Canad. Math. Bull. Vol. **61** (2), 2018 pp. 292–300
http://dx.doi.org/10.4153/CMB-2017-015-0

A Note on a Unicity Theorem for the Gauss Maps of Complete Minimal Surfaces in Euclidean Four-space

Dedicated to Professor Miyuki Koiso on the occation of her sixtieth birthday

Pham Hoang Ha and Yu Kawakami

Abstract. The classical result of Nevanlinna states that two nonconstant meromorphic functions on the complex plane having the same images for five distinct values must be identically equal to each other. In this paper, we give a similar uniqueness theorem for the Gauss maps of complete minimal surfaces in Euclidean four-space.

1 Introduction

The Gauss map of a complete minimal surface in Euclidean space have some properties similar to the results in value distribution theory of a meromorphic function on the complex plane **C**. One of the most notable results in this area is the Fujimoto theorem [3, Theorem I], which states that the Gauss map of a nonflat complete minimal surface in Euclidean 3-space $\mathbf{R}^3$ can omit at most four values. He also obtained the sharp estimate [3, Theorem II] for the number of exceptional values of the Gauss map of a complete minimal surface in Euclidean 4-space $\mathbf{R}^4$. Recently, the second author [11] (for $\mathbf{R}^3$) and Aiyama, Akutagawa, Imagawa, and the second author [1] (for $\mathbf{R}^4$) gave geometric interpretations of these results. Moreover, Dethloff and the first author [7] proved ramification theorems for the Gauss maps of complete minimal surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$ on annular ends. Their results extended a result of Kao [10].

Another famous result is on uniqueness and value sharing, and is called the *unicity theorem.* For meromorphic functions on **C**, Nevanlinna [14] proved that two meromorphic functions on **C** sharing five distinct values must be identically equal to each other. Here we say that two meromorphic functions (or maps) f and $\widehat{f}$ share the value α (ignoring multiplicity) when $f^{-1}(\alpha) = \widehat{f}^{-1}(\alpha)$. Fujimoto [5] obtained the following analogue of this theorem for the Gauss maps of complete minimal surfaces in $\mathbf{R}^3$.

Theorem 1.1 ([5, Theorem I]) *Let $X\colon \Sigma \to \mathbf{R}^3$ and $\widehat{X}\colon \widehat{\Sigma} \to \mathbf{R}^3$ be two nonflat minimal surfaces and let $g\colon \Sigma \to \overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and $\widehat{g}\colon \widehat{\Sigma} \to \overline{\mathbf{C}}$ be the Gauss maps of $X(\Sigma)$ and*

Received by the editors December 5, 2016; revised February 25, 2017.
Published electronically April 13, 2017.
Author Y. K. is supported by the Grant-in-Aid for Scientific Research (C), No. 15K04840, Japan Society for the Promotion of Science.
AMS subject classification: **53A10**, 30D35, 53C42.
Keywords: minimal surface, Gauss map, unicity theorem.

$\widehat{X}(\widehat{\Sigma})$, respectively. Assume that there exists a conformal diffeomorphism $\Psi: \Sigma \to \widehat{\Sigma}$ and either $X(\Sigma)$ or $\widehat{X}(\widehat{\Sigma})$ is complete. If g and $\widehat{g} \circ \Psi$ share 7 distinct values, then $g \equiv \widehat{g} \circ \Psi$.

We remark that the second author [12] gave a unified explanation for the unicity theorems of the Gauss maps of several classes of surfaces in 3-dimensional space forms including minimal surfaces in $\mathbf{R}^3$.

The purpose of this paper is to give a similar uniqueness theorem for the Gauss maps of complete minimal surfaces in $\mathbf{R}^4$. The main theorem is stated as follows.

Theorem 1.2 *Let $X: \Sigma \to \mathbf{R}^4$ and $\widehat{X}: \widehat{\Sigma} \to \mathbf{R}^4$ be two nonflat minimal surfaces, and let $G = (g_1, g_2): \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ and $\widehat{G} = (\widehat{g}_1, \widehat{g}_2): \widehat{\Sigma} \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ be the Gauss maps of $X(\Sigma)$ and $\widehat{X}(\widehat{\Sigma})$, respectively. We assume that there exists a conformal diffeomorphism $\Psi: \Sigma \to \widehat{\Sigma}$ and either $X(\Sigma)$ or $\widehat{X}(\widehat{\Sigma})$ is complete.*

(i) *Assume that $g_1, g_2, \widehat{g}_1, \widehat{g}_2$ are nonconstant, and for each i $(i = 1, 2)$, g_i and $\widehat{g}_i \circ \Psi$ share $p_i > 4$ distinct values. If $g_1 \not\equiv \widehat{g}_1 \circ \Psi$ and $g_2 \not\equiv \widehat{g}_2 \circ \Psi$, then we have*

$$\frac{1}{p_1 - 4} + \frac{1}{p_2 - 4} \geq 1. \tag{1.1}$$

In particular, if $p_1 \geq 7$ and $p_2 \geq 7$, then either $g_1 \equiv \widehat{g}_1 \circ \Psi$, or $g_2 \equiv \widehat{g}_2 \circ \Psi$, or both hold.

(ii) *Assume that $g_1, \widehat{g}_1$ are nonconstant, and g_1 and $\widehat{g}_1 \circ \Psi$ share p distinct values. If $g_1 \not\equiv \widehat{g}_1 \circ \Psi$ and $g_2 \equiv \widehat{g}_2 \circ \Psi$ is constant, then we have $p \leq 5$. In particular, if $p \geq 6$, then $G \equiv \widehat{G} \circ \Psi$.*

The paper is organized as follows. In Section 2, to reveal the geometric interpretation of Theorem 1.2, we give a unicity theorem for the holomorphic map $G = (g_1, \dots, g_n)$ into

$$(\overline{\mathbf{C}})^n := \underbrace{\overline{\mathbf{C}} \times \cdots \times \overline{\mathbf{C}}}_{n}$$

on open Riemann surfaces with the conformal metric $ds^2 = \prod_{i=1}^n (1 + |g_i|^2)^{m_i} |\omega|^2$, where ω is a holomorphic 1-form on Σ and each m_i $(i = 1, \dots, n)$ is a positive integer (Theorem 2.1). By virtue of the result, Theorem 1.2 deeply depends on the induced metric from $\mathbf{R}^4$. Moreover, we give examples (Example 2.2) that ensure that Theorem 1.2 is optimal. The proof and some remarks of Theorem 1.2 are given at the end of Section 2. Section 3 provides the proof of Theorem 2.1. The main idea of the proof is to construct some flat pseudo-metric on Σ and compare it with the Poincaré metric.

2 Main Results

To elucidate the geometric interpretation of Theorem 1.2, we give the following theorem.

Theorem 2.1 *Let Σ be an open Riemann surface with the conformal metric*

$$ds^2 = \prod_{i=1}^{n} (1 + |g_i|^2)^{m_i} |\omega|^2$$

https://doi.org/10.4153/CMB-2017-015-0 Published online by Cambridge University Press

and let $\widehat{\Sigma}$ be another open Riemann surface with the conformal metric

$$d\widehat{s}^2 = \prod_{i=1}^{n}(1+|\widehat{g}_i|^2)^{m_i}|\widehat{\omega}|^2,$$

where ω and $\widehat{\omega}$ are holomorphic 1-forms, G and $\widehat{G}$ are holomorphic maps into

$$(\overline{\mathbf{C}})^n := \underbrace{\overline{\mathbf{C}}\times\cdots\times\overline{\mathbf{C}}}_{n}$$

on Σ and $\widehat{\Sigma}$ respectively, and each m_i $(i=1,\dots,n)$ is a positive integer. We assume that there exists a conformal diffeomorphism $\Psi\colon\Sigma\to\widehat{\Sigma}$, and $g_{i_1},\dots,g_{i_k}$ and $\widehat{g}_{i_1},\dots,\widehat{g}_{i_k}$ $(1\le i_1<\cdots<i_k\le n)$ are nonconstant and the others are constant. For each i_l $(l=1,\dots,k)$, we suppose that g_{i_l} and $\widehat{g}_{i_l}\circ\Psi$ share $q_{i_l}>4$ distinct values and $g_{i_l}\not\equiv\widehat{g}_{i_l}\circ\Psi$. If either ds^2 or $d\hat{s}^2$ is complete, then we have

$$\sum_{l=1}^{k}\frac{m_{i_l}}{q_{i_l}-4}\ge 1. \tag{2.1}$$

We remark that Theorem 2.1 also holds for the case where at least one of $m_1,\dots,m_n$ is positive and the others are zeros. For example, we assume that $g := g_{i_1}$ and $\widehat{g} := \widehat{g}_{i_1}$ are nonconstant and the others are constant. If $m := m_{i_1}$ is a positive integer and the others are zeros, then the inequality (2.1) coincides with

$$\frac{m}{q-4}\ge 1 \Longleftrightarrow q\le m+4,$$

where $q := q_{i_1}$. The result corresponds with [12, Theorem 2.9].

Theorem 2.1 is optimal because of the following examples.

Example 2.2 For positive integers $m_1,\dots,m_n$ whose the sum $M := \sum_{l=1}^{k} m_{i_l}$ of the subset $\{i_1,\dots,i_k\}$ in $\{1,\dots,n\}$ is even, we take $M/2$ distinct points $\alpha_1,\dots,\alpha_{M/2}$ in $\mathbf{C}\backslash\{0,\pm1\}$. Let Σ be either the complex plane punctured at $M+1$ distinct points $0,\alpha_1,\dots,\alpha_{M/2},1/\alpha_1,\dots,1/\alpha_{M/2}$ or the universal covering of the punctured plane. We set

$$\omega = \frac{dz}{z\prod_{i=1}^{M/2}(z-\alpha_i)(\alpha_i z-1)},$$

and the map $G=(g_1,\dots,g_n)$ is given by

$$g_{i_1}=\cdots=g_{i_k}=z \quad (1\le i_1<\cdots<i_k\le n),$$

and the others are constant. In a similar manner, we set

$$\widehat{\omega}(=\omega) = \frac{dz}{z\prod_{i=1}^{M/2}(z-\alpha_i)(\alpha_i z-1)},$$

and the map $\widehat{G}=(\widehat{g}_1,\dots,\widehat{g}_n)$ is given by

$$\widehat{g}_{i_1}=\cdots=\widehat{g}_{i_k}=1/z \quad (1\le i_1<\cdots<i_k\le n),$$

and the others are constant. We can easily show that the identity map $\Psi\colon\Sigma\to\Sigma$ is a conformal diffeomorphism, and the metric $ds^2=\prod_{i=1}^{n}(1+|g_i|^2)^{m_i}|\omega|^2$ is complete. Then for each i_l, the maps g_{i_l} and $\widehat{g}_{i_l}$ $(l=1,\dots,k)$ share the $M+4$ distinct values

$0, \infty, 1, -1, \alpha_1, \ldots, \alpha_{M/2}, 1/\alpha_1, \ldots, 1/\alpha_{M/2}$ and $g_{i_l} \not\equiv \widehat{g}_{i_l} \circ \Psi$. These show that Theorem 2.1 is optimal.

We will apply Theorem 2.1 to the Gauss maps of complete minimal surfaces in $\mathbf{R}^4$. We first recall some basic facts of minimal surfaces in $\mathbf{R}^4$. For more details, we refer the reader to [2, 8, 9, 15]. Let $X = (x^1, x^2, x^3, x^4): \Sigma \to \mathbf{R}^4$ be an oriented minimal surface in $\mathbf{R}^4$. By associating a local complex coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), Σ is considered as a Riemann surface whose conformal metric is the induced metric ds^2 from $\mathbf{R}^4$. Then

$$\triangle_{ds^2} X = 0 \tag{2.2}$$

holds; that is, each coordinate function x^i is harmonic. With respect to the local coordinate z of the surface, (2.2) is given by $\bar{\partial}\partial X = 0$, where $\partial = (\partial/\partial u - \sqrt{-1}\partial/\partial v)/2$, $\bar{\partial} = (\partial/\partial u + \sqrt{-1}\partial/\partial v)/2$. Hence, each $\phi_i := \partial x^i dz$ $(i = 1, 2, 3, 4)$ is a holomorphic 1-form on Σ. If we set

$$\omega = \phi_1 - \sqrt{-1}\phi_2, \qquad g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \qquad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2},$$

then ω is a holomorphic 1-form, and g_1 and g_2 are meromorphic functions on Σ. Moreover, the holomorphic map $G := (g_1, g_2): \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ coincides with the Gauss map of $X(\Sigma)$. We remark that the Gauss map of $X(\Sigma)$ in $\mathbf{R}^4$ is the map from each point of Σ to its oriented tangent plane, the set of all oriented (tangent) planes in $\mathbf{R}^4$ is naturally identified with the quadric

$$\mathbf{Q}^2(\mathbf{C}) = \left\{ [w^1 : w^2 : w^3 : w^4] \in \mathbf{P}^3(\mathbf{C}) \,;\, (w^1)^2 + \cdots + (w^4)^2 = 0 \right\}$$

in $\mathbf{P}^3(\mathbf{C})$, and $\mathbf{Q}^2(\mathbf{C})$ is biholomorphic to the product of the Riemann spheres $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. Furthermore the induced metric from $\mathbf{R}^4$ is given by

$$ds^2 = (1 + |g_1|^2)(1 + |g_2|^2)|\omega|^2. \tag{2.3}$$

Applying Theorem 2.1 to the induced metric, we obtain Theorem 1.2.

Proof of Theorem 1.2 We first show case (i). Since $m_1 = m_2 = 1$ from (2.3), we can prove the inequality (1.1) by Theorem 2.1. Next we show case (ii). By Theorem 2.1, we obtain $1/(p-4) \geq 1$. Thus, we have $p \leq 4 + 1 = 5$. ∎

Remark 2.3 Fujimoto [6] obtained the unicity theorem for the Gauss maps $G: \Sigma \to \mathbf{P}^{m-1}(\mathbf{C})$ of complete minimal surfaces in $\mathbf{R}^m$ $(m \geq 3)$. Recently, Park and Ru [16] showed the result that is an improvement of this theorem. However, these results do not contain Theorem 1.2, because corresponding hyperplanes in $\mathbf{P}^3(\mathbf{C})$ are not necessary located in general position (for more details, see [13, p. 353]).

3 Proof of Theorem 2.1

We first recall the notion of chordal distance between two distinct points in $\overline{\mathbf{C}}$. For two distinct points $\alpha, \beta \in \overline{\mathbf{C}}$, we set

$$|\alpha, \beta| := \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2}\sqrt{1 + |\beta|^2}}$$

if $\alpha \neq \infty$ and $\beta \neq \infty$, and $|\alpha, \infty| = |\infty, \alpha| := 1/\sqrt{1+|\alpha|^2}$. We note that if we take v_1, $v_2 \in \mathbf{S}^2$ with $\alpha = \varpi(v_1)$ and $\beta = \varpi(v_2)$, we have $|\alpha, \beta|$ is half of the chordal distance between v_1 and v_2, where ϖ denotes the stereographic projection of the 2-sphere $\mathbf{S}^2$ onto $\overline{\mathbf{C}}$.

We next review the following three lemmas used in the proof of Theorem 2.1.

Lemma 3.1 ([5, Proposition 2.1]) *Let g_{i_l} and $\widehat{g}_{i_l}$ be mutually distinct nonconstant meromorphic functions on a Riemann surface Σ. Let q_{i_l} be a positive integer and $\alpha^l_1, \dots, \alpha^l_{q_{i_l}} \in \overline{\mathbf{C}}$ be distinct. Suppose that $q_{i_l} > 4$ and $g^{-1}_{i_l}(\alpha^l_j) = \widehat{g}^{-1}_{i_l}(\alpha^l_j)$ $(1 \le j \le q_{i_l})$. For $a^l_0 > 0$ and ε with $q_{i_l} - 4 > q_{i_l}\varepsilon > 0$, we set*

$$\xi_{i_l} := \Big(\prod_{j=1}^{q_{i_l}} |g_{i_l}, \alpha^l_j| \log\Big(\frac{a^l_0}{|g_{i_l}, \alpha^l_j|^2} \Big) \Big)^{-1+\varepsilon}, \quad \widehat{\xi}_{i_l} := \Big(\prod_{j=1}^{q_{i_l}} |\widehat{g}_{i_l}, \alpha^l_j| \log\Big(\frac{a^l_0}{|\widehat{g}_{i_l}, \alpha^l_j|^2} \Big) \Big)^{-1+\varepsilon},$$

and define

$$d\tau^2_{i_l} := \Big(|g_{i_l}, \widehat{g}_{i_l}|^2 \xi_{i_l} \, \widehat{\xi}_{i_l} \frac{|g'_{i_l}|}{1+|g_{i_l}|^2} \frac{|\widehat{g}'_{i_l}|}{1+|\widehat{g}_{i_l}|^2} \Big) |dz|^2 \tag{3.1}$$

outside the set $E := \bigcup_{j=1}^{q} g^{-1}_{i_l}(\alpha^l_j)$ and $d\tau^2_{i_l} = 0$ on E. Then for a suitably chosen a_0, $d\tau^2_{i_l}$ is continuous on Σ and has strictly negative curvature on the set $\{d\tau^2_{i_l} \neq 0\}$.

Lemma 3.2 ([5, Corollary 2.4]) *Let g_{i_l} and $\widehat{g}_{i_l}$ be meromorphic functions on $\triangle_R$ satisfying the same assumption as in Lemma 3.1. Then for the metric $d\tau^2$ defined by (3.1), there exists a constant $C > 0$ such that*

$$d\tau^2_{i_l} \le C \frac{R^2}{(R^2 - |z|^2)^2} |dz|^2.$$

Lemma 3.3 ([4, Lemma 1.6.7]) *Let $d\sigma^2$ be a conformal flat-metric on an open Riemann surface Σ. Then, for each point $p \in \Sigma$, there exists a local diffeomorphism Φ of a disk $\Delta_R = \{z \in \mathbf{C}; |z| < R\}$ $(0 < R \le +\infty)$ onto an open neighborhood of p with $\Phi(0) = p$ such that Φ is an isometry; that is, the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard Euclidean metric ds^2_E on Δ_R and that, for a specific point a_0 with $|a_0| = 1$, the Φ-image Γ_{a_0} of the curve $L_{a_0} = \{w := a_0 s; 0 < s < R\}$ is divergent in Σ.*

Proof of Theorem 2.1 Since the given map Ψ provides a biholomorphic isomorphism between Σ and $\widehat{\Sigma}$, we denote the function $\widehat{g}_{i_l} \circ \Psi$ by $\widehat{g}_{i_l}$ $(l = 1, \dots, k)$ for brevity. For each i_l, we assume that g_{i_l} and $\widehat{g}_{i_l}$ share the q_{i_l} distinct values $\alpha^l_1, \dots, \alpha^l_{q_{i_l}}$. After suitable Möbius transformations for g_{i_l} and $\widehat{g}_{i_l}$, we can assume that

$$\alpha^1_{q_{i_1}} = \cdots = \alpha^k_{q_{i_k}} = \infty.$$

Moreover, we assume that either ds^2 or $d\hat{s}^2$, say ds^2, is complete and $g_{i_l} \not\equiv \widehat{g}_{i_l} \circ \Psi$ for each l $(1 \le l \le k)$. Thus, for each local complex coordinate z defined on a simply connected open domain U, we can find a nonzero holomorphic function h_z such that

$$ds^2 = |h_z|^2 \prod_{i=1}^{n} (1+|g_i|^2)^{m_i/2} (1+|\widehat{g}_i|^2)^{m_i/2} |dz|^2. \tag{3.2}$$

https://doi.org/10.4153/CMB-2017-015-0 Published online by Cambridge University Press

Suppose that each $q_{i_l} > 4$ and

$$\sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 4} < 1. \tag{3.3}$$

Then by (3.3), we can suppose that $q_{i_l} > m_{i_l} + 4$ for each i_l $(l = 1, \dots, k)$. Taking some positive number η_0 with

$$0 < \eta_0 < \frac{q_{i_l} - 4 - m_{i_l}}{q_{i_l}}$$

for each i_l $(l = 1, \dots, k)$ and

$$\Lambda_0 := \sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 4 - q_{i_l}\eta_0} = 1. \tag{3.4}$$

For a positive number η with $\eta < \eta_0$, we set

$$\lambda_{i_l} := \frac{m_{i_l}}{q_{i_l} - 4 - q_{i_l}\eta} \quad (l = 1, \dots, k).$$

By (3.4) we get

$$\Lambda := \sum_{l=1}^{k} \lambda_{i_l} = \sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 4 - q_{i_l}\eta} < \sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 4 - q_{i_l}\eta_0} = \Lambda_0 = 1. \tag{3.5}$$

Now we can choose a positive number $\eta (< \eta_0)$ sufficiently near η_0 satisfying

$$\Lambda_0 - \Lambda < \min_{1 \le t \le k} \Big\{ \frac{m_{i_t}}{q_{i_t} - 4 - q_{i_t}\eta}, \frac{m_{i_t}\eta}{q_{i_t} - 4 - q_{i_t}\eta} \Big\}. \tag{3.6}$$

Using (3.5) and (3.6), we have

$$\frac{\lambda_{i_l}}{1 - \Lambda} > 1 \quad \text{and} \quad \frac{\eta\lambda_{i_l}}{1 - \Lambda} > 1 \quad (l = 1, \dots, k).$$

Now we define a new metric

$$d\sigma^2 = |h_z|^{\frac{4}{1-\Lambda}} \prod_{l=1}^{k} \left(\frac{\prod_{j=1}^{q_{i_l}-1} (|g_{i_l} - \alpha_j^l||\widehat{g}_{i_l} - \alpha_j^l|)^{1-\eta}}{|g_{i_l} - \widehat{g}_{i_l}|^2 |g'_{i_l}||\widehat{g}'_{i_l}| \prod_{j=1}^{q_{i_l}-1} (1 + |\alpha_j^l|^2)^{1-\eta}} \right)^{\frac{2\lambda_{i_l}}{1-\Lambda}} |dz|^2 \tag{3.7}$$

on the set $\Sigma' = \Sigma \setminus E$, where

$$E = \big\{ p \in \Sigma; g'_{i_l}(p) = 0, \widehat{g}'_{i_l}(p) = 0 \text{ or } g_{i_l}(p) (= \widehat{g}_{i_l}(p)) = \alpha_j^l \text{ for some } l \big\}.$$

On the other hand, setting $\varepsilon := \eta/2$, we can define another pseudo-metric $d\tau_{i_l}^2$ on Σ given by (3.1) for each l, which has strictly negative curvature on Σ'. Take a point $p \in \Sigma'$. Since the metric $d\sigma^2$ is flat on Σ', by Lemma 3.3, there exists an isometry Φ satisfying $\Phi(0) = p$ from a disk $\triangle_R = \{z \in \mathbf{C}; |z| < R\}$ $(0 < R \le +\infty)$ with the standard metric ds_E^2 on an open neighborhood of p in Σ' with the metric $d\sigma^2$. We denote the functions $g_{i_l} \circ \Phi$ and $\widehat{g}_{i_l} \circ \Phi (= \widehat{g}_{i_l} \circ \Psi \circ \Phi)$ by g_{i_l} and $\widehat{g}_{i_l}$ respectively $(l = 1, \dots, k)$ in the following. Moreover, for each i_l, the pseudo-metric $d\sigma_{i_l}^2$ on $\triangle_R$ has strictly negative curvature. Since there exists no metric with strictly negative curvature on $\mathbf{C}$ (see [4, Corollary 4.2.4]), we obtain that the radius R is finite. Furthermore, by Lemma 3.3, we can choose a point a_0 with $|a_0| = 1$ such that, for the line segment

$L_{a_0} := \{w := a_0 s; 0 < s < R\}$, the Φ-image Γ_{a_0} tends to the boundary of Σ' as s tends to R. Then Γ_{a_0} is divergent in Σ. Indeed, if not, then Γ_{a_0} must tend to a point $p_0 \in E$. Then we consider the following two possible cases:

Case 1: $g_{i_l}(p_0)(=\widehat{g}_{i_l}(p_0)) = \alpha_j^l$ for some l.

Since $g'_{i_l}(p_0) = (g_{i_l} - \alpha_j^l)'(p_0)$ and $\widehat{g}'_{i_l}(p_0) = (\widehat{g}_{i_l} - \alpha_j^l)'(p_0)$, the function

$$|h_z|^{\frac{2}{1-\Lambda}} \prod_{l=1}^{k} \left(\frac{\prod_{j=1}^{q_{i_l}-1} (|g_{i_l} - \alpha_j^l||\widehat{g}_{i_l} - \alpha_j^l|)^{1-\eta}}{|g_{i_l} - \widehat{g}_{i_l}|^2 |g'_{i_l}||\widehat{g}'_{i_l}| \prod_{j=1}^{q_{i_l}-1} (1 + |\alpha_j^l|^2)^{1-\eta}} \right)^{\frac{\lambda_{i_l}}{1-\Lambda}}$$

has a pole of order at least $2\eta\lambda_{i_l}/(1-\Lambda)$ at p_0. Taking a local complex coordinate ζ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write the metric $d\sigma^2$ as

$$d\sigma^2 = |\zeta|^{-4\eta\lambda_{i_l}/(1-\Lambda)} w |d\zeta|^2$$

with some positive function w. Since $\eta\lambda_{i_l}/(1-\Lambda) > 1$, we have

$$R = \int_{\Gamma_{a_0}} d\sigma > C_1 \int_{\Gamma_{a_0}} |d\zeta|/|\zeta|^{2\eta\lambda_{i_l}/(1-\Lambda)} = +\infty.$$

This contradicts that R is finite.

Case 2: $g'_{i_l}(p_0)\widehat{g}'_{i_l}(p_0) = 0$ for some i_l.

Without loss of generality, we may assume that $g'_{i_l}(p_0) = 0$ for some i_l. Taking a local complex coordinate $\zeta := g'_{i_l}$ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write the metric $d\sigma^2$ as

$$d\sigma^2 = |\zeta|^{-2\lambda_{i_l}/(1-\Lambda)} w |d\zeta|^2$$

with some positive function w. Since $\lambda_{i_l}/(1-\Lambda) > 1$, we have

$$R = \int_{\Gamma_{a_0}} d\sigma > C_2 \int_{\Gamma_{a_0}} |d\zeta|/|\zeta|^{\lambda_{i_l}/(1-\Lambda)} = +\infty.$$

This also contradicts that R is finite.

Since $\Phi^* d\sigma^2 = |dz|^2$, we get by (3.7)

$$|h_z|^2 = \prod_{l=1}^{k} \left(\frac{|g_{i_l} - \widehat{g}_{i_l}|^2 |g'_{i_l}||\widehat{g}'_{i_l}| \prod_{j=1}^{q_{i_l}-1} (1 + |\alpha_j^l|^2)^{1-\eta}}{\prod_{j=1}^{q_{i_l}-1} (|g_{i_l} - \alpha_j^l||\widehat{g}_{i_l} - \alpha_j^l|)^{1-\eta}} \right)^{\lambda_{i_l}}.$$

By (3.2), we have

$$\begin{aligned} \Phi^* ds &= |h_z|^2 \prod_{i=1}^{n} (1 + |g_i|^2)^{m_i/2} (1 + |\widehat{g}_i|^2)^{m_i/2} |dz|^2 \\ &\le C_3 \prod_{l=1}^{k} \left(\frac{ \begin{array}{r} |g_{i_l} - \widehat{g}_{i_l}|^2 |g'_{i_l}||\widehat{g}'_{i_l}| (1 + |g_{i_l}|^2)^{m_{i_l}/2\lambda_{i_l}} (1 + |\widehat{g}_{i_l}|^2)^{m_{i_l}/2\lambda_{i_l}} \\ \times \prod_{j=1}^{q_{i_l}-1} (1 + |\alpha_j^l|^2)^{1-\eta} \end{array} }{\prod_{j=1}^{q_{i_l}-1} (|g_{i_l} - \alpha_j^l||\widehat{g}_{i_l} - \alpha_j^l|)^{1-\eta}} \right)^{\lambda_{i_l}} |dz|^2 \\ &= C_3 \prod_{l=1}^{k} \left(\mu_{i_l}^2 \prod_{j=1}^{q_{i_l}} \Big(|g_{i_l}, \alpha_j^l||\widehat{g}_{i_l}, \alpha_j^l| \Big)^{\varepsilon} \Big(\log\Big(\frac{a_0^l}{|g_{i_l}, \alpha_j^l|} \Big) \log\Big(\frac{a_0^l}{|\widehat{g}_{i_l}, \alpha_j^l|} \Big) \Big)^{1-\varepsilon} \right)^{\lambda_{i_l}} |dz|^2, \end{aligned}$$

https://doi.org/10.4153/CMB-2017-015-0 Published online by Cambridge University Press

where μ_{i_l} is the function with $d\tau_{i_l}^2 = \mu_{i_l}^2|dz|^2$. Since the function $x^{\varepsilon}\log^{1-\varepsilon}(a_0^l/x)$ $(0 < x \le 1)$ is bounded, we obtain that

$$ds^2 \le C_4 \prod_{l=1}^{k} \Big(\frac{|g_{i_l}, \widehat{g}_{i_l}|^2 |g'_{i_l}||\widehat{g}'_{i_l}|\xi_{i_l}\widehat{\xi}_{i_l}}{(1+|g_{i_l}|^2)(1+|\widehat{g}_{i_l}|^2)} \Big)^{\lambda_{i_l}} |dz|^2$$

for some C_4. By Lemma 3.2, we have

$$ds^2 \le C_5 \prod_{l=1}^{k} \Big(\frac{R}{R^2 - |z|^2} \Big)^{\lambda_{i_l}} |dz|^2 = C_5 \Big(\frac{R^2}{R^2 - |z|^2} \Big)^{\Lambda} |dz|^2$$

for some constant C_5. Thus, we obtain that

$$\int_{\Gamma_{a_0}} ds \le (C_5)^{1/2} \int_{L_{a_0}} \Big(\frac{R^2}{R^2 - |z|^2} \Big)^{\Lambda/2} |dz| < C_6 \frac{R^{(2-\Lambda)/2}}{1 - (\Lambda/2)} < +\infty,$$

because $0 < \Lambda < 1$. However, it contradicts the assumption that the metric ds^2 is complete. ∎

Acknowledgment The authors gratefully acknowledge the useful comments of Professor Yasuhiro Nakagawa during the preparation of this paper.

References

[1] R. Aiyama, K. Akutagawa, S. Imagawa and Y. Kawakami, *Remarks on the Gauss images of complete minimal surfaces in Euclidean four-space.* Annali di Matematica (2017), to appear. http://dx.doi.org/10.1007/s10231-017-0643-6

[2] S. S. Chern, *Minimal surfaces in an Euclidean space of N dimensions.* In: Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, NJ, 1965, pp. 187–198.

[3] H. Fujimoto, *On the number of exceptional values of the Gauss maps of minimal surfaces.* J. Math. Soc. Japan **40**(1988), no. 2, 235–247. http://dx.doi.org/10.2969/jmsj/04020235

[4] ———, *Value distribution theory of the Gauss map of minimal surfaces in* $\mathbf{R}^m$. Aspects of Mathematics, E21, Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1993. http://dx.doi.org/10.1007/978-3-322-80271-2

[5] ———, *Unicity theorems for the Gauss maps of complete minimal surfaces.* J. Math. Soc. Japan **45**(1993), no. 3, 481–487. http://dx.doi.org/10.2969/jmsj/04530481

[6] ———, *Unicity theorems for the Gauss maps of complete minimal surfaces. II.* Kodai Math. J. **16**(1993), no. 3, 335–354. http://dx.doi.org/10.2996/kmj/1138039844

[7] G. Dethloff and P. H. Ha, *Ramification of the Gauss map of complete minimal surfaces in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$ *on annular ends.* Ann. Fac. Sci. Toulouse Math. **23**(2014), 829–846. http://dx.doi.org/10.5802/afst.1426

[8] D. A. Hoffman and R. Osserman, *The geometry of the generalized Gauss map.* Mem. Amer. Math. Soc. **28**(1980), no. 236. http://dx.doi.org/10.1090/memo/0236

[9] ———, *The Gauss map of surfaces in* $\mathbf{R}^3$ *and* $\mathbf{R}^4$. Proc. London Math. Soc. (3) **50**(1985), no. 1, 27–56. http://dx.doi.org/10.1112/plms/s3-50.1.27

[10] S. J. Kao, *On values of Gauss maps of complete minimal surfaces on annular ends.* Math. Ann. **291**(1991), no. 2, 315–318. http://dx.doi.org/10.1007/BF01445210

[11] Y. Kawakami, *On the maximal number of exceptional values of Gauss maps for various classes of surfaces.* Math. Z. **274**(2013), no. 3–4, 1249–1260. http://dx.doi.org/10.1007/s00209-012-1115-8

[12] ———, *Function-theoretic properties for the Gauss maps for various classes of surfaces.* Canad. J. Math. **67**(2015), no. 6, 1411–1434. http://dx.doi.org/10.4153/CJM-2015-008-5

[13] X. Mo and R. Osserman, *On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto's theorem.* J. Differential Geom. **31**(1990), no. 2, 343–355.

[14] R. Nevanlinna, *Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen.* Acta Math. **48**(1926), no 3-4, 367–391. http://dx.doi.org/10.1007/BF02565342

[15] R. Osserman, *Global properties of minimal surfaces in E^3 and E^n*. Ann. of Math. (2) **80**(1964), 340–364. http://dx.doi.org/10.2307/1970396
[16] J. Park and M. Ru, *Unicity results for Gauss maps of minimal surfaces immersed in $\mathbb{R}^m$*. J. Geom. to appear. http://dx.doi.org/10.1007/s00022-016-0353-z

Department of Mathematics, Hanoi National University of Education, 136, XuanThuy str., Hanoi, Vietnam
e-mail: ha.ph@hnue.edu.vn

Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kanazawa, 920-1192, Japan
e-mail: y-kwkami@se.kanazawa-u.ac.jp

https://doi.org/10.4153/CMB-2017-015-0 Published online by Cambridge University Press