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INTEGRAL EQUATION METHODS IN CHANGE-POINT DETECTION PROBLEMS

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In a standard formulation, a change-point detection (CPD) problem consists in detecting spontaneous changes in the distribution function of sequential random observations, at some unknown points.

In many applications, observations of a random process in discrete or continuous time are received sequentially, and, at a certain moment, random or not but unknown, some probabilistic characteristics of this process are changing. An observer should decide as quickly as possible whether change-points occur or not. Besides, the observer should not raise too many 'false alarms', that is, make decisions about detecting change-points when they are not presented.

Control charts are sequential procedures used to detect a change-point in sequentially observed data. In this thesis, we analyse cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts and their main characteristics: the average run length (ARL) and the average delay time (AD). CUSUM and EWMA are widely used in a great variety of practical applications, such as economics, finance, medicine, and engineering, to name but a few. For an introduction to CUSUM and EWMA charts with their applications see Brodsky and Darkhovsky [2], and Basseville and Nikiforov [1].

The main results of this thesis are as follows.

(1) Implementation of Fredholm type integral equations to derive analytical solutions for the ARL and AD for some classes of distribution functions. In particular, we derive a closed-form representation for the ARL of CUSUM charts when the random observations are hyperexponentially distributed;

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see Mititelu *et al.* [8]. We apply Banach's fixed point theorem to analyse the existence and uniqueness of solutions for the ARL integral equations [9].

(2) In the case of EWMA, we solve the corresponding ARL integral equation in explicit form when observations are Laplace distributed, and derive a closed-form expression for the AD [7]. Based on the same technique, we derive the solution for the corresponding integral equation of the generating function [8]. Our new result extends the recent result of Larralde [6].

(3) In practical applications it is very important to have analytical solutions, to detect change-points as quickly as possible when they occur. Analytical solutions for the ARL and AD are useful tools for practitioners to design quick and optimal EWMA and CUSUM charts. We present several numerical examples to compare the computational time and the accuracy of our closed-form solutions with Monte Carlo simulations.

(4) For some distribution functions, such as Weibull, gamma or Pareto, it is not always possible to solve the ARL integral equation in a closed form. Therefore, our results can be implemented to approximate the ARL and AD of the CUSUM procedure for these distributions.

More precisely, it is well known that any completely monotone probability density function (pdf) can be approximated by hyperexponential distributions (see [3, 4]). This covers a very large class of pdf functions; in particular, Pareto, Weibull and gamma distributions are completely monotone, and as a result can be approximated by hyperexponentials. To illustrate this, we approximate the Weibull distribution by a mixture of six exponentials with the fitting parameters obtained via the matching moments method presented in [4].

As directions for future research, we are planning to study the robustness (stability) of the approximation by hyperexponential distributions for different types of distributions such as Pareto and gamma. In a recent paper Jacobsen [5] implemented the partial eigenfunctions method and obtained the expectation of the first exit time of an autoregressive process of order one, AR(1). We are planning to apply this result to calculate the characteristics of the EWMA procedure in the case of hyperexponential distributions. Recently the Shiryayev–Roberts chart has come under increasing attention [10]. We will consider the integral equation methods approach to derive the main characteristics of the Shiryayev–Roberts chart.

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