UNIVALENT SOLUTIONS OF $W'' + pW = 0$

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1. Introduction. Consider the differential equation

$$W''(z) + p(z)W(z) = 0 \tag{1.1}$$

where

$$z^2 p(z) = p_0 + p_1 z + \ldots + p_n z^n + \ldots \tag{1.2}$$

is regular in $|z| < R$.

The indicial equation associated with (1.1) is of the form

$$\lambda^2 - \lambda + p_0 = 0.$$

We shall denote the two roots of this equation by α and β, where $\Re\{\alpha\} \geqslant \frac{1}{2}$. Corresponding to the root α there exists a unique solution of (1.1) of the following form

$$W_\alpha(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1. \tag{1.3}$$

In those cases for which $0 < \Re\{\beta\} < \frac{1}{2}$ there exists a unique second solution of (1.1) of the form

$$W_\beta(z) = z^\beta \sum_{n=0}^{\infty} b_n z^n, \quad b_0 = 1. \tag{1.4}$$

In 1955 Robertson **(5)** obtained conditions on $p(z)$ in (1.1) sufficient to ensure the univalence in $|z| < 1$ of the function $F(z) = [W_\alpha(z)]^{1/\alpha}$, where that branch of $F(z)$ is chosen for which $F'(0) = 1$. In this same paper Robertson posed the problem of obtaining similar conditions for the univalence of the functions $[W_\beta(z)]^{1/\beta}$, where $0 < \Re\{\beta\} < \frac{1}{2}$. It is towards the solution of this problem that we direct our attention in §§ 3 and 4 of this paper. In § 3 we also obtain bounds on the radii of zero-free regions of the regular singular point $z = 0$.

Section 2 contains all pertinent definitions as well as the statement of a known theorem to which we shall refer in § 3.

In § 5 we employ some of the results of § 3 to obtain two theorems concerning the univalence of normalized Bessel functions and their products.

2. Preliminaries. We state in this section several definitions and a theorem of which we shall have need in the succeeding sections.

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DEFINITION 2.1. *A function* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, *regular in* $|z| < R$, *is said to be univalently star-like in* $|z| < R$ *if for every* r *in the range* $0 \leqslant r < R$

(a) $$\Re\left\{\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} > 0, 0 \leqslant \theta < 2\pi,$$

(b) $$\int_0^{2\pi} \Re\left\{\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta = 2\pi.$$

DEFINITION 2.2 *A function* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, *regular in* $|z| < R$, *is said to be spiral-like in* $|z| < R$ *if for some real* α, $|\alpha| < \pi/2$,

$$\Re\left\{\frac{e^{i\alpha}zf'(z)}{f(z)}\right\} > 0$$

or all $|z| < R$. *It was shown by Špaček* **(6)** *that this condition is sufficient for the univalence of* $f(z)$ *in* $|z| < R$.

Following the usage of Hille **(3)** we make the following

DEFINITION 2.3. *Given a particular zero* $z = c$ *of a particular solution* $W(z)$ *of* (1.1) *then we call* G *a zero-free region about* c *provided that* G *is a region containing* c *in which* $W(z)$ *does not vanish again.*

THEOREM 2.4. (*Beesack* **(1)**.) *Let* $p(z)$ *be analytic in a region* D, *and let the line* $z = a + re^{i\theta}$, $0 \leqslant r \leqslant R$, *lie in the region* D. *Set*

$$e^{2i\theta}p(a + re^{i\theta}) = q_1(r;\theta) + iq_2(r;\theta).$$

Let $Q(r)$ *be continuous in* $0 \leqslant r \leqslant R$, *and suppose the differential equation*

$$y'' + Q(r)y = 0$$

has a real solution $y(r)$ *which does not vanish on* $0 < r < R$. *If there exist real numbers* $\lambda \geqslant 0$, μ, $(\lambda^2 + \mu^2 \neq 0)$, *such that*

$$\lambda q_1(r;0) + \mu q_2(r;0) \leqslant \lambda Q(r), \qquad 0 \leqslant r \leqslant R, \tag{2.5}$$

then any non-trivial solution $w(z)$ *of* $w''(z) + p(z)w(z) = 0$ *having* $w(a) = 0$ *has no other zeros on the open line segment* $(a, a + Re^{i\theta})$ *unless* $q_2(r;\theta) \equiv 0$. *Even if* $q_2(r;\theta) \equiv 0$, *the conclusion holds provided that* $\lambda \neq 0$. *Moreover, if strict inequality holds in* (2.5) *for a single point, then* $w(z)$ *has no zeros on the half-closed segment* $(a, a + Re^{i\theta}]$.

3. Univalence of $[W_\beta(z)]^{1/\beta}$, β real.

Let

$$z^2p^*(z) = p_0^* + p_1^*z + \ldots + p_n^*z^n + \ldots, \quad p_0^* \leqslant \tfrac{1}{4}, \tag{3.1}$$

be regular in $|z| < R$ and real on the real axis. Consider the differential equation

$$W''(z) + \left\{C\left[p^*(z) - \frac{p_0^*}{z^2}\right] + \frac{p_0^*}{z^2}\right\}W(z) = 0 \tag{3.2}$$

where C is any non-negative constant. Let us denote the roots of the associated indicial equation by α^* and β^* where $\alpha^* \geqslant \frac{1}{2}$. Corresponding to the root α^* there exists a unique solution of (3.2) of the form

$$(3.3) \qquad W_{\alpha^*}(z) = z^{\alpha^*} \sum_{n=0}^{\infty} a_n^*(C) z^n, \; a_0^* = 1,$$

and whenever $0 < \beta^* < \frac{1}{2}$ there corresponds to the root β^* a unique second solution of the form

$$(3.4) \qquad W_{\beta^*}(z) = z^{\beta^*} \sum_{n=0}^{\infty} b_n^*(C) z^n, \; b_0^* = 1.$$

We are now prepared to prove the following lemma.

LEMMA 3.5. *Let $y(\rho)$, $y'(\rho) = dy(\rho)/d\rho$ be real functions, continuous in the real variable ρ for $0 < \rho < R$. Let $W_k(z)$, $k = \alpha^*, \beta^*$, represent one of the solutions* (3.3) *or* (3.4) *of equation* (3.2). *Then*

$$(3.6) \qquad \int_{r_1}^{r_2} \{C[\rho^2 p^*(\rho) - p_0^*] + p_0^*\} \cdot \frac{y^2(\rho)}{\rho^2}\, d\rho \leqslant \int_{r_1}^{r_2} [y'(\rho)]^2 d\rho - \left[\frac{W_k'(\rho)}{W_k(\rho)} \cdot y^2(\rho)\right]_{r_1}^{r_2}$$

whenever $0 < r_1 < r_2 < \min(\bar{\rho}_k, R)$, where $\bar{\rho}_k$ is the smallest positive zero of the function $W_k(\rho)$. Equality holds in (3.6) *when, and only when, $y(\rho) = AW_k(\rho)$, where A is an arbitrary real constant. For the case $k = \alpha^*$ if $y(\rho) = O(\rho^\delta)$, $y'(\rho) = O(\rho^{\delta-1})$, $\delta > \frac{1}{2}$, for small ρ then the inequality* (3.6) *is valid even when* $r_1 = 0$, **(5)**.

Proof. The proof follows immediately from the expansion and integration by parts of the integrand in the inequality

$$\int_{r_1}^{r_2} \left[y'(\rho) - \frac{W_k'(\rho)}{W_k(\rho)} y(\rho)\right]^2 d\rho \geqslant 0,$$

$0 < r_1 < r_2 < \min(\bar{\rho}_k, R)$.

We shall need the following lemma for the proof of Theorem 3.8.

LEMMA 3.7. *Let $p(z)$, $W_\alpha(z)$, $W_\beta(z)$, $p^*(z)$, $W_{\alpha^*}(z)$* and *$W_{\beta^*}(z)$ be defined as in* (1.2), (1.3), (1.4), (3.1), (3.3), *and* (3.4) *respectively. Let*

$$\Re\{z^2 p(z)\} \leqslant C[|z|^2 p^*(|z|) - p_0^*] + p_0^*, \; C > 0,$$

for all $|z| < R$. Then neither $W_\alpha(z)$ nor $W_\beta(z)$ can have more than one zero on any ray $\arg z =$ *constant*, $0 < |z| < \min(\rho, R)$ *where* $\rho = \max(\bar{\rho}_{\alpha^*}, \bar{\rho}_{\beta^*})$ *and $\bar{\rho}_{\alpha^*}$, $\bar{\rho}_{\beta^*}$ are respectively the smallest positive zeros of the functions $W_{\alpha^*}(\rho)$ and $W_{\beta^*}(\rho)$.*

Proof. This lemma is a particular case of Theorem 2.4.

With the aid of Lemmas 3.5 and 3.7 we now prove the following theorem.

THEOREM 3.8. *Let* $0 < p^*_0 < \frac{1}{4}$ *and define* $p(z)$, $W_\beta(z)$, $p^*(z)$, *and* $W_{\beta^*}(z)$ *as in* (1.2), (1.4), (3.1), *and* (3.4) *respectively. Let* $p_0 = p^*_0$ *and*

$$\Re\{z^2p(z)\} < C[|z|^2p^*(|z|) - p^*_0] + p^*_0,\ C > 0, \tag{3.9}$$

for all $0 < |z| < R$. *Then* $W_\beta(z)$ *has no zeros in the annulus* $0 < |z| < \min(\bar{\rho}_{\beta^*}, R)$ *where* $\bar{\rho}_{\beta^*}$ *is the smallest positive zero of the function* $W_{\beta^*}(\rho)$. *This theorem remains valid if we replace* $W_\beta(z)$ *by* $W_\alpha(z)$, $W_{\beta^*}(z)$ *by* $W_{\alpha^*}(z)$, *and* $\bar{\rho}_{\beta^*}$ *by* $\bar{\rho}_{\alpha^*}$, *where* $\bar{\rho}_{\alpha^*}$ *is the smallest positive zero of the function* $W_{\alpha^*}(\rho)$.

Proof. If we multiply (1.1) by $\overline{W(z)dz}$ and integrate from z_1 to z_2 in $0 < |z| < R$ we obtain the Green's transform of (1.1) in the form

$$\left[\overline{W(z)}W'(z)\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |W(z)|^2\overline{dz} + \int_{z_1}^{z_2} p(z)|W(z)|^2 dz = 0. \tag{3.10}$$

It is readily seen that (3.10) is valid for either of the solutions (1.3) or (1.4) of (1.1).

Let us suppose that z_1 and z_2 lie on the same ray $\theta =$ constant, $0 < r < \min(\bar{\rho}_{\beta^*}, R)$. Furthermore, let z_2 be a zero of $W_\beta(z)$. Then it follows from Lemma 3.7 that there are no other zeros of $W_\beta(z)$ on the open segment $\theta =$ constant, $0 < r < |z_2|$.

If we multiply (3.10) by $e^{i\theta}$ and choose our path of integration to be the line segment joining $z_1 = r_1e^{i\theta}$ to $z_2 = r_2e^{i\theta}$ we obtain the equality

$$e^{i\theta}\overline{W_\beta(r_1e^{i\theta})}W'_\beta(r_1e^{i\theta}) = \int_{r_1}^{r_2} e^{2i\theta}p(re^{i\theta})|W_\beta|^2 dr - \int_{r_1}^{r_2} |W'_\beta|^2 dr. \tag{3.11}$$

From (3.11) it follows that

$$\frac{|W_\beta(z_1)|^2}{|z_1|}\Re\left\{\frac{z_1W'_\beta(z_1)}{W_\beta(z_1)}\right\} = \int_{r_1}^{r_2} \Re\{z^2p(z)\}\frac{|W_\beta|^2}{r^2}dr - \int_{r_1}^{r_2} |W'_\beta|^2 dr. \tag{3.12}$$

Next we apply Lemma 3.5 to $\Re\{W_\beta(z)\}$ and $\Im\{W_\beta(z)\}$ and add the resulting inequalities to obtain the following inequality.

$$|W_\beta(z_1)|^2\frac{W'_{\beta^*}(|z_1|)}{W_{\beta^*}(|z_1|)} \geqslant \int_{r_1}^{r_2}\{C[|z|^2p^*(|z|) - p^*_0] + p^*_0\}\frac{|W_\beta|^2}{r^2}dr - \int_{r_1}^{r_2}|W'_\beta|^2 dr. \tag{3.13}$$

If we now subtract (3.12) from (3.13) we obtain

$$\frac{|W_\beta(z_1)|^2}{|z_1|}\left[\frac{|z_1|W'_{\beta^*}(|z_1|)}{W_{\beta^*}(|z_1|)} - \Re\left\{\frac{z_1W'_\beta(z_1)}{W_\beta(z_1)}\right\}\right] \geqslant \int_{r_1}^{r_2}\{C[|z|^2p^*(|z|) - p^*_0] + p^*_0 - \Re[z^2p(z)]\}\frac{|W_\beta|^2}{r^2}dr. \tag{3.14}$$

Now from (3.9) the right member of (3.14) is positive, and from the condition $p_0 = p^*_0$ we obtain the limiting relation

$$\lim_{|z_1|\to 0} \frac{|z_1| W'_{\beta^*}(|z_1|)}{W_{\beta^*}(|z_1|)} = \lim_{|z_1|\to 0} \Re\left\{\frac{z_1 W'_\beta(z_1)}{W_\beta(z_1)}\right\} = \beta^*. \tag{3.15}$$

Therefore, inasmuch as $|W_\beta(z)| \to 0$ as $z \to 0$ it follows that the left member of the inequality (3.14) approaches zero as z approaches zero along the ray θ = constant. However, the right member of (3.14) is positive and increasing as r_1 approaches zero. This contradiction leads to the conclusion that z_2 cannot be a zero of $W_\beta(z)$ and, therefore, to the proof of Theorem 3.8.

The proof of Theorem 3.8 for $W_\alpha(z)$ is identical to the above proof and is valid even when $p_0 \neq p^*_0$ as long as condition (3.9) is satisfied.

It is of interest to note that the condition (3.9) of Theorem 3.8 may be replaced by the somewhat weaker condition

$$\Re\{z^2 p(z)\} \leqq C[|z|^2 p^*(|z|) - p_0^*] + p_0^*, \ (C > 0) \tag{3.16}$$

with strict inequality holding for at least one point on every ray θ = constant, $0 < |z| < \epsilon$, for every $0 < \epsilon < R$.

The following corollary follows readily from Theorem 3.8.

COROLLARY. 3.17 *The conclusions of Theorem* 3.8 *remain valid when* $z^2 p(z) \equiv C[z^2 p^*(z) - p^*_0] + p^*_0$ *provided that* (3.9) *is valid only for all* $0 < |z| < R$, $\arg z \neq 0$.

Proof. The proof is identical to that of Theorem 3.8 on every ray except $\theta = 0$ in which case the right member of (3.14) is zero. However, since $W_\beta(z) = W_{\beta^*}(z)$ for all z we have $W_\beta(r) = W_{\beta^*}(r)$ and the conclusion of Theorem 3.8 is, therefore, valid on $\theta = 0$ also.

Theorem 3.8 and Corollary 3.17 give us criteria for the determination of the radii of zero-free regions of the solutions (1.3) and (1.4) of (1.1) in the neighbourhood of the regular singular point $z = 0$. For further results of this nature see **(1)** and **(3)**.

THEOREM 3.18. *Define* $p(z)$, $W_\beta(z)$, $p^*(z)$, *and* $W_{\beta^*}(z)$ *as in* (1.2), (1.4), (3.1), *and* (3.4) *respectively and let* $p_0 = p^*_0$, $0 < p_0 < \frac{1}{4}$. *If for all* $0 < |z| < R$

$$\Re\{z^2 p(z)\} < C[|z|^2 p^*(|z|) - p_0^*] + p_0^*, \ C > 0, \tag{3.19}$$

then

$$\frac{|W_\beta(z)|^2}{|z|}\left[\Re\left\{\frac{zW'_\beta(z)}{W_\beta(z)}\right\} - \frac{|z| W'_{\beta^*}(|z|)}{W_{\beta^*}(|z|)}\right]$$

is a non-decreasing function of $r = |z|$ *on every ray* θ = *constant for all* $0 < r < \min(\bar{\rho}_{\beta^*}, R)$, *where* $\bar{\rho}_{\beta^*}$ *is the smallest positive zero of the function* $W_{\beta^*}(\rho)$.

Proof. As in the proof of Theorem 3.8 we first obtain the Green's transform of (1.1) in the form (3.10). We then multiply equation (3.10) by $z_2 = r_2 e^{i\theta}$, replace z by $re^{i\theta}$, and integrate along the segment joining $z_1 = r_1 e^{i\theta}$ to $z_2 = r_2 e^{i\theta}$, $(z_1 \cdot z_2 \neq 0)$.

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This yields the equation

$$z_2\overline{W_\beta(z_2)}W_\beta'(z_2) - z_2\overline{W_\beta(z_1)}W_\beta'(z_1)$$
$$= r_2\int_{r_1}^{r_2} |W_\beta'|^2 dr - \int_{r_1}^{r_2} r_2 \,.\, e^{2i\theta}p(re^{i\theta})|W_\beta|^2 dr.$$

Since by Theorem 3.8 $W_\beta(z) \neq 0$ in $0 < |z| < \min(\bar{\rho}_{\beta^*}, R)$ we may take the real part of both members of this equation and write the result in the following form

$$\text{(3.20)}\quad \frac{|W_\beta(z_2)|^2}{|z_2|}\,\Re\left\{\frac{z_2W_\beta'(z_2)}{W_\beta(z_2)}\right\} - \frac{|W_\beta(z_1)|^2}{|z_1|}\,\Re\left\{\frac{z_1W_\beta'(z_1)}{W_\beta(z_1)}\right\}$$
$$= \int_{r_1}^{r_2} |W_\beta'|^2 dr - \int_{r_1}^{r_2} \Re\{z^2p(z)\}\,\frac{|W_\beta|^2}{r^2}\,dr.$$

Now if we apply Lemma 3.5 to $\Re\{W_\beta(z)\}$ and $\Im\{W_\beta(z)\}$ and add the resulting inequalities, we obtain

$$\text{(3.21)}\quad \int_{r_1}^{r_2} |W_\beta'|^2 dr - \int_{r_1}^{r_2} \{C[|z|^2p^*(|z|) - p_0^*] + p_0^*\}\,\frac{|W_\beta|^2}{r^2}\,dr$$
$$\geqslant \frac{|W_\beta(z_2)|^2}{|z_2|}\cdot\frac{|z_2|W_{\beta^*}'(|z_2|)}{W_{\beta^*}(|z_2|)} - \frac{|W_\beta(z_1)|^2}{|z_1|}\cdot\frac{|z_1|W_{\beta^*}'(|z_1|)}{W_{\beta^*}(|z_1|)}$$

where $0 < r_1 < \min(\bar{\rho}_{\beta^*}, R)$.

Thus it follows from (3.19), (3.20), and (3.21) that

$$\frac{|W_\beta(z_2)|^2}{|z_2|}\cdot\left[\Re\left\{\frac{z_2W_\beta'(z_2)}{W_\beta(z_2)}\right\} - \frac{|z_2|W_{\beta^*}'(|z_2|)}{W_{\beta^*}(|z_2|)}\right]$$
$$\geqslant \frac{|W_\beta(z_1)|^2}{|z_1|}\cdot\left[\Re\left\{\frac{z_1W_\beta'(z_1)}{W_\beta(z_1)}\right\} - \frac{|z_1|W_{\beta^*}'(|z_1|)}{W_{\beta^*}(|z_1|)}\right].$$

This proves Theorem 3.18.

COROLLARY 3.22. *The conclusions of Theorem* 3.18 *remain valid if* $z^2p(z) \equiv C[z^2p^*(z) - p^*_0] + p^*_0$ *provided that inequality* (3.19) *is valid for all* $0 < |z| < R$ *for which* $\arg z \neq 0$.

Proof. The proof follows immediately from Corollary 3.17 and the method of proof in Theorem 3.18.

We are now prepared to give sufficient conditions for the univalence of $[W_\beta(z)]^{1/\beta}$ when $0 < \beta < \frac{1}{2}$.

THEOREM 3.23. *Let* $0 < p_0 < \frac{1}{4}$ *and define* $p(z)$, $W_\beta(z)$, $p^*(z)$, *and* $W_{\beta^*}(z)$ *as in* (1.2), (1.4), (3.1), *and* (3.4) *respectively. Let* $p_0 = p^*_0$ *and*

$$\text{(3.24)}\qquad \Re\{z^2p(z)\} < C[|z|^2p^*(|z|) - p_0^*] + p_0^*,\ C > 0,$$

for all $0 < |z| < R$. *Then*

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$$\Re\left\{\frac{zW'_\beta(z)}{W_\beta(z)}\right\} \geqslant \frac{|z|W'_{\beta^*}(|z|)}{W_{\beta^*}(|z|)} \tag{3.25}$$

for all $|z| < \min(\bar{\rho}_{\beta^*}, R)$, *and* $[W_\beta(z)]^{1/\beta}$ *is univalently star-like in the circle* $|z| < \min(\rho_{\beta^*}, R)$ *where* $\bar{\rho}_{\beta^*}$, ρ_{β^*} *are the smallest positive zeros of the functions* $W_{\beta^*}(\rho)$ *and* $W'_{\beta^*}(\rho)$ *respectively.*

If, in addition,

$$\Re\{z^2p^*(z)\} < |z|^2p^*(|z|) \tag{3.26}$$

for all $0 < |z| < R$ *for which* $\arg z \neq 0$ *then*

$$\Re\left\{\frac{zW'_{\beta^*}(z)}{W_{\beta^*}(z)}\right\} \geqslant \frac{|z|W'_{\beta^*}(|z|)}{W_{\beta^*}(|z|)} \tag{3.27}$$

for all $|z| < \min(\bar{\rho}_{\beta^*}, R)$, *and the radius of univalence is sharp whenever* $\rho_{\beta^*} < R$.

Proof. Since $p_0 = p^*_0$ we obtain (3.25) directly from (3.15) and Theorem 3.18. To establish the univalence of $[W_\beta(z)]^{1/\beta}$ in $|z| < \min(\rho_{\beta^*}, R)$ we first set $F(z) \equiv [W_\beta(z)]^{1/\beta}$, $0 < \beta < \frac{1}{2}$. Then since

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} = \frac{1}{\beta}\Re\left\{\frac{zW'_\beta(z)}{W_\beta(z)}\right\}$$

it follows from (3.25) and definition 2.1 that $F(z)$ is univalently star-like for all $|z| < \min(\rho_{\beta^*}, R)$.

If we know in addition that (3.26) is valid then we may apply Corollary 3.22 and (3.15) to obtain (3.27).

The sharpness of the radius of univalence when $\rho_{\beta^*} < R$ follows by taking $z^2p(z) \equiv C[z^2p^*(z) - p^*_0] + p^*_0$ subject to condition (3.26) and noting that the derivative of $[W_{\beta^*}(z)]^{1/\beta^*}$ vanishes for $z = \rho_{\beta^*}$.

This completes the proof of Theorem 3.23.

We note here that Theorem 3.23 is the analogue of Robertson's Main Theorem **(5)** for the solution (1.4) of (1.1) in the case where $0 < p_0 < \frac{1}{4}$.

4. Univalence of $[W_\beta(z)]^{1/\beta}$, β **complex.** We shall now consider the case where β is complex and $0 < \Re\{\beta\} < \frac{1}{2}$. In this case the method of proof employed in Theorem 3.23 is not applicable since the critical inequalities

$$\Re\{z^2p(z)\} \leqslant C[|z|^2p^*(|z|) - p^*_0] + p^*_0, \quad |z| < R, \tag{4.1}$$

$$\lim_{r\to 0}\Re\left\{\frac{zW'_\beta(z)}{W_\beta(z)}\right\} \geqslant \lim_{r\to 0}\frac{|z|W'_{\beta^*}(|z|)}{W_{\beta^*}(|z|)} \tag{4.2}$$

are always incompatible throughout some neighbourhood of the origin. This follows from the fact that if $\beta = \beta_1 + i\beta_2$ and $r \to 0$ then (4.1) and (4.2) approach the limiting inequalities

$$\beta_1 - \beta_1^2 + \beta_2^2 \leqslant \beta^* - \beta^{*2} \tag{4.3}$$

$$\beta_1 \geqslant \beta^*, 0 < (\beta_1, \beta^*) < \tfrac{1}{2}, \tag{4.4}$$

which are clearly incompatible unless $\beta_2 = 0$ and $\beta_1 = \beta^*$. Indeed, this was the reason for the restriction $p_0 = p^*_0$ in Theorem 3.23. This incompatibility of (4.1) and (4.2) in the neighbourhood of the origin prevents us from obtaining the zero-free regions of the origin as we did in Theorem 3.8. We can, however, find functions $p(z)$ which satisfy (4.2) and also (4.1) for all $0 < r_1 < |z| < R$. Thus our theorem for the case where β is complex takes the following form.

THEOREM 4.5. *Let the expressions $p(z)$, $W_\beta(z)$, $p^*(z)$, ρ_{β^*}, and $W_{\beta^*}(z)$ be defined as in Theorem 3.23 and let*

$$(4.6) \quad \Re\{z^2 p(z)\} < C[|z|^2 p^*(|z|) - p_0^*] + p_0^*,$$

$$C > 0, 0 < r_1 \leqslant |z| < \min(\rho_{\beta^*}, R).$$

If there exists a ρ, $r_1 \leqslant \rho < \min(\rho_{\beta^*}, R)$, such that

$$(4.7) \quad \Re\left\{\frac{zW_\beta'(z)}{W_\beta(z)}\right\} \geqslant \rho\frac{W_{\beta^*}'(\rho)}{W_{\beta^*}(\rho)}$$

for all $|z| = \rho$, and if

$$(4.8) \quad W_\beta(z) \neq 0, \qquad 0 < |z| \leqslant \rho,$$

then it follows that $[W_\beta(z)]^{1/\beta}$ is univalent and spiral-like in $|z| < \min(\rho_{\beta^*}, R)$.

Proof. If, as in the proof of Theorem 3.8, we assume that $W_\beta(r_2 e^{i\theta}) = 0$, where $\rho < r_2 < \min(\bar{\rho}_{\beta^*}, R)$ then using Lemma 3.7 we obtain inequality (3.14) for $z = r_1 e^{i\theta}$, $\rho < r_1 < r_2$. Then allowing $z_1 \to \rho e^{i\theta}$ along the ray $\arg z_1 = \theta$ we see as in Theorem 3.8 that the left member of (3.14) becomes non-positive while the right member remains positive by (4.6). Thus $W_\beta(z)$ has no zeros in the annulus $\rho < |z| < \min(\bar{\rho}_{\beta^*}, R)$. Now by applying the method of proof of Theorem 3.18 to $W_\beta(z)$ in the annulus $\rho < |z| < \min(\bar{\rho}_{\beta^*}, R)$ we readily obtain the proof that

$$(4.9) \quad \frac{|W_\beta(z)|^2}{|z|}\cdot\left[\Re\left\{\frac{zW_\beta'(z)}{W_\beta(z)}\right\} - \frac{|z|W_{\beta^*}'(|z|)}{W_{\beta^*}(|z|)}\right]$$

is a non-decreasing function of $r = |z|$ on every ray in the annulus. Then since $\Re\{zW'_\beta(z)/W_\beta(z)\}$ is harmonic in $|z| < \min(\bar{\rho}_{\beta^*}, R)$ and by (4.7) positive on $|z| = \rho$ it must be positive for $|z| \leqslant \rho$. This result along with (4.7) and the monotonicity of (4.9) along any ray proves that

$$(4.10) \quad \Re\left\{\frac{zW_\beta'(z)}{W_\beta(z)}\right\} > 0$$

for all $|z| < \min(\rho_{\beta^*}, R)$.

We now set $F(z) \equiv [W_\beta(z)]^{1/\beta}$, where that branch of the function is chosen for which $F'(0) = 1$. Then

$$\Re\left\{\frac{\beta zF'(z)}{F(z)}\right\} = \Re\left\{\frac{zW_\beta'(z)}{W_\beta(z)}\right\}$$

and the univalence of $[W_\beta(z)]^{1/\beta}$ in $|z| < \min(\rho_{\beta^*}, R)$ follows immediately from (4.10) and Definition 2.2.

As mentioned previously, since $0 < \Re\{\beta\} < \frac{1}{2}$, we cannot allow r_1 to be zero in Theorem 4.5. However, if we replace $W_\beta(z)$, $W_{\beta^*}(z)$, and ρ_{β^*} by $W_\alpha(z)$, $W_{\alpha^*}(z)$, and ρ_{α^*} respectively then we may set $r_1 = 0$ and Theorem 4.5 yields the result on the univalence of $[W_\alpha(z)]^{1/\alpha}$ in the Main Theorem of **(5)**. Thus our Theorem 4.5 is in a sense a generalization of Robertson's Main Theorem and is applicable to either of the solutions (1.3) or (1.4) of (1.1).

It would be very desirable to replace condition (4.6) of Theorem 4.5 by a condition which would imply (4.8). This would give us a theorem of the type of Theorem 3.8 for complex β. So far as the author is aware no such result appears in the literature.

5. Applications. In **(4)** it was proved that for all $\nu > -1$ the normalized Bessel functions $z^{1-\nu}J_\nu(z)$ are univalent in $|z| < \rho^*_\nu$, where ρ^*_ν is the smallest positive zero of the function $\rho J'_\nu(\rho) + (1-\nu)J_\nu(\rho)$. In Theorem 3 of **(2)** it was shown that for all $\nu \geqslant 0$ these functions are not only univalent but are also star-like in $|z| < \rho^*_\nu$. With the aid of Theorem 3.18 we shall now prove the following theorem which extends the range of ν in Theorem 3 of **(2)** to include all $\nu > -\frac{1}{2}$.

THEOREM 5.1. *For all $-\frac{1}{2} < \nu < 0$ the normalized Bessel functions $z^{1-\nu}J_\nu(z)$ are star-like in the circle $|z| < \rho^*_\nu$, where ρ^*_ν is the smallest positive zero of the function*

$$\rho J'_\nu(\rho) + (1-\nu)J_\nu(\rho).$$

This result is sharp.

Proof. As in **(2)** we consider the differential equation

$$W''(z) + \left[1 - \frac{1}{z^2}\left(\nu^2 - \frac{1}{4}\right)\right]W(z) = 0, 0 < \nu^2 < \frac{1}{4} \tag{5.2}$$

and its solution

$$W_\nu(z) = 2^\nu\Gamma(\nu+1)z^{1/2}J_\nu(z) \tag{5.3}$$

corresponding to $-\frac{1}{2} < \nu < 0$.

Now we set $z^2p(z) \equiv z^2p^*(z) = z^2 - (\nu^2 - \frac{1}{4})$. With this definition of $z^2p(z)$ we see that $\Re\{z^2p(z)\} < |z|^2p^*(|z|)$ when and only when $r^2\cos 2\theta < r^2$, $(z = re^{i\theta})$. Thus, except for the ray $\theta = \pi$, we may apply Corollary 3.17 to prove that $W_\nu(z)$ has no zeros on any ray $\theta =$ constant $(\theta \neq \pi)$, $0 < r < \min(\bar\rho_\nu, R)$ where $\bar\rho_\nu$ is the smallest positive zero of the function $W_\nu(\rho)$. However, since $J_\nu(-z) = (-1)J_\nu(z)$, it follows immediately from (5.3) that $W_\nu(z)$ has no zeros on the ray $\theta = \pi$, $0 < r < \min(\bar\rho_\nu, R)$. Thus, as in the proof of Corollary 3.22 we obtain the inequality

$$\Re\left\{\frac{zW'_\nu(z)}{W_\nu(z)}\right\} \geqslant \frac{|z|W'_\nu(|z|)}{W_\nu(|z|)} \tag{5.4}$$

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for all $|z| < \bar{\rho}_\nu$ and hence also for all $|z|$ less than the smallest positive zero of the function $\rho^{1-\nu}J_\nu(\rho)$. If we now set $F(z) = z^{1-\nu}J_\nu(z)$ it follows directly from (5.3) and (5.4) that

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} > 0 \tag{5.5}$$

for all $|z| < \rho^*_\nu$. This proves Theorem 5.1.

The sharpness of the radius of univalence follows from the fact that $F'(\rho^*_\nu) = 0$.

With the aid of Theorem 5.1 we are now able to prove the following theorem.

THEOREM 5.5. *For all* $-\frac{1}{2} < \nu < 3/2$ *the product of Bessel functions* $J_\nu(z)J_{1-\nu}(z)$ *is univalently star-like in the circle* $|z| < \rho_\nu$ *where* $\rho_\nu = \min(\rho^*_\nu, \rho^*_{1-\nu})$.

Proof. Since $J_\nu(z)J_{1-\nu}(z) = z + \sum_{n=2}^{\infty} c_n z^n$ no further normalization is necessary. If we set $F(z) = J_\nu(z)J_{1-\nu}(z)$ we find that

$$\frac{zF'(z)}{F(z)} = \frac{zJ'_\nu(z)}{J_\nu(z)} + \frac{zJ'_{1-\nu}(z)}{J_{1-\nu}(z)}$$

and, therefore, it follows from **(5)** and Theorem 5.1 that

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} \geqslant \frac{|z|J'_\nu(|z|)}{J_\nu(|z|)} + \frac{|z|J'_{1-\nu}(|z|)}{J_{1-\nu}(|z|)} > 0$$

for all $|z| < \min(\rho^*_\nu, \rho^*_{1-\nu})$ when $-\frac{1}{2} < \nu < 3/2$. This proves Theorem 5.5.

For $\nu = \frac{1}{2}$ the result is sharp since then $\rho^*_\nu = \rho^*_{1-\nu}$ and $F'(\rho_\nu) = 0$.

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