $\mathcal{K}_2^{\boldsymbol{v}}$ can be written uniquely in the form $f + gT$, where f and g are restricted. We refer to this last fact as the 'basis theorem' for $\mathcal{K}_2^{\boldsymbol{v}}$.

1.4. We start with a technical lemma. Let D be the difference operator on S given by

$$D(f) = \frac{f - {}^s f}{1 - X_1 X_2^{-1}}, \quad f \in S,$$

so that the relation (1.1) becomes $Tf = {}^s f T + QD(f)$. Let D_s be the composition $f \mapsto {}^s(-D(f))$, that is,

$$D_s(f) = \frac{f - {}^s f}{1 - X_1^{-1} X_2}.$$

Lemma 1.2. *The operators D and D_s preserve S_m and thus induce A-linear maps on R_m. Moreover,*

$$D(f) = D_s(f)$$

if and only if $f = {}^s f$.

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Proof. The proof that D and D_s preserve R_m is a direct calculation: observe that if $f = X_1^i X_2^j$, then we have

$$D(f) = \begin{cases} X_1^i \displaystyle\sum_{k=0}^{j-i-1} X_1^k X_2^{j-k} & \text{if } j > i, \\ -X_1^j \displaystyle\sum_{k=0}^{i-j-1} X_1^k X_2^{i-k} & \text{if } j < i. \end{cases}$$

Thus, as the highest power of X_1 and X_2 occurring in these expressions is $\max\{i,j\}$, it is clear that if p is any m-restricted polynomial, so is $D(p)$. Since ${}^sD$ is the composition of $-D$ with s, it clearly also preserves restricted polynomials.

Moreover, note that in $D(f)$, where f is the monomial above, X_1 never occurs to the power $\max\{i,j\}$, whereas X_2 does. Thus, for any restricted polynomial p, if $D(p) \neq 0$, it has a higher power of X_2 occurring than occurs as a power of X_1. Thus, similarly $D_s(p)$, if non-zero, has a higher power of X_1 occurring than occurs as a power of X_2. It follows that $D(p) = D_s(p)$ if and only if $D(p) = 0$, and this occurs only if $p = {}^sp$ as claimed. □

Lemma 1.3. *Let $z \in \mathcal{K}_2^v$ and suppose that $z = f + gT$, where $f, g \in R_m$. Then z commutes with T if and only if $f, g \in R_m^W$.*

Proof. The sufficiency of the condition is clear. To see the necessity, we have

$$\begin{aligned} T(f + gT) &= {}^sfT + QD(f) + {}^sgT^2 + QD(g)T \\ &= ({}^sf + Q\,{}^sg + QD(g))T + QD(f) + q\,{}^sg \end{aligned}$$

(where we write D for the operator on R_m given by the previous lemma). On the other hand, we have

$$(f + gT)T = (f + Qg)T + qg.$$

Since f and g are restricted, it follows from the basis theorem for cyclotomic Hecke algebras that we must have

$$({}^sf + Q\,{}^sg + QD(g)) = f + Qg,$$

and

$$QD(f) + q\,{}^sg = qg$$

Thus, after rearranging, the second of these equations becomes

$$QD(f) = q(g - {}^sg).$$

Now note that the right-hand side is an eigenvector for the action of s with eigenvalue -1, and thus so is the left-hand side, whence we get ${}^s(D(f)) = -D(f)$, or, equivalently, $D(f) = D_s(f)$. By the previous lemma, this is possible only if $f = {}^sf$ and $D(f) = 0$. But then we must also have $g - {}^sg = 0$, and so f and g are symmetric, as required. □

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1.5. Let $\mathcal{Z}$ denote the centre of $\mathcal{K}_2^{\boldsymbol{v}}$. From the previous lemma, we see that if $z = f + gT \in \mathcal{Z}$, then $f, g \in R_m^W$. Since $f \in R_m^W$ is already central, we see that $\mathcal{Z} = \mathcal{Z} \cap R_m \oplus \mathcal{Z} \cap R_m T$, and we are reduced to calculating when gT is central. For this we introduce the following operator.

Definition 1.4. Let $d\colon S \to S$ be the linear map given on monomials by

$$d(X_1^i X_2^j) = \begin{cases} X_1^i X_2^j & \text{if } i < j, \\ -X_1^j X_2^i & \text{if } i > j, \\ 0 & \text{if } i = j. \end{cases}$$

Clearly, d preserves S_m, and so we may transport it to a map on R_m (which we will also denote by d). Clearly, the kernel $\ker(d)$ of its action on R_m is R_m^W and since $d^2 = d$, $R_m = R_m^W \oplus d(R_m)$.

Lemma 1.5. *$\mathcal{Z} \cap R_m T$ is a free A-module of rank m.*

Proof. Suppose that $gT \in \mathcal{Z} \cap R_m T$. We must have $X_1 gT = gTX_1$ and $TgT = gT^2$, and these conditions are sufficient. Since T is invertible (indeed $T^{-1} = q^{-1}(T+1-q)$), the second equation is equivalent to $Tg = gT$. By Lemma 1.3 this implies that $g \in R^W$, and hence the first equation becomes $(X_1 g)T = T(X_1 g)$. But then, again using Lemma 1.3, we see that $X_1 g \in R_m^W$. Let M be the space of such restricted symmetric polynomials:

$$M = \{g \in R_m^W : X_1 g \in R_m^W\}.$$

We have shown that $\mathcal{Z} \cap R_m T = MT$. It is now a linear algebra exercise to check that M is a free A-module of rank m. By the paragraph preceding the lemma, $R_m = R_m^W \oplus d(R_m)$ as an A-module. Thus, if we let $\phi\colon R_m^W \to d(R_m)$ be the map $g \mapsto d(X_1 g)$, we see that $M = \ker(\phi)$. Let $m_{ij} = X_1^i X_2^j + X_1^j X_2^i$ for $i < j$ and $m_{ii} = X_1^i X_2^i$ be the monomial symmetric functions, and let R_{m-1}^W be the span of $\{m_{ij} : 0 \leqslant i \leqslant j < m-1\}$. Then we claim that $\phi\colon R_{m-1}^W \to d(R_m)$ is an isomorphism of A-modules. Indeed, for $j < m-1$ we have

$$\phi(m_{ij}) = \begin{cases} -X_1^i X_2^{j+1} & \text{if } j - i \leqslant 1, \\ -X_1^i X_2^{j+1} + X_1^{i+1} X_2^j & \text{if } j - i > 1. \end{cases}$$

If we use reverse lexicographical ordering on the bases $\{m_{ij} : 0 \leqslant i \leqslant j < m-1\}$ and $\{X_1^i X_2^j : 0 \leqslant i < j \leqslant m-1\}$, then the above equations show that the matrix of $\phi|_{R_{m-1}^W}$ with respect to these ordered bases is triangular with diagonal entries equal to -1; thus, $\phi|_{R_{m-1}^W}$ is an isomorphism as claimed.

This immediately implies that M is a free A-module of rank m. However, we can be more precise and even specify a basis of M by considering the images of $\phi(m_{i(m-1)})$, $0 \leqslant i \leqslant m-1$: set

$$p_i = m_{i(m-1)} - (\phi_{R_{m-1}^W})^{-1}(\phi(m_{i(m-1)})).$$

Then $\{p_0 T, p_1 T, \dots, p_{m-1} T\}$ is an A-basis of $\mathcal{Z} \cap R_m T$. □

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Example 1.6. Let $m=3$. In this case we find that the space $\mathcal{Z}\cap R^W T$ is spanned by $\{p_0T, p_1T, p_2T\}$, where

$$\left.\begin{aligned} p_0 &= e_2 - (e_3 - e_1(X_1+X_2) + X_1X_2 + X_1^2 + X_2^2),\\ p_1 &= (e_3 - e_1X_1X_2 + X_1X_2(X_1+X_2)),\\ p_2 &= (e_3(X_1+X_2) - e_2X_1X_2 + X_1^2X_2^2). \end{aligned}\right\} \tag{1.3}$$

Thus, we have shown that the centre $\mathcal{Z}$ of $\mathcal{K}_2^{\boldsymbol{v}}$ (with $m=3$) is a nine-dimensional free submodule spanned by R_3^W and $\{p_0T, p_1T, p_2T\}$ (this is exactly as stated in [**2**, § 2]).

Remark 1.7. It is easy to check directly that Lemma 1.5 implies that the rank of the centre is the number of m-multipartitions of 2. This also follows by passing to the field of fractions of A, and using the result of [**3**] which shows that in the semisimple case (for any n) the centre has rank equal to the number of m-multipartitions of n.

2. On the image of $Z(H_2^{\mathrm{aff}})$

2.1. Next we do some simple computations. Let

$$H_k = \sum_{j=0}^{k} X_1^j X_2^{k-j}$$

be the complete symmetric function in X_1 and X_2 of degree k, and let $\mathcal{I}$ denote the image of the centre of H_2^{aff} in $\mathcal{K}_2^{\boldsymbol{v}}$. Recall that $f_1 = f_{\boldsymbol{v}}(X_1)$.

Definition 2.1. Let $f_2 = Tf_1T$. Thus, $f_2 \in J_v$.

Lemma 2.2.

(i) *In* H_2^{aff}, *for any* $k \geqslant 2$,

$$TX_1^kT = qX_2^k - Q(X_1X_2)H_{k-2}T.$$

(ii) *We have*

$$qf_{\boldsymbol{v}}(X_2) = f_2 + QzT,$$

where

$$z = (-1)^{m+1}e_m + (X_1X_2)\biggl(\sum_{j=0}^{m-2}(-1)^j e_j H_{m-2-j}\biggr) \in S_m^W.$$

Proof. The proof of (i) is a direct calculation using (1.1). For (ii) we have, using (i),

$$qX_2^k = TX_1^kT + Q(X_1X_2)H_{k-2}T, \quad k \geqslant 2.$$

Moreover, $qX_2 = TX_1T$, and $q = T^2 - QT$, so that

$$qf_{\boldsymbol{v}}(X_2) = f_2 + Q\biggl((-1)^{m+1}e_m + (X_1X_2)\biggl(\sum_{j=2}^{m}(-1)^{m-j}e_{m-j}H_{j-2}\biggr)\biggr)T,$$

as claimed. □

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Remark 2.3. Note that one has the well-known identity of symmetric functions

$$\sum_{r=o}^{n}(-1)^r e_{n-r}h_r = 0$$

for elementary and complete symmetric functions in the *same* set of variables. In the preceding lemma the e_j are symmetric functions in the v_i, while the H_k are symmetric in the X_i.

Proposition 2.4. *The elements $QX_1^k zT$ lie in $\mathcal{I}$ for all $k \in \mathbb{Z}$, and, moreover, the elements $\{X_1^{k-1}z\colon 0 \leqslant k \leqslant m-1\}$ are linearly independent.*

Proof. From the previous lemma we have

$$\begin{aligned} QX_1^k zT &= qX_1^k f_{\boldsymbol{v}}(X_2) - X_1^k f_2 \\ &= qX_1^k f_{\boldsymbol{v}}(X_2) + qX_2^k f_{\boldsymbol{v}}(X_1) - (qX_2^k f_1 + X_1^k f_2) \\ &\in qX_1^k f_{\boldsymbol{v}}(X_2) + qX_2^k f_{\boldsymbol{v}}(X_1) + J_{\boldsymbol{v}}. \end{aligned}$$

Hence, we see that $QX_1^k zT \in \mathcal{K}_2^{\boldsymbol{v}} = H_2^{\mathrm{aff}}/J_{\boldsymbol{v}}$ is in the image of the centre of H_2^{aff}, as required.

It remains to show that the elements $\{X_1^{k-1}z\colon 0 \leqslant k \leqslant m-1\} \subset \mathcal{K}_2^{\boldsymbol{v}}$ are linearly independent. Since A is an integral domain, we see that $X_1^{k-1}zT \in \mathcal{Z}$, and hence $X_1^{k-1}z \in R_m^W$ by Lemma 1.3. We have

$$X_1^{k-1}z = (-1)^{m+1}e_m X_1^{k-1} - (X_1^k X_2)\Bigg(\sum_{j=0}^{m-2}(-1)^{j+1}e_j H_{m-2-j}\Bigg).$$

Now consider this expression for $X_1^{k-1}z$ as a linear combination of the monomial symmetric functions m_{ij} lying in R_m. While this may require using the equation $f_{\boldsymbol{v}}(X_1) = 0$ (e.g. for the first term when $k = 0$), the powers of X_2 occurring are already in the restricted range. Thus, by considering the terms involving X_2^{m-1} it is easy to see that the coefficient of $m_{j(m-1)}$ is 0, unless $j = k$, in which case it is 1. It follows immediately that the $X_1^{k-1}z$ in the range $0 \leqslant k \leqslant m-1$ are linearly independent. □

2.2. We can now combine the above results to establish our main theorem.

Theorem 2.5. *Let B denote the localization of A where $Q = q-1$ is inverted. Then, over B, the centre of H_2^{aff} surjects onto the centre of $\mathcal{K}_2^{\boldsymbol{v}}$.*

Proof. It is clearly sufficient to show that $p_i T$ lies in $\mathcal{I}$, where $p_i \in R_m^W$, $0 \leqslant i \leqslant m-1$, is as in Lemma 1.5. Since we have inverted Q, Proposition 2.4 shows that we have $X_1^k zT \in \mathcal{I}$ for all $k \in \mathbb{Z}$. Now by the proof of the previous proposition, we also know that the coefficient of $m_{j(m-1)}$ in $X_1^{k-1}z$, $0 \leqslant k \leqslant m-1$, in the basis $\{m_{ij}\}$ of restricted monomial symmetric functions is δ_{jk}, and the same is true, by definition, for the p_k. Since $\mathcal{I} \subset \mathcal{Z}$, we can write $X_1^k zT$ as a linear combination of the elements $p_i T$; hence, it follows immediately that $p_k T = X_1^{k-1}zT$, and we are done. □

Example 2.6. We consider again the case $m = 3$, keeping the notation of the previous example. Then $z = p_1$, and it is easy to check that $X_1 p_1 = p_2$, and similarly

$$\begin{aligned} X_1^{-1} p_1 &= e_3 X_1^{-1} - e_1 X_2 + X_2(X_1 + X_2) \\ &= (e_2 - e_1 X_1 + X_1^2) - e_1 X_2 + X_2(X_1 + X_2) \\ &= e_2 - e_1(X_1 + X_2) + (X_1^2 + X_1 X_2 + X_2^2) \\ &= p_0, \end{aligned}$$

so that $X_1^{k-1} z = p_k$ for $0 \leqslant k \leqslant 2$.

3. Comments on the general case

3.1. Let H_n^{aff} denote the affine Hecke algebra attached to $\mathfrak{gl}_n$, and let $\mathcal{K}_n^{\boldsymbol{v}}$ denote the cyclotomic Hecke algebra, the quotient of H_n^{aff} by the two-sided ideal $J_{\boldsymbol{v}}$ generated by $f_{\boldsymbol{v}}(X_1)$. We wish to consider the following conjecture.

Conjecture 3.1. *Let* H_n^{aff} *be the affine Hecke algebra with coefficients extended to* B*, the ring*

$$A = \mathbb{Z}[q^{\pm 1}, v_1^{\pm 1}, \dots, v_m^{\pm 1}]$$

with $Q = q - 1$ *inverted. Let* $\psi_m \colon H_n^{\mathrm{aff}} \to \mathcal{K}_n^{\boldsymbol{v}}$ *be the quotient map. Then*

$$\psi_m(Z(H_n^{\mathrm{aff}})) = Z(\mathcal{K}_n^{\boldsymbol{v}}).$$

In fact, it may be easier (and as useful) to show this in the case where H_n^{aff} is defined over a field F, and the parameter q is not equal to 1.

Remark 3.2. As pointed out in [**2**], if we specialize to $q = 1$, i.e. $Q = 0$, then the image of the centre of the affine Hecke algebra does *not* necessarily surject onto the centre of the specialized Ariki–Koike algebra (for $\mathcal{K}_2^{\boldsymbol{v}}$, if we say require X_1 to satisfy $X_1^3 - 1 = 0$, then at $q = 1$ this is just the group algebra of the complex reflection group $G(3,1,2)$ which has a nine-dimensional centre (the specialization of the centre of $\mathcal{K}_2^{\boldsymbol{v}}$), whereas the images of $X_1 + X_2$, $X_1 X_2$ only generate a six-dimensional subalgebra). Of course at $q = 1$ one should instead consider the degenerate algebra.

While the above conjecture is certainly not new, it does not seem to be explicitly stated in the literature, and some of existing literature is unclear as to its status: the counterexample of [**2**] is quoted in [**10**] in a fashion which makes it appear it is more general than [**2**] intended to imply.*

We list the following evidence for the conjecture:

(i) Ariki and Koike [**3**] show that the conjecture holds in the semisimple case (they also show explicitly the conditions on the parameters under which the Ariki–Koike algebras is semisimple);

* The author thanks Professors Ariki and Mathas for helping him sort out this confusion.

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(ii) the present paper establishes the case $n = 2$;

(iii) in an orthogonal direction, Francis and Graham [**7**] have verified the conjecture for the case of the finite Hecke algebra of type A, i.e. the case $m = 1$.

Remark 3.3. Unfortunately, the direct approach to the calculation of the centre taken in this paper seems not to generalize readily. A number of new issues present themselves. Firstly, the explicit construction of elements in the image of the centre of the affine Hecke algebra becomes more subtle because of the braid relations, while, secondly, the strategy to describe the centre relies on relating monomials X^α to monomials X^λ (in the normal multi-index notation) where λ is dominant, which becomes noticeably more complicated for $n > 2$.

3.2. We end with another result which supports the conjecture when we work over F, an algebraically closed field of characteristic zero. Let $\mathfrak{H}_n$ be the degenerate, or graded, affine Hecke algebra of type A. As a vector space it is isomorphic to $\mathbb{C}[S_n] \otimes \mathbb{C}[r][x_1, \dots, x_n]$. Let $\mathfrak{T}$ denote the algebraic torus with regular functions $\mathcal{O} = \mathbb{C}[q^{\pm 1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, and let $\mathfrak{t} \oplus \mathbb{C}$ denote the vector space with functions $\bar{\mathcal{O}} = \mathbb{C}[r][x_1, x_2, \dots, x_n]$ and note that we may use the group structure on $\mathfrak{T}$ to identify $\mathfrak{t} \oplus \mathbb{C}$ with the tangent space of $\mathfrak{T}$ at any point.

Now let I be a maximal ideal of the centre $\mathcal{Z}$ of H_n^{aff}. Thus, I corresponds to an S_n-orbit Σ in $\mathfrak{T}$. We want to consider the completions $\hat{\mathcal{Z}}$ and $\hat{H}_n^{\mathrm{aff}}$ with respect to I. Assume that all of the coordinates of the elements of Σ are equal to a power of q. In this case, by choosing a logarithm of q we may attach to Σ a maximal ideal $\mathfrak{I}$ of the centre $\mathfrak{Z} = \mathbb{C}[r][x_1, \dots, x_n]^{S_n}$ of $\mathfrak{H}_n$, and consider the corresponding completions $\hat{\mathfrak{H}}_n$ and $\hat{\mathfrak{Z}}$. The algebras $\hat{\mathcal{Z}}$ and $\hat{\mathfrak{Z}}$ are then naturally isomorphic and Lusztig [**8**, § 9.3] has shown that there is an isomorphism $\theta \colon \hat{H}_n^{\mathrm{aff}} \to \hat{\mathfrak{H}}_n$ of the I-adic completion $\hat{H}_n^{\mathrm{aff}}$ of H_n^{aff} with the corresponding completion $\hat{\mathfrak{H}}_n$ of $\mathfrak{H}_n$ as algebras over $\hat{\mathcal{Z}} \cong \hat{\mathfrak{Z}}$. Moreover, θ restricts to give an isomorphism between the (completed) commutative subalgebras $\hat{\mathcal{O}}$ and $\hat{\bar{\mathcal{O}}}$.

Let $\mathcal{K}_n$ be the cyclotomic Hecke algebra over F, where we moreover assume that $v_i = q^{a_i}$ for some integers a_i, $1 \leqslant i \leqslant m$, and that $q \in F$ has infinite order. Note that, while it is known [**6**] that representation theory of a general cyclotomic quotient can be reduced to the case where the v_i are of this form, the requirement that q must have infinite order is genuinely restrictive. Recently, Brundan [**4**] has established the analogue of Conjecture 3.1 for the degenerate cyclotomic Hecke algebras $\mathfrak{K}_n$, where $\mathfrak{K}_n$ is the quotient of $\mathfrak{H}_n$ by the two-sided ideal generated by $f_r(x_1)$, where $f_r(t) = \prod_{i=1}^m (t - a_i)$.

Now we may decompose a cyclotomic Hecke algebra according to the spectrum of the image of $\mathcal{Z}$, which is certainly a central subalgebra (in fact, it follows from the work of Lyle and Mathas [**9**] that this decomposition yields the blocks of the cyclotomic algebra, but we do not need that here). For each such central character with corresponding maximal ideal I, the above discussion shows that the corresponding summands $\mathcal{b}$ and $\mathfrak{b}$ of the cyclotomic and degenerate cyclotomic Hecke algebras are isomorphic. Indeed, since they are finite-dimensional and Lusztig's isomorphism is a map of $\hat{\mathcal{Z}} \cong \hat{\mathfrak{Z}}$-algebras, they are both annihilated by a power of some maximal ideal I (respectively, $\mathcal{I}$) of the centre

$\mathcal{Z}$ (respectively, $\mathfrak{Z}$) where I and $\mathcal{I}$ are identified under θ. Thus, the ideal $J_{\boldsymbol{v},I} \subset H_n^{\mathrm{aff}}$ (respectively, $\mathfrak{J}_{\boldsymbol{v},\mathcal{I}} \subset \mathfrak{H}_n$) defining $\mathcal{b}$ (respectively, $\mathfrak{b}$) contains a sufficiently large power of I (respectively, $\mathfrak{I}$). It follows that if we write $I_H = IH_n^{\mathrm{aff}}$ and $\mathfrak{I}_H = \mathfrak{I}\mathfrak{H}_n$ and $\hat{I}$ for the unique maximal ideal of $\hat{\mathfrak{Z}}$, then for large enough N we have

$$\mathcal{b} \cong \frac{H_n^{\mathrm{aff}}}{J_{\boldsymbol{v},I}} \cong \frac{H_n^{\mathrm{aff}}/I_H^N}{J_{\boldsymbol{v},I}/I_H^N} \cong \frac{\hat{H}_n^{\mathrm{aff}}/(\hat{I}\hat{H}_n^{\mathrm{aff}})^N}{J_{\boldsymbol{v},I}/(\hat{I}\hat{H}_n^{\mathrm{aff}})^N}$$

and, similarly,

$$\mathfrak{b} \cong \frac{\mathfrak{H}_n}{\mathfrak{J}_{\boldsymbol{v},\mathcal{I}}} \cong \frac{\mathfrak{H}_n/(\mathfrak{I}_H)^N}{\mathfrak{J}_{\boldsymbol{v},\mathcal{I}}/\mathfrak{I}_H^N} \cong \frac{\hat{\mathfrak{H}}_n/(\hat{I}\hat{\mathfrak{H}}_n)^N}{\mathfrak{J}_{\boldsymbol{v},\mathcal{I}}/(\hat{I}\hat{\mathfrak{H}}_n)^N}.$$

Now the map induced by θ on the quotients sends $J_{\boldsymbol{v},I}/I^N$ to $\mathfrak{J}_{\boldsymbol{v},\mathcal{I}}/\mathfrak{I}^N$ because $f_{\boldsymbol{v}}$ is sent to f_r so that $\mathfrak{b} \cong \mathcal{b}$. Hence, using Brundan's work we obtain the following result.

Proposition 3.4. *Let F be an algebraically closed field of characteristic zero, $q \in F$ of infinite order and $\mathcal{K}_n$ a cyclotomic Hecke algebra with $v_i = q^{a_i}$ for some integers a_i. Then the centre of $\mathcal{K}_n$ is equal to the image of the centre of H_n^{aff}.*

Remark 3.5. It is shown in [**1**] that the cyclotomic Hecke algebra is semisimple precisely when the polynomial

$$P(q,\boldsymbol{v}) = \prod_{1\leqslant i<j\leqslant m} \left(\prod_{-n<a<n} (q^a v_i - v_j) \right) \prod_{k=1}^{n} (1 + q + \cdots + q^{k-1})$$

is non-vanishing. Thus, the cyclotomic Hecke algebra need not be semisimple even when q is not a root of unity, so this result includes cases which are not covered by the results of [**3**]. It should also be noted that Brundan and Kleshchev [**5**] have recently shown that if F is any field of characteristic zero and q is not a root of unity, then the cyclotomic and degenerate cyclotomic Hecke algebras are isomorphic. One can presumably use their results to extend the above proposition to this more general situation.

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