

A NEW MINIMAX INEQUALITY ON H -SPACES WITH APPLICATIONS

XIE PING DING, WON KYU KIM AND KOK-KEONG TAN

A new minimax inequality on H -spaces is obtained together with six equivalent formulations. As applications, some results on fixed point theorems and system of inequalities are proved. Our results generalise the corresponding results on (1) minimax inequalities due to Fan, Yen, Tan, Shih-Tan and Ding-Tan, (2) fixed point theorems due to Browder, Tarafdar, Shih-Tan and Ding-Tan, (3) convex inequalities due to Fan, (4) systems of inequalities due to Granas-Liu and (5) a minimax theorem due to Kneser.

1. INTRODUCTION

In 1961, Fan [13] gave an extension of the classical Knaster-Kuratowski-Mazurkiewicz theorem [20] to an arbitrary Hausdorff topological vector space. Since then this result has been widely used in non-linear functional analysis and is known as Fan's Lemma. Among various applications of Fan's Lemma, an application (both simple and non-trivial) is the celebrated 1972 minimax inequality of Fan [14]. The minimax inequality of Fan is of a topological nature and has important repercussions both in convex analysis and non-linear functional analysis.

Since 1972, there have been numerous generalisations of Ky Fan's minimax inequality by weakening the compactness assumption or the convexity assumption in topological vector spaces, for example see [1, 2, 4, 7, 15, 26, 27, 29, 30]. In [17, 18, 19], Horvath obtained some minimax inequalities by replacing convexity with pseudo-convexity [17] or contractibility [18, 19] in topological spaces but only in a compact setting. In [3], using Horvath's approach in [19], Bardaro and Ceppitelli introduced the notions of " H -space", "weakly H -convex" and " H -compact" to obtain some minimax inequalities in non-compact setting for mappings taking values in an ordered vector space.

In this paper, we first give two versions of a new generalisation of the Ky Fan minimax inequality to non-compact H -spaces which include a generalisation of Ky

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Fan's minimax inequality by Shih-Tan [26]. Next five equivalent formulations of our minimax inequality are given, extending some earlier results of Fan [13], Browder [5, 6], Shih-Tan [24, 26], Ding-Tan [10] and Tarafdar [28]. As applications, some results on system of inequalities are proved which generalise those of Granas-Liu [16] and Kneser [21].

2. MINIMAX INEQUALITIES

Let X be a non-empty set; we shall denote by 2^X the family of all non-empty subsets of X and by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X . If $F: X \rightarrow 2^X$, we define F^{-1} , F^* , F^c by

$$F^{-1}(y) = \{x \in X : y \in F(x)\}, \quad F^*(y) = \{x \in X : y \notin F(x)\} \quad \text{and} \\ F^c(x) = \{y \in X : y \notin F(x)\}.$$

The following notions, which were introduced by Bardaro and Ceppitelli in [3], were motivated by an earlier work of Horvath [19].

A pair $(X, \{F_A\})$ is said to be an H -space if X is a topological space (which need not be Hausdorff) and $\{F_A\}$ is a family of non-empty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $F_A \subset F_{A'}$ whenever $A \subset A'$. Let $(X, \{F_A\})$ be an H -space. A subset D of X is called H -convex if $F_A \subset D$ for each $A \in \mathcal{F}(D)$. A map $F: X \rightarrow 2^X$ is called H -KKM if $F_A \subset \bigcup_{x \in A} F(x)$ for each $A \in \mathcal{F}(X)$.

The following lemma is a slight improvement of Corollary 1.1 of Horvath in [19].

LEMMA 1. *Let $(X, \{F_A\})$ be an H -space and $G: X \rightarrow 2^X$ be such that*

- (a) G is an H -KKM map;
- (b) for some $x_0 \in X$, $G(x_0)$ is compact and for each $x \in X$, $G(x)$ is closed in X .

Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

PROOF: From Theorem 1 of Horvath in [19], it follows that the family $\{G(x): x \in X\}$ has the finite intersection property. By (b), $\bigcap_{x \in X} G(x) \neq \emptyset$. \square

THEOREM 1. *Let $(X, \{F_A\})$ be an H -space and $\phi, \psi: X \times X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

- (a) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ and $\psi(x, x) \leq 0$ for each $x \in X$;
- (b) for each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on X ;
- (c) the map $x \rightarrow F(x) = \{y \in X : \psi(x, y) \leq 0\}$ is H -KKM;

- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

PROOF: For each $x \in X$, let

$$F(x) = \{y \in X : \psi(x, y) \leq 0\}, \quad G(x) = \{y \in X : \phi(x, y) \leq 0\}.$$

Then we have

- (i) for each $x \in X$, $F(x) \subset G(x)$ by (a),
- (ii) for each $x \in X$, $G(x)$ is closed in X by (b),
- (iii) by (a) and (d), $F(x_0) \subset K$ so that $\overline{F(x_0)} \subset \overline{K} = K$ and hence $\overline{F(x_0)}$ is compact.

By Lemma 1, $\bigcap_{x \in X} \overline{F(x)} \neq \emptyset$. Hence, it follows from (i) and (ii) that $\bigcap_{x \in X} G(x) \neq \emptyset$.

Take any $\hat{y} \in \bigcap_{x \in X} G(x)$; then $\phi(x, \hat{y}) \leq 0$ for all $x \in X$. □

LEMMA 2. Let $(X, \{F_A\})$ be an H -space and $\psi : X \times X \rightarrow \mathbb{R}$ be such that $\psi(x, x) \leq 0$ for all $x \in X$.

Then the following two conditions are equivalent:

- (1) the map $x \rightarrow F(x) = \{y \in X : \psi(x, y) \leq 0\}$ is $H - KKM$;
- (2) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, $\min_{z \in A} \psi(x, y) \leq 0$.

PROOF: (1) \Rightarrow (2): Suppose (2) does not hold; then there exist $A \in \mathcal{F}(X)$ and $y \in F_A$ such that $\min_{z \in A} \psi(x, y) > 0$; it follows that $y \notin f(x)$ for all $x \in A$ so that $y \notin \bigcup_{z \in A} F(x)$. This contradicts (1) that $F_A \subset \bigcup_{z \in A} F(x)$ and $y \in F_A$.

(2) \Rightarrow (1): Suppose (1) does not hold; then there exists $A \in \mathcal{F}(X)$ such that $F_A \not\subset \bigcup_{z \in A} F(x)$. Choose any $y \in F_A$ such that $y \notin \bigcup_{z \in A} F(x)$; it follows that $\psi(x, y) > 0$ for all $x \in A$ so that $\min_{z \in A} \psi(x, y) > 0$ which contradicts (2). □

The condition (2) in Lemma 2 generalises the notion of 0-diagonally quasi-concave in y introduced in [31]. In view of Lemma 2, we have the following result which is equivalent to Theorem 1.

THEOREM 2. Let $(X, \{F_A\})$ be an H -space and $\phi, \psi : X \times X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that

- (a) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ and $\psi(x, x) \leq 0$ for each $x \in X$;
- (b) for each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on X ;

- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, $\min_{z \in A} \psi(x, y) \leq 0$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

Theorem 2 generalises Theorem 1 of Shih-Tan [26] to H -spaces.

COROLLARY 1. Let X be a non-empty convex subset of a topological vector space and $\phi, \psi : X \times X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that

- (a) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ and $\psi(x, x) \leq 0$ for each $x \in X$;
- (b) for each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in \text{co}(A)$, $\min_{z \in A} \psi(x, y) \leq 0$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

PROOF: For each $A \in \mathcal{F}(X)$, let $F_A = \text{co}(A)$; then $(X, \{F_A\})$ forms an H -space so that the conclusion follows from Theorem 2. □

We emphasise here that, in Theorem 1 (also Theorem 2), the coercive condition (d) is imposed on ψ while in other non-compact generalisations of Fan’s minimax inequality, the coercive condition is imposed on ϕ , for example see Allen [1], Tan [27], Bae-Kim-Tan [2] and Ding-Tan [7]. For the purpose of comparison, we shall state the minimax inequality of Ding-Tan [7, Theorem 2] as follows:

THEOREM A. Let X be a non-empty convex set in a topological vector space and $\phi, \psi : X \times X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that

- (a) $\phi(x, y) \leq \psi(x, y)$ for all $(x, y) \in X \times X$ and $\psi(x, x) \leq 0$ for all $x \in X$;
- (b) for each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on C for each non-empty compact subset C of X ;
- (c) for each $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ is convex;
- (d) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in \text{co}(X_0 \cup \{y\})$ with $\phi(x, y) > 0$.

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

For generalisations of Theorem A to H -spaces, we refer to Ding-Tan [8, 9].

The following examples show that the coercive conditions (d) of Theorem 1 and (d) of Theorem A are independent of each other:

EXAMPLE 1. Let $X = [0, \infty)$ and $\phi, \psi : X \times X \rightarrow \mathbb{R}$ be defined by

$$\phi(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y < x + \frac{1}{1+x} \text{ for } 0 \leq x < 1\}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi(x, y) = \begin{cases} 0 & \text{if } y \leq x, \\ 1 & \text{otherwise.} \end{cases}$$

Then all the conditions (a), (b), (c) and (d) in Theorem 1 are satisfied with $K = \{0\} = \{x_0\}$ so that there exists $\hat{y} = 0 \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$. However, there does not exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists a point $x \in \text{co}(X_0 \cup \{y\})$ with $\phi(x, y) > 0$. Thus Theorem A is not applicable.

EXAMPLE 2. Let $X = [0, \infty)$ and $\phi, \psi : X \times X \rightarrow \mathbb{R}$ be defined by

$$\phi(x, y) = \psi(x, y) = \begin{cases} 0 & \text{if } y \leq x \text{ or } x + \frac{1}{1+x} \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then all the conditions (a), (b), (c) and (d) in Theorem A are satisfied with $X_0 = K = \{0\}$ so that there exists $\hat{y} = 0 \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$. However, there does not exist a non-empty compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$. Therefore Theorem 1 is not applicable.

3. EQUIVALENT FORMULATIONS OF THE MINIMAX INEQUALITY

The following Theorems 3, 4, 5, 6 and 7 are all equivalent to our minimax inequality Theorem 2 (and hence also Theorem 1):

THEOREM 3. (First Geometric Form) *Let $(X, \{F_A\})$ be an H -space and $B, C \subset X \times X$ be such that*

- (a) $B \subset C$;
- (b) for each $x \in X$, the set $\{y \in X : (x, y) \in B\}$ is open in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $(x, y) \notin C$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $(x_0, y) \in C$ for all $y \in X \setminus K$.

Then either there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \in C$ or there exists $\hat{y} \in X$ such that $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$.

THEOREM 4. (Second Geometric Form) *Let $(X, \{F_A\})$ be an H -space and $M, N \subset X \times X$ be such that*

- (a) $N \subset M$;

- (b) for each $x \in X$, the set $\{y \in X : (x, y) \in M\}$ is closed in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $(x, y) \in N$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $(x_0, y) \notin N$ for all $y \in X \setminus K$.

Then either there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \notin N$ or there exists $\hat{y} \in X$ such that $X \times \{\hat{y}\} \subset M$.

THEOREM 5. (Fixed Point Version) Let $(X, \{F_A\})$ be an H -space and $F, G : X \rightarrow 2^X \cup \{\emptyset\}$ be such that

- (a) for each $x \in X$, $F(x) \subset G(x)$;
- (b) for each $x \in X$, $F(x)$ is open in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $y \notin G(x)$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset G(x_0)$;
- (e) for each $y \in X$, $F^{-1}(y) \neq \emptyset$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

THEOREM 6. (Fixed Point Version) Let $(X, \{F_A\})$ be an H -space and $S, T : X \rightarrow 2^X$ be such that

- (a) for each $x \in X$, $S(x) \subset T(x)$;
- (b) for each $y \in X$, $S^{-1}(y)$ is open in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $x \notin T(y)$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset T^{-1}(x_0)$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

THEOREM 7. (Maximal Element Version) Let $(X, \{F_A\})$ be an H -space and $P, Q : X \rightarrow 2^X \cup \{\emptyset\}$ be such that

- (a) for each $x \in X$, $P(x) \subset Q(x)$ and $x \notin Q(x)$;
- (b) for each $y \in X$, $P^{-1}(y)$ is open in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $x \notin Q(y)$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset Q^{-1}(x_0)$.

Then there exists $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$.

PROOF OF "THEOREM 2 \Rightarrow THEOREM 3":

Suppose that for each $x \in X$, $(x, x) \notin C$. Let ϕ and ψ be the characteristic functions of B and C respectively, then

- (i) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ by (a) and $\psi(x, x) \leq 0$ by assumption;
- (ii) for each $x \in X$ and for each $\lambda \in \mathbb{R}$,

$$\{y \in X : \phi(x, y) > \lambda\} = \begin{cases} X, & \text{if } \lambda < 0, \\ \{y \in X : (x, y) \in B\}, & \text{if } 0 \leq \lambda < 1, \\ \emptyset, & \text{if } \lambda \geq 1, \end{cases}$$

is open in X by (b) so that $\phi(x, y)$ is a lower semicontinuous function of y on X ;

- (iii) by (c) and (d), the conditions (c) and (d) of Theorem 2 hold.

By Theorem 2, there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$. It follows that $(x, \hat{y}) \notin B$ for all $x \in X$; that is, $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$. □

PROOF OF "THEOREM 3 \Rightarrow THEOREM 2":

Let $B, C \subset X \times X$ be defined by

$$B = \{(x, y) \in X \times X : \phi(x, y) > 0\}, \\ C = \{(x, y) \in X \times X : \psi(x, y) > 0\}.$$

Then the conditions (a), (b), (c) and (d) of Theorem 3 are all satisfied and $(x, x) \notin C$ for all $x \in X$. Hence by Theorem 3, there exists $\hat{y} \in X$ such that $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$; that is $\phi(x, \hat{y}) \leq 0$ for all $x \in X$. □

PROOF OF "THEOREM 3 \Rightarrow THEOREM 4":

Let $B = X \times X \setminus M$ and $C = X \times X \setminus N$; apply Theorem 3, the conclusion follows. □

PROOF OF "THEOREM 4 \Rightarrow THEOREM 3":

Let $M = X \times X \setminus B$ and $N = X \times X \setminus C$; apply Theorem 4, the conclusion follows. □

PROOF OF "THEOREM 3 \Rightarrow THEOREM 5":

Let $B, C \subset X \times X$ be defined by

$$B = \{(x, y) \in X \times X : y \in F(x)\}, \\ C = \{(x, y) \in X \times X : y \in G(x)\}.$$

Then the conditions (a), (b), (c) and (d) in Theorem 3 are all satisfied and by (e), for each $y \in X$, there exists $x \in X$ such that $(x, y) \in B$ so that the set $\{x \in X :$

$(x, y) \in B\} \neq \emptyset$. Hence by Theorem 3, there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \in C$; that is, $\hat{x} \in G(\hat{x})$. □

PROOF OF "THEOREM 5 \Rightarrow THEOREM 3":

Define $F, G: X \rightarrow 2^X \cup \{\emptyset\}$ by

$$F(x) = \{y \in X : (x, y) \in B\},$$

$$G(x) = \{y \in X : (x, y) \in C\}$$

for each $x \in X$. Then the conditions (a), (b), (c) and (d) in Theorem 5 are all satisfied. Hence by Theorem 5, either there exists $\hat{y} \in X$ such that $F^{-1}(\hat{y}) = \emptyset$ or there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$; that is, either there exists $\hat{y} \in X$ such that $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$ or there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \in C$. □

PROOF OF "THEOREM 5 \Rightarrow THEOREM 6":

Define $F, G: X \rightarrow 2^X \cup \{\emptyset\}$ by $F = S^{-1}$ and $G = T^{-1}$; then by (a), (b), (c) and (d), the conditions (a), (b), (c) and (d) of Theorem 5 are all satisfied. Since $S(x) \neq \emptyset$ for all $x \in X$, the condition (e) of Theorem 5 is also satisfied. Hence by Theorem 5, there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$; that is, $\hat{x} \in T(\hat{x})$. □

PROOF OF "THEOREM 6 \Rightarrow THEOREM 5":

Define $S, T: X \rightarrow 2^X \cup \{\emptyset\}$ by $S = F^{-1}$ and $T = G^{-1}$; then by (a), (b), (c) (d), the conditions (a), (b), (c) and (d) of Theorem 6 are all satisfied. By (e), $S(x) \neq \emptyset$ for each $x \in X$ so that $S, T: X \rightarrow 2^X$. By Theorem 6, there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$; that is, $\hat{x} \in G(\hat{x})$. □

The proof of "Theorem 6 \Leftrightarrow Theorem 7" is obvious.

As an immediate consequence of Theorem 3, we have the following result which generalises Theorem 3 of Shih-Tan [24] in several aspects:

COROLLARY 2. *Let X be a non-empty convex subset of a topological vector space and $B, C \subset X \times X$ be such that*

- (a) $B \subset C$;
- (b) for each $x \in X$, the set $\{y \in X : (x, y) \in B\}$ is open in X ;
- (c) for each $A \in \mathcal{F}(X)$ and for each $y \in \text{co}(A)$, there exists $x \in A$ such that $(x, y) \notin C$;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $(x_0, y) \in C$ for all $y \in X \setminus K$.

Then either there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \in C$ or there exists $\hat{y} \in X$ such that $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$.

Similarly, Theorem 4 generalises Theorem 4 of Shih-Tan [24] to H -spaces in non-compact setting.

COROLLARY 3. Let $(X, \{F_A\})$ be an H -space and $F, G: X \rightarrow 2^X \cup \{\emptyset\}$ be such that

- (a) for each $x \in X$, $F(x) \subset G(x)$;
- (b) for each $x \in X$, $F(x)$ is open in X ;
- (c) for each $y \in X$, $F^{-1}(y) \neq \emptyset$ and $G^{-1}(y)$ is H -convex;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset G(x_0)$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$.

PROOF: Suppose there exist $A \in \mathcal{F}(X)$ and $y \in F_A$ such that $y \in G(x)$ for all $x \in A$, then $A \subset G^{-1}(y)$ so that by (c), $F_A \subset G^{-1}(y)$ and hence $y \in F_A \subset G^{-1}(y)$; thus $y \in G(y)$. Suppose for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $y \notin G(x)$, then the condition (c) of Theorem 5 is also satisfied. By Theorem 5, there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$. \square

Corollary 3 generalises Theorem 5 of Shih-Tan [24] and its non-compact generalisation by Shih-Tan [26, Theorem 2] to non-convex setting.

We remark here that Theorem 7, the maximal element version of minimax inequality, has applications to mathematical economics, for example see Ding-Tan [10].

The following result is an easy consequence of Theorem 6:

COROLLARY 4. Let $(X, \{F_A\})$ be an H -space and $S, T: X \rightarrow 2^X$ be such that

- (a) for each $x \in X$, $S(x) \subset T(x)$;
- (b) for each $y \in X$, $S^{-1}(y)$ is open in X ;
- (c) for each $x \in X$, $T(x)$ is H -convex;
- (d) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset T^{-1}(x_0)$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

PROOF: Suppose there exist $A \in \mathcal{F}(X)$ and $y \in F_A$ such that $x \in T(y)$ for all $x \in A$, then $A \subset T(y)$ so that by (c), $F_A \subset T(y)$ and hence $y \in T(y)$ as $y \in F_A$. If for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $x \notin T(y)$, then the condition (c) of Theorem 6 is also satisfied. By Theorem 6, there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$. \square

Corollary 4 generalises Theorem 1 of Browder in [5] (which is equivalent to Lemma 4 of Fan in [13]) and its non-compact generalisation by Browder [6, Theorem 2] in several aspects. Clearly Corollary 4 is equivalent to Corollary 3. We shall now give another equivalent formulation of Corollary 4 as follows:

COROLLARY 5. Let $(X, \{F_A\})$ be an H -space and $T: X \rightarrow 2^X \cup \{\emptyset\}$ be such

that

- (i) for each $x \in X$, $T(x)$ is H -convex;
- (ii) for each $y \in X$, $T^{-1}(y)$ contains an open subset O_y (which may be empty) of X such that $\bigcup_{y \in X} O_y = X$;
- (iii) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $X \setminus K \subset T^{-1}(x_0)$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

PROOF OF "COROLLARY 4 \Rightarrow COROLLARY 5":

Define $S: X \rightarrow 2^X$ by $S(x) = \{y \in X : x \in O_y\}$ for each $x \in X$. Then we have

- (1) for each $x \in X$, $S(x) \neq \emptyset$ and $S(x) \subset T(x)$ by (ii);
- (2) for each $y \in X$, $S^{-1}(y) = O_y$ is open in X by (ii).

Thus all the hypotheses of Corollary 4 are satisfied and hence the conclusion follows from Corollary 4. □

PROOF OF "COROLLARY 5 \Rightarrow COROLLARY 4":

For each $y \in X$, let $O_y = S^{-1}(y)$; then

- (1) for each $y \in X$, O_y is open in X by (b) and $O_y = S^{-1}(y) \subset T^{-1}(y)$ by (a);
- (2) by the assumption, for each $x \in X$, $S(x) \neq \emptyset$, we have $\bigcup_{y \in X} O_y = \bigcup_{y \in X} S^{-1}(y) = X$.

By Corollary 5, the conclusion of Corollary 4 holds. □

Corollary 5 generalises Theorem 2 of Ding-Tan [10] which in turn generalises Theorem 1 of Tarafdar in [28] (also Theorem 1 of Mehta-Tarafdar in [22]) to non-compact H -spaces.

4. SYSTEMS OF INEQUALITIES

According to Pietsch [23, p.40], a collection \mathcal{F} of real-valued functions f defined on a set X is called *concave* if, given any finite subset $\{f_1, \dots, f_n\}$ of \mathcal{F} and $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, there exists $f \in \mathcal{F}$ such that $f(x) \geq \sum_{i=1}^n \alpha_i f_i(x)$ for all $x \in X$.

Given any two collections \mathcal{F} and \mathcal{G} of real-valued functions on a set X , we shall write $\mathcal{F} \leq \mathcal{G}$ if for any $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f(x) \leq g(x)$ for all $x \in X$.

THEOREM 8. *Let $(X, \{F_A\})$ be a normal H -space. Let \mathcal{F}, \mathcal{G} and \mathcal{H} be three collections of real-valued functions on X such that*

- (i) $\mathcal{F} \leq \mathcal{G} \leq \mathcal{H}$;

- (ii) for each $f \in \mathcal{F}$, f is lower semicontinuous on X ;
- (iii) for any finite subset $\{g_1, \dots, g_n\}$ of \mathcal{G} and for any $\beta_1, \dots, \beta_n: X \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(x) = 1$ for each $x \in X$, the following property holds: for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that

$$\sum_{i=1}^n \beta_i(y)g_i(y) \leq \sum_{i=1}^n \beta_i(y)g_i(x);$$

- (iv) there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that for any finite subset $\{g_1, \dots, g_n\}$ of \mathcal{G} and for any $\beta_1, \dots, \beta_n: X \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(x) = 1$ for each $x \in X$,

$$\sum_{i=1}^n \beta_i(y)g_i(y) > \sum_{i=1}^n \beta_i(y)g_i(x_0) \text{ for all } y \in X \setminus K;$$

- (v) the collection \mathcal{H} is concave.

Then given any $\rho \in \mathbb{R}$, one of the following properties holds:

- (1) there exists an $h \in \mathcal{H}$ such that $\inf_{x \in X} h(x) > \rho$;
- (2) there exists a point $\hat{y} \in X$ such that $f(\hat{y}) \leq \rho$ for all $f \in \mathcal{F}$.

PROOF: Without loss of generality, we may assume that $\rho = 0$. For each $f \in \mathcal{F}$, let $Q(f) = \{x \in K : f(x) \leq 0\}$; then $Q(f)$ is closed in K by (ii). If the family $\{Q(f) : f \in \mathcal{F}\}$ has the finite intersection property, then by the compactness of K we obtain the alternative (2). Suppose $\{Q(f) : f \in \mathcal{F}\}$ does not have the finite intersection property. Then there are $f_1, \dots, f_n \in \mathcal{F}$ such that $\bigcap_{i=1}^n Q(f_i) = \emptyset$. For each $i = 1, \dots, n$, let $V_i = X \setminus Q(f_i)$; then each V_i is open in X and $\{V_1, \dots, V_n\}$ is an open covering of the normal space X . Let $\{\beta_1, \dots, \beta_n\}$ be a continuous partition of unity, subordinate to this open covering. Thus for each $i = 1, \dots, n$, $\beta_i: X \rightarrow [0, 1]$ is continuous and $\text{Supp } \beta_i \subset V_i$ such that $\sum_{i=1}^n \beta_i(x) = 1$ for each $x \in X$. Choose $g_1, \dots, g_n \in \mathcal{G}$ and $h_1, \dots, h_n \in \mathcal{H}$ such that $f_i \leq g_i \leq h_i$ on X for each $i = 1, \dots, n$. Define $\phi, \psi: X \times X \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} \phi(x, y) &= \sum_{i=1}^n \beta_i(y)f_i(y) - \sum_{i=1}^n \beta_i(y)g_i(x), \\ \psi(x, y) &= \sum_{i=1}^n \beta_i(y)g_i(y) - \sum_{i=1}^n \beta_i(y)g_i(x). \end{aligned}$$

Then we have

- (1) $\phi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times X$ and $\psi(x, x) = 0$ for all $x \in X$;
- (2) for each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on X by (ii);
- (3) by (iii), for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ such that $\psi(x, y) \leq 0$; thus $\min_{x \in A} \psi(x, y) \leq 0$;
- (4) by (iv), there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.

Thus all hypotheses of Theorem 2 are satisfied so that there exist $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; that is,

$$\sum_{i=1}^n \beta_i(\hat{y}) f_i(\hat{y}) \leq \sum_{i=1}^n \beta_i(\hat{y}) g_i(x) \text{ for all } x \in X.$$

By (v), there is an $h \in \mathcal{H}$ satisfying $h(x) \geq \sum_{i=1}^n \beta_i(\hat{y}) h_i(x)$ for all $x \in X$. Therefore for all $x \in X$,

$$0 < \sum_{i=1}^n \beta_i(\hat{y}) f_i(\hat{y}) \leq \sum_{i=1}^n \beta_i(\hat{y}) g_i(x) \leq \sum_{i=1}^n \beta_i(\hat{y}) h_i(x) \leq h(x).$$

This proves the alternative (1) □

Theorem 8 generalises Fan’s Lemma [23, p.40] (see also Theorem 6 in [25] which is equivalent to Theorem 1 of Fan in [12]) to H -spaces.

Let h be a real-valued function defined on the product set $X \times Y$ of two arbitrary non-empty sets X and Y . According to Fan [11], h is said to be *concave on X* if for any two elements $x_1, x_2 \in X$ and two numbers $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$, there exists $x_0 \in X$ such that

$$h(x_0, y) \geq \alpha_1 h(x_1, y) + \alpha_2 h(x_2, y) \text{ for all } y \in Y.$$

THEOREM 9. *Let X be an arbitrary non-empty set and $(Y, \{F_A\})$ be a normal H -space. Let $f, g, h: X \times Y \rightarrow \mathbb{R}$ be such that*

- (i) $f \leq g \leq h$ on $X \times Y$;
- (ii) for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on Y ;
- (iii) for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y$, the following property holds: for each

$A \in \mathcal{F}(Y)$ and for each $y \in F_A$, there exists $z \in A$ such that

$$\sum_{i=1}^n \beta_i(y)g(x_i, y) \leq \sum_{i=1}^n \beta_i(y)g(x_i, z);$$

- (iv) there exist a non-empty closed and compact subset K of Y and $z_0 \in Y$ such that for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y$,

$$\sum_{i=1}^n \beta_i(y)g(x_i, y) > \sum_{i=1}^n \beta_i(y)g(x_i, z_0) \text{ for all } y \in Y \setminus K;$$

- (v) h is concave on X .

Then there exists $\hat{y} \in Y$ such that

$$\sup_{z \in X} f(x, \hat{y}) \leq \sup_{z \in X} \inf_{y \in Y} h(x, y).$$

PROOF: Let $\rho = \sup_{z \in X} \inf_{y \in Y} h(x, y)$. Applying Theorem 8 with X being the index set, there exists $\hat{y} \in Y$ such that $f(x, \hat{y}) \leq \rho$ for all $x \in X$. Hence the conclusion $f(x, \hat{y}) \leq \rho$ follows. □

Theorem 9 generalises the corresponding result due to Granas-Liu [16] to non-convex and non-compact setting.

When $f = g = h$ in Theorem 9, we obtain the following new minimax theorem:

THEOREM 10. Let X be an arbitrary non-empty set and $(Y, \{F_A\})$ be a normal H -space. Let $g: X \times Y \rightarrow \mathbb{R}$ be such that

- (i) for each fixed $x \in X$, $g(x, y)$ is a lower semicontinuous function of y on Y ;
- (ii) for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y$, the following property holds: for each $A \in \mathcal{F}(Y)$ and for each $y \in F_A$, there exists $z \in A$ such that

$$\sum_{i=1}^n \beta_i(y)g(x_i, y) \leq \sum_{i=1}^n \beta_i(y)g(x_i, z);$$

- (iii) there exist a non-empty closed and compact subset K of Y and $z_0 \in Y$ such that, for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y$,

$$\sum_{i=1}^n \beta_i(y)g(x_i, y) > \sum_{i=1}^n \beta_i(y)g(x_i, z_0) \text{ for all } y \in Y \setminus K;$$

- (iv) g is concave on X .

Then

$$\inf_{y \in Y} \sup_{z \in X} g(x, y) = \sup_{z \in X} \inf_{y \in Y} g(x, y).$$

PROOF: By Theorem 9, we have $\inf_{y \in Y} \sup_{z \in X} g(x, y) \leq \sup_{z \in X} \inf_{y \in Y} g(x, y)$. The reverse inequality always holds. □

In Theorem 10, if Y is also compact, by taking $K = Y$ the condition (iii) is satisfied trivially. Hence we have the following immediate consequence of Theorem 10:

THEOREM 11. *Let X be an arbitrary non-empty set and $(Y, \{F_A\})$ be a compact H -space. Let $g: X \times Y \rightarrow \mathbb{R}$ be such that*

- (i) *for each fixed $x \in X$, $g(x, y)$ is a lower semicontinuous function of y on Y ;*
- (ii) *for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y$, the following property holds: for each $A \in \mathcal{F}(Y)$ and for each $y \in F_A$, there exists $z \in A$ such that*

$$\sum_{i=1}^n \beta_i(y)g(x_i, y) \leq \sum_{i=1}^n \beta_i(y)g(x_i, z);$$

- (iii) *g is concave on X .*

Then

$$\min_{y \in Y} \sup_{z \in X} g(x, y) = \sup_{z \in X} \inf_{y \in Y} g(x, y).$$

Theorem 11 (and hence Theorem 10) generalises a well-known minimax theorem of Kneser [21].

Note that $\min_{y \in Y} \sup_{z \in X} g(x, y)$ is attained at some point in Y since $y \rightarrow \sup_{z \in X} g(x, y)$ is also lower semicontinuous and Y is compact.

REMARK. The condition (ii) of Theorem 10 (respectively Theorem 11) can be replaced by the following condition (ii)' without affecting the conclusion:

- (ii)' *for any finite subset $\{x_1, \dots, x_n\}$ of X and for any $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, the following property holds: for any $\lambda \in \mathbb{R}$ and for any*

$$A \in \mathcal{F}(Y), \text{ if } A \subset \{y \in Y: \sum_{i=1}^n \alpha_i g(x_i, y) \leq \lambda\}, \text{ then}$$

$$F_A \subset \{y \in Y: \sum_{i=1}^n \alpha_i g(x_i, y) \leq \lambda\}.$$

PROOF: It suffices to show that (ii)' \Rightarrow (ii). If (ii) were not true, then there exist $\{x_1, \dots, x_n\} \subset X, \beta_1, \dots, \beta_n: Y \rightarrow [0, 1]$ with $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in Y, A \in \mathcal{F}(Y)$ and $\hat{y} \in F_A$ such that

$$(*) \quad \sum_{i=1}^n \beta_i(\hat{y})g(x_i, \hat{y}) > \sum_{i=1}^n \beta_i(\hat{y})g(x_i, z) \text{ for all } z \in A.$$

Let $\lambda = \max_{z \in A} \sum_{i=1}^n \beta_i(\hat{y})g(x_i, z)$, then

$$A \subset \{y \in Y : \sum_{i=1}^n \beta_i(\hat{y})g(x_i, y) \leq \lambda\}.$$

By (ii)', $F_A \subset \{y \in Y : \sum_{i=1}^n \beta_i(\hat{y})g(x_i, y) \leq \lambda\}$; as $\hat{y} \in F_A$, we must have $\sum_{i=1}^n \beta_i(\hat{y})g(x_i, \hat{y}) \leq \lambda$ which contradicts (*). □

Similarly, the condition (iii) of Theorem 8 (respectively, Theorem 9) can be replaced by the following condition (iii)' without affecting the conclusion:

(iii)' for each finite subset $\{g_1, \dots, g_n\}$ of G and for any $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, the following property holds: for any $\lambda \in \mathbb{R}$ and for any $A \in \mathcal{F}(X)$, if $A \subset \{x \in X : \sum_{i=1}^n \alpha_i g_i(y) \leq \lambda\}$, then

$$F_A \subset \{x \in X : \sum_{i=1}^n \alpha_i g_i(x) \leq \lambda\}.$$

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Department of Mathematics
Sichuan Normal University
Chengdu, Sichuan
China

Department of Mathematics Education
Chungbuk National University
Cheongju
Korea

Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5