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THE MARKOFF SPECTRUM OF AN ALGEBRAIC NUMBER FIELD

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To Kurt Mahler on his seventy-fifth birthday

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Abstract

The Markoff spectrum of an algebraic number field is defined and it is proved that the spectrum of $Q(\sqrt{5})$ is not discrete.

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Let K be a finite extension of the rational numbers $\mathbf{Q}$ find let M be a full module of K. Further, denote by $n(M)$ the least value of $|N(m)|$ as m runs over all nonzero elements of M and by $D(M)$ the discriminant of M. Put

$$\mu(M) = n(M)/\sqrt{(|D(M)|)}.$$

The number $\mu(M)$ is an invariant of the similarity class of modules containing M. In analogy to the Markoff spectrum of real indefinite quadratic forms the set of such numbers $\mu(M)$ is called the Markoff spectrum of K. The first question concerning this set of numbers is whether it possesses any finite limit points. Here we prove the following:

THEOREM. *The Markoff spectrum of* $\mathbf{Q}(\sqrt{5})$ *has at least one limit point.*

PROOF. Let $u_0 = 1$, $u_1 = 1$ and, in general, $u_{n+2} = u_{n+1} + u_n$, $n \geqslant 0$, be the Fibonacci numbers. Denote by L_n the sublattice of L of basis

$$(u_n, u_n), ((2u_{n-1}+u_n)\,w, (2u_{n-1}+u_n)\,w').$$

It suffices to prove $m(L_n) = u_n^2$, for $n \geqslant 3$, as in this case, since

$$d(L_n) = u_n(2u_{n-1}+u_n)\sqrt{5},$$

it follows that

$$\mu(L_n) = 1/(2(u_{n-1}/u_n)+1)\sqrt{5} > 1/3\sqrt{5}.$$

Since the ratios u_{n-1}/u_n are all distinct, the theorem follows.

It remains to show $m(L_n) = u_n^2$ for $n \geqslant 3$. In fact this is also true when $n = 1, 2$ but we do not need this.

The values taken by the quadratic form

$$\begin{aligned} F_n(x,y) &= (u_n x + w(2u_{n-1}+u_n)y)(u_n x + w'(2u_{n-1}+u_n)y) \\ &= u_n^2 x^2 + (u_n^2 + 2u_{n-1}u_n)xy - (2u_{n-1}+u_n)^2 y^2, \end{aligned}$$

as (x, y) ranges over all pairs of rational integers other than $(0, 0)$, are precisely the values of $x_1 x_2$ as (x_1, x_2) ranges over all points of L_n other than 0. Hence it suffices to show that the minimal value of $|F_n(x,y)|$ is u_n^2 when (x, y) ranges over all pairs of rational integers other than $(0, 0)$. We use the classical theory of reduction of binary indefinite quadratic forms to complete the proof, see for example Dickson (1930).

The standard notation for continued fractions is employed, so that $(a_1, \ldots, a_m)$ denotes the simple continued fraction

$$a_1 + 1/(a_2 + 1/(a_3 + \ldots + 1/a_n)\ldots)$$

of finite length, whereas $(a_1, \overline{a_2, \ldots, a_n})$ denotes the simple continued fraction of infinite length obtained when the block $a_2, \ldots, a_n$ is repeated a countable number of times.

LEMMA. *For* $n \geqslant 1$, *if* $\theta = (1, \overline{2, \underbrace{1, \ldots, 1}_{n-1}, 3, \underbrace{1, \ldots, 1}_{n}, 4})$ *then* θ *is a root of* $F_{n+2}(x, 1)$.

PROOF. All rational numbers appearing in this proof will be in lowest terms. For $n \geqslant 1$, we have $(\overbrace{1, \ldots, 1}^{n}) = u_n/u_{n-1}$, and so

$$(\overbrace{1, \ldots, 1}^{n}, 4) = (4u_n + u_{n-1})/(u_n + 3u_{n-1}),$$

$$(\overbrace{1, \ldots, 1}^{n-1}, 3) = (u_n + 2u_{n-1})/(2u_n - u_{n-1}),$$

and therefore also

$$z = (\overbrace{1, \ldots, 1}^{n-1}, 3, \overbrace{1, \ldots, 1}^{n}, 4) = (4u_n^2 + 10u_n u_{n-1} + 5u_{n-1}^2)/(9u_n^2 - 4u_{n-1}^2).$$

Hence

$$\begin{aligned} (2, z) &= (17u_n^2 + 20u_n u_{n-1} + 6u_{n-1}^2)/(4u_n^2 + 10u_n u_{n-1} + 5u_{n-1}^2) \\ &= (3u_{n+2}^2 + 2u_{n+2}u_{n+1} + u_{n+1}^2)/(4u_{n+1}^2 + 2u_{n+1}u_{n+2} - u_{n+2}^2). \end{aligned}$$

Next,

$$w = (\overset{\longleftarrow n-1 \longrightarrow}{1, \dots, 1}, 3, \overset{\longleftarrow n \longrightarrow}{1, \dots, 1})$$

$$= (u_n^2 + 2u_n u_{n-1} + u_{n-1}^2)/(2u_n^2 - u_{n-1}^2),$$

and so,

$$(2, w) = u_{n+2}^2/u_{n+1}^2.$$

Hence, if $\psi = (2, z, \psi)$ then ψ satisfies the equation

$$\psi = \frac{(3u_{n+2}^2 + 2u_{n+2}u_{n+1} + u_{n+1}^2)\psi + u_{n+2}^2}{(4u_{n+1}^2 + 2u_{n+1}u_{n+2} - u_{n+2}^2)\psi + u_{n+1}^2},$$

and therefore also $\theta = 1 + 1/\psi$ satisfies $F_{n+2}(\theta, 1) = 0$, which proves the lemma.

It now follows that the ordered set of integers

$$2, 1, \dots, 1, 3, 1, \dots, 1, 4$$

is a period of the form $F_{n+2}(x, y)$. Hence, applying the classical theory of reduction of indefinite binary quadratic forms, see for example Dickson (1930), it is easy to see that the required minimum value of $F_{n+2}(x, y)$ is attained for $(x, y) = (p, q)$ where p and q are relatively prime integers for which

$$p/q = (1, 2, 1, \dots, 1, 3, 1, \dots, 1).$$

As $p/q = (1, 2, w) = (u_{n+2}^2 + u_{n+1}^2)/u_{n+2}^2$, so the required minimum value is

$$|F_{n+2}(u_{n+2}^2 + u_{n+1}^2, u_{n+2}^2)| = u_{n+2}^2$$

since

$$u_{n+2}^2 + 2u_{n+2}u_{n+1} - u_{n+1}^2 = (-1)^n.$$

This completes the proof of the theorem.

References

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