Can. J. Math., Vol. XXIX, No. 5, 1977, pp. 914–927

THE COMPLETE QUOTIENT RING OF IMAGES OF SEMILOCAL PRÜFER DOMAINS

JOHN CHUCHEL AND NORMAN EGGERT

Introduction. It is well known that the complete quotient ring of a Noetherian ring coincides with its classical quotient ring, as shown in Akiba [**1**]. But in general, the structure of the complete quotient ring of a given ring is largely unknown. This paper investigates the structure of the complete quotient ring of certain Prüfer rings. Boisen and Larsen [**2**] considered conditions under which a Prüfer ring is a homomorphic image of a Prüfer domain and the properties inherited from the domain. We restrict our investigation primarily to homomorphic images of semilocal Prüfer domains. We characterize the complete quotient ring of a semilocal Prüfer domain in terms of complete quotient rings of local rings and a completion of a topological ring. Further, if the kernel of the homomorphism has an irredundant primary decomposition, we characterize the elements of the complete quotient ring.

Throughout the paper, all rings are commutative and have identity 1. If S is a multiplicatively closed set in a ring R, we let R_S be a ring of quotients of R. For S the set of regular elements of R, R_S is $Q_{cl}(R)$, the classical quotient ring of R. If S is the complement of a prime ideal P of R, R_S is also written as R_P, the localization of R to P. Among the conditions which are equivalent to R being a Prüfer domain, one which we will find particularly useful is:

A domain R is Prüfer if and only if for every proper prime ideal P of R, the localization R_P is a valuation ring (Theorem 22.1, (1), p. 276, Gilmer [**5**]).

An ideal A in a commutative ring R is *dense* if $rA = 0$ implies $r = 0$ for all $r \in R$. It follows immediately that the finite intersection of dense ideals is dense. The notation of Lambek [**6**] is used in the discussion of complete quotient rings. If R is a ring, then $Q(R)$ denotes the complete quotient ring of R. For $f \in Q(R)$ we define the domain of f or $\operatorname{dom}_R f$ to be $R:_R f$, a dense ideal in R. We will write $\operatorname{dom} f$ for $\operatorname{dom}_R f$ provided that R is clearly specified. If R is an arbitrary commutative ring, then $R \subseteq Q_{cl}(R) \subseteq Q(R)$.

Let $A_1, \ldots, A_n$ be primary ideals of a ring R and let $P_i = \sqrt{A_i}$, $i = 1, \ldots, n$. A representation $A = \bigcap A_i$ of an ideal A is said to be an *irredundant primary decomposition* if:

(i) No A_i contains the intersection of the other primary ideals, and
(ii) $P_i \neq P_j$ for $i \neq j$.

Received June 8, 1976 and in revised form, March 4, 1977.

If $x \in R$, we will write $\bar{x}$ as the image of x under the canonical map $R \to R/A$. Let R be an integral domain. For the monomorphism $R \to R_P$, e and c will represent the extension and contraction, respectively, of ideals; namely $I^e = IR_P$ and $J^c = J \cap R$.

1. Semilocal Prüfer domains. Let R be a semilocal Prüfer domain and let T be an ideal of R. Let η be the epimorphism of R onto R/T. Let $S = \{x \in R | \eta(x)$ is a regular element of $R/T\}$. By Theorem 2 in [**2**], η can be extended to an epimorphism μ from R_S to $Q_{cl}(R/T)$. The ring of quotients $D = R_S$ is an integral domain satisfying the following properties, where $A = TR_S$:

(i) D is a semilocal Prüfer domain,

(ii) A is an ideal of D where D/A is its own classical quotient ring $Q_{cl}(D/A)$, and

(iii) x is not a unit of D if and only if $\bar{x}$ is a zero divisor in D/A.

In the following we assume that the integral domain D and ideal A satisfy the above three properties.

Notice that R need not be semilocal in order that R_S be semilocal. If R is a Prüfer domain and T has an irredundant primary decomposition $\bigcap T_P$ with $P = \sqrt{T_P}$, then R_S is semilocal where $S = R \backslash \bigcup P$, and R_S, TR_S satisfy the three conditions.

Let the set of maximal ideals of D be denoted by Δ. We will establish that there is a unique smallest dense ideal in D/A.

We let v_P denote the valuation associated with the maximal ideal P of D.

THEOREM 1.1. *For each maximal ideal $P \in \Delta$ there is a prime ideal $P^* \subsetneq P$ such that whenever $x_P \in D \backslash P^*$, $P \in \Delta$, then there exists $x \in D$ such that $v_P(x) = v_P(x_P)$ for all $P \in \Delta$.*

Proof. Suppose for each $P \in \Delta$ there is a $y_P \in P$, but y_P is not in any other maximal ideal of D. We define

$$P^* = \{x \in D | v_P(x) > m \cdot v_P(y_P) \text{ for all } m \in \mathbf{Z}^+\}.$$

For each $P \in \Delta$, choose $x_P \in D \backslash P^*$. Then there is an $m \in \mathbf{Z}^+$, independent of P, such that $m \cdot v_P(y_P) > v_P(x_P)$. Let $z_P = \prod y_{P'}{}^m$, where the product is over all maximal ideals but P. Then for maximal ideals $P' \neq P$, we have $z_P \in P' \backslash P$; further, $v_P(z_P) = 0$ and $v_{P'}(z_P) = v_{P'}(y_{P'}{}^m) > v_{P'}(x_{P'})$. Let $x = \sum x_P z_P$, where the sum is over all maximal ideals. Since $v_P(x_{P'} z_{P'}) > v_P(x_P)$ for $P \neq P'$ and $v_P(x_P z_P) = v_P(x_P)$, we have $v_P(x) = v_P(x_P)$. The existence of the y_P's follows since the elements in Δ are maximal ideals. For maximal ideals $P \neq P'$, let $y_{PP'} \in P \backslash P'$. Define y_P by:

$$y_P = \sum_{P' \neq P} \left(\prod_{P'' \neq P'} y_{P''P'} \right).$$

Then y_P is in P, but in no other maximal ideals.

For a maximal ideal P, if the ideal A^{ec} is not P-primary, then $\sqrt{A^{ec}}$ is a prime ideal properly contained in P, and the prime ideal P^* of the theorem can be chosen to contain $\sqrt{A^{ec}}$.

Let B be an ideal of D containing A. Then $\bar{B}$ is a dense ideal of $\bar{D}$ if and only if $A: B = A$. If $\bar{B}$ is a dense ideal of $\bar{D}$, we say that the ideal B is A-dense of D. We note that there are no proper principal A-dense ideals of D.

LEMMA 1.2. *Let B be an A-dense ideal of D and let Δ_B be the set of maximal ideals of D which contain B. Then $B = \sqrt{B} = \bigcap \Delta_B$.*

Proof. Let P_0 be a prime ideal containing B. Then there is an ideal $P \in \Delta_B \subseteq \Delta$ which contains P_0. Suppose $x \in P \backslash P_0$ and thus $x \notin B$. By Theorem 1.1 there is a $w \in D$ such that $v_P(x) \geqq v_P(w) > 0$ and $v_{P'}(w) = 0$ for $P' \in \Delta$ and $P' \neq P$. Since the ideals of $D_{P'}$ are linearly ordered, $BD_{P'} \subseteq wD_{P'} = D_{P'}$ for $P' \neq P$ and $BD_P \subseteq P_0 D_P \subseteq wD_P \neq D_P$. Thus $B = \bigcap_\Delta B^e \subseteq \bigcap_\Delta wD^e = wD \neq D$. But since B is A-dense, so is wD, a contradiction. Thus $P_0 = P$ is a maximal ideal.

If $P \in \Delta_B$ and $BD_P \cap D \neq P$, then by Theorem 1.1, there is a $y \in D \backslash P'$ for $P' \in \Delta$ and $P' \neq P$, and $y \in P \backslash BD_P$. Again $B = \bigcap_\Delta B^{ec} \subseteq \bigcap_\Delta yD^e = yD$. Thus yD is A-dense, a contradiction. For $P \in \Delta_B$, we have $BD_P \cap D = P$, and for $P \in \Delta \backslash \Delta_B$, we have $BD_P \cap D = D$. Thus all primes containing a dense ideal B are maximal and $B = \bigcap_\Delta B^{ec} = \bigcap_{\Delta_B} P^{ec} = \bigcap \Delta_B$.

We now fix B by letting B be the intersection of all A-dense ideals P of Δ. Since Δ is a finite set, $\bar{B}$ is a dense ideal in $\bar{D}$.

COROLLARY 1.3. *The ideal B is the unique smallest A-dense ideal of D. Furthermore $Q(\bar{D}) = \mathrm{Hom}_{\bar{D}}(\bar{B}, \bar{B})$.*

Proof. By Lemma 1.2, B is the smallest A-dense ideal of D and by Corollary 3, page 97, Lambek [**6**], the complete quotient ring of D is $\mathrm{Hom}_{\bar{D}}(\bar{B}, \bar{B})$.

Next we relate the complete quotient ring of a homomorphic image of a semilocal Prüfer domain to a certain product of complete quotient rings of homomorphic images of valuation domains.

LEMMA 1.4. *Let V be a valuation domain with maximal ideal P and suppose that A is an ideal of V such that $Q_{\mathrm{cl}}(V/A) = V/A$. Then P is not A-dense if and only if there exists a $y \in V \backslash A$ such that $A = yP = \{x \in D | v(x) > v(y)\}$.*

Remark. If P is A-dense, then $Q(\bar{V}) = \mathrm{Hom}_{\bar{V}}(\bar{P}, \bar{P})$, whereas if P is not A-dense, then $Q(\bar{V}) = Q_{\mathrm{cl}}(\bar{V}) = \bar{V}$.

Proof. The property that P is not A-dense is equivalent to the existence of a $y \in V$ where $y \in (A : P) \backslash A$. Since the ideals of V are linearly ordered, we have that $A \subsetneqq yV$. But $yP \subseteq A$ and since P is a maximal ideal of V, there are no ideals properly between yP and yV. Thus $A = yP = \{x \in D | v(x) > v(y)\}$.

We next consider a necessary condition for a prime ideal of Δ to be A-dense. First suppose that $P \in \Delta$ and $P^2 \neq P$. Then, for some $x \in D$, $x \in P \backslash P^2$. By Theorem 1.1 we may assume that $v_{P'}(x) = 0$ for $P' \in \Delta$, $P' \neq P$, and $v_P(x) > 0$. Thus $xD_{P'} = D_{P'} = PD_{P'}$ for $P' \neq P$ and $xD_P = PD_P$. Hence $xD = \bigcap_\Delta xD^e = P$. If P is A-dense, then x is a unit, contrary to $x \in P$. Thus, in order that P in Δ be A-dense, we must have $P = P^2$. Of course, P need not be A-dense even though $P = P^2$.

Suppose that $P \in \Delta$ is A-dense, but PD_P is not AD_P-dense in D_P. By Lemma 1.4 there is a $w \in P$ such that $AD_P = wPD_P = \{y \in D_P | v(y) > v(w)\}$ and AD_P is a proper subset of $wD_P = AD_P : PD_P$. We denote by $A^{(P)}$ the ideal $AD_P : PD_P$ of D_P. Note that $A^{(P)} : PD_P = (AD_P : PD_P) : PD_P = A^{(P)}$ and thus PD_P is an $A^{(P)}$-dense ideal of D_P.

If $P \in \Delta$ implies that both P is A-dense and PD_P is AD_P-dense, we again define $A^{(P)} = AD_P : PD_P$, but now $A^{(P)} = AD_P$. If $P \in \Delta$ is not A-dense, let $A^{(P)} = AD_P$.

For $P \in \Delta$ let $Q^{(P)}$ be the complete ring of quotients of $D_P/A^{(P)}$. Thus if $P \in \Delta$ we have the following possibilities:

(i) $P \in \Delta$ and P is A-dense. Then PD_P is $A^{(P)} = AD_P : PD_P$-dense and the complete ring of quotients of $D_P/A^{(P)}$ is $Q^{(P)} = \text{Hom}\,(PD_P/A^{(P)}, PD_P/A^{(P)})$, as given by the remark after Lemma 1.4.

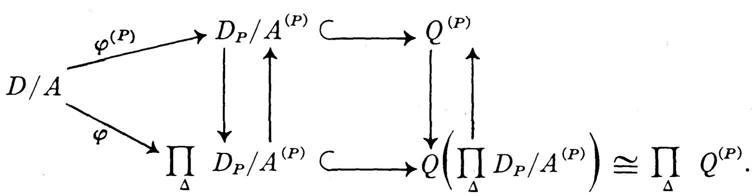
(ii) $P \in \Delta$ and P is not A-dense. Then $A^{(P)} = AD_P$ and PD_P could be $A^{(P)}$-dense. In this case we are more concerned with the factor ring $D_P/A^{(P)}$ than with the quotient ring $Q^{(P)}$.

LEMMA 1.5. $A = \bigcap_\Delta A^{(P)}$.

Proof. Since $AD_P \subseteq A^{(P)}$, we have $A \subseteq \bigcap_\Delta A^{(P)}$. Conversely, suppose $x \in (\bigcap_\Delta A^{(P)}) \backslash A$. Let $\Delta' = \{P \in \Delta | x \in A^{(P)} \backslash AD_P\}$. Since $\bigcap_\Delta AD_P = A$, Δ' is not empty. By definition of $A^{(P)}$, the maximal ideals in Δ' are A-dense, but PD_P is not AD_P-dense. Thus $C = \bigcap \Delta'$ is A-dense. By the definition of $A^{(P)}$, if $P \in \Delta'$ then $AD_P = xPD_P = \{y \in D | v_P(y) > v_P(x)\}$. Let $c \in C$. For $P \in \Delta'$, $c \in P$ and $v_P(c) > 0$; hence $v_P(cx) > v_P(x)$ and $cx \in AD_P$. Thus $Cx \subseteq \bigcap_\Delta AD_P = A$ or $x \in (A : C) \backslash A$, contrary to C being dense. Thus $\bigcap_\Delta A^{(P)} \subseteq A$.

Consider the natural homomorphism from D into $\prod_\Delta D_P/A^{(P)}$. By Lemma 1.5 the kernel of this homomorphism is A. Thus we have a monomorphism $\varphi : D/A \rightarrow \prod_\Delta D_P/A^{(P)}$.

Let $\varphi^{(P)}$ be the natural homomorphism $D/A \rightarrow D_P/A^{(P)}$. Then we have the commutative diagram:



By Proposition 8, p. 41, Lambek [**6**], $Q(\prod_\Delta D_P/A^{(P)}) \cong \prod_\Delta Q^{(P)}$. These mappings will be considered further in the next section.

Next we show that φ can be extended to a monomorphism from $Q(D/A)$ to $\prod_\Delta Q^{(P)}$. In fact, if $P \in \Delta$ is not A-dense, we can replace $Q^{(P)}$ by $D_P/A^{(P)}$.

LEMMA 1.6. *If $P \in \Delta$ and P is A-dense, then $z \in A^{(P)} \cap D$ if and only if $P \subseteq (AD_P \cap D) : z$.*

THEOREM 1.7. *Let D be a semilocal Prüfer domain with an ideal A such that $Q_{\text{cl}}(D/A) = D/A$. Then, there exists a monomorphism $\eta : Q(D/A) \to \prod_\Delta Q^{(P)}$ which extends $\varphi : D/A \to \prod_\Delta D_P/A^{(P)}$. In particular for $f \in Q(D/A)$, if $x \in B$ and $y \in D$ such that $f\bar{x} = \bar{y}$ then $\eta(f) = (f^{(P)})$, where $f^{(P)} \in Q^{(P)}$ and $f^{(P)}(x + A^{(P)}) = y + A^{(P)}$.*

Proof. Let B be the smallest A-dense ideal of D. For $f \in Q(D/A) = \mathrm{Hom}_{\bar{D}}(\bar{B}, \bar{B})$, we define $f^{(P)} \in Q^{(P)}$. If $P \in \Delta$ is not A-dense, then there is an $x \in B\backslash P$ such that $BD_P = xD_P = D_P$. We define $f^{(P)} \in D_P/A^{(P)} \subseteq Q^{(P)}$ as $f^{(P)} = z/x + A^{(P)}$, where $f\bar{x} = \bar{z}$. Notice that $f^{(P)}$ is independent of the choice of x and z.

If $P \in \Delta$ is A-dense, then $BD_P = PD_P$ is $A^{(P)}$-dense and we define $f^{(P)} \in Q^{(P)}$ by $f^{(P)}(x/y + A^{(P)}) = z/y + A^{(P)}$, where $y \in D\backslash P$, $x \in B$, and $f\bar{x} = \bar{z}$. We need to show that $f^{(P)}$ is, in fact, well-defined. Suppose $x/y + A^{(P)} = x'/y' + A^{(P)}$ for $x, x' \in B$ and $y, y' \in D\backslash P$. Then $xy' - x'y \in A^{(P)} \cap D$. By Lemma 1.6, if $w \in P$ then $w(xy' - x'y) \in AD_P \cap D$ and there is a $t \in D\backslash P$, where $wt(xy' - x'y) \in A$. Applying f, where $f\bar{x} = \bar{z}$ and $f\bar{x}' = \bar{z}'$, we obtain $wt(zy' - z'y) \in A$, or $w(zy' - z'y) \in AD_P \cap D$. Since this holds for all $w \in P$, $P \subseteq (AD_P \cap D) : (zy' - z'y)$ and by Lemma 1.6,

$$zy' - z'y \in A^{(P)} \cap D \quad \text{or} \quad z/y + A^{(P)} = z'/y' + A^{(P)}.$$

Thus $f^{(P)}$ is well-defined and clearly a $D_P/A^{(P)}$ endomorphism of $PD_P + A^{(P)}$; that is, $f^{(P)} \in Q^{(P)}$.

We define $\eta : Q(D/A) \to \prod_\Delta Q^{(P)}$ by $\eta(f)(P) = f^{(P)}$; η is a homomorphism which extends φ. Further, η is a monomorphism. Suppose $\eta(f) = 0$. Then for all $P \in \Delta$, $f^{(P)} = 0$. Let $x \in B$ and $f\bar{x} = \bar{z}$. Since $f^{(P)}(x + A^{(P)}) = A^{(P)}$, $z \in A^{(P)}$ for all $P \in \Delta$. That is, $z \in A$ and f is the zero element of $Q(D/A)$.

2. Irredundant primary decomposition. We next characterize $Q(V/A)$, where V is a valuation ring with maximal ideal P and A is a P-primary ideal not equal to P. The result is then used to characterize $Q(D/A)$, where D is a semilocal Prüfer domain with maximal ideals P and A has an irredundant primary decomposition $\bigcap A_P$, with $P = \sqrt{A_P}$.

Let V be a valuation ring with associated valuation v and with ideals P and A as described above. The intersection of all P-primary ideals of V is a prime ideal P_0 of V, and associated with this prime ideal is a rank one subgroup of the value group of V. By Proposition 5.15, p. 110 of [**7**], this subgroup is isomorphic to an additive subgroup of the real numbers. Thus the group $H = \{\pm v(x) | x \in V \backslash P_0\}$ will be considered a subgroup of the real numbers. Let $s = \text{glb}(v(A) \cap H) \in \mathbf{R}$. Since $A \neq P$, we have $s > 0$. Either $A = \{x \in V | v(x) > r\}$ or $A = \{x \in V | v(x) \geqq r\}$ with $r = s$. If H is discrete we write $A = \{x \in V | v(x) > r\}$, where r is the immediate predecessor of s. If $s \notin H$, we write $A = \{x \in V | v(x) \geqq r\}$, where $r = s$. By Lemma 1.4 and the above convention, P is A-dense if and only if $A = \{x \in V | v(x) \geqq r\}$.

Notice that if $x, x' \notin A$ and $\bar{x} = \bar{x}'$ in V/A, then $v(x) = v(x')$.

LEMMA 2.1. *Let $f \in Q(V/A)$ and if $\bar{x} \in \text{dom } f$, let $\bar{y} = f\bar{x}$, where x and y are preimages of $\bar{x}$ and $\bar{y}$. Then, if $v(y) < v(x)$, $x \in A$.*

Proof. The domain of f is either V/A or P/A. If $v(y) < v(x)$, then $x \in yV$ and there is a $b \in V$ such that $x = by$. Hence $b \in P$, since $v(x) = v(b) + v(y)$ and $v(b) > 0$. By the remark after Lemma 1.4, $f\bar{b} \in P/A$ and there is a $c \in P$ where $\bar{c} = f\bar{b}$. Then there is a $q \in \mathbf{Z}^+$ such that $q \cdot v(c) = v(c^q) > r$, and $c^q \in A$. Thus in V/A, $(f\bar{b})^q = \bar{0}$ and $f\bar{b}$ is nilpotent. Since $x = by$, we have

$$f\bar{x} = f\overline{by} = f\bar{x}f\bar{b} \quad \text{and} \quad f\bar{x}(\bar{1} - f\bar{b}) = \bar{0}.$$

With $\bar{1} - f\bar{b}$ a unit in V/A, we must have $\bar{y} = f\bar{x} = \bar{0}$ in V/A or $y \in A$. Hence $x \in A$.

Equivalently, $v(y) \geqq v(x)$ for each $x \in P \backslash A \neq \emptyset$, where $\bar{y} = f\bar{x}$.

LEMMA 2.2. *For $f \in Q(V/A)$, let $\bar{x}, \overline{x^*} \in \text{dom } f$, where $\bar{y} = f\bar{x}$ and $\overline{y^*} = f\overline{x^*}$. If $x, x^*, y, y^* \notin A$, then $v(y^*) - v(x^*) = v(y) - v(x)$.*

Proof. By Lemma 2.1 above, $v(y^*) - v(x^*) \geqq 0$ and $v(y) - v(x) \geqq 0$. Without loss of generality we may assume that $v(x) \leqq v(x^*)$. Then there is a $c \in D$ such that $x^* = cx$. But $\overline{y^*} = f\bar{c}\bar{x} = \bar{c}f\bar{x} = \bar{c}\bar{y}$. Since y^* and $cy \notin A$, $v(y^*) = v(cy)$. Hence $v(y^*) - v(x^*) = v(y) + v(c) - (v(x) + v(c)) = v(y) - v(x)$. Therefore $v(y^*) - v(x^*) = v(y) - v(x) \geqq 0$, showing the invariance of this difference for all pairs $\langle \bar{x}, \bar{y} \rangle$ associated with f, where $x, y \notin A$.

For each $f \in Q(V/A)$, $f \neq \bar{0}$, let $v[f]$ be the above constant $v(y) - v(x)$ associated with f. We come now to a fundamental theorem characterizing the elements of $Q(V/A)$.

THEOREM 2.3. *Let V be a valuation ring with valuation v and maximal ideal P and let A be a P-primary ideal of V. If P is A-dense and $f \in Q(V/A)$, $f \neq \bar{0}$, then there exist $N \in \mathbf{Z}^+$, $r \in \mathbf{R}^+$, $a \in V$, and $\langle b_j \rangle_{j \geqq N}$ in V such that b_j is a unit in V, $1/N < r - v(a)$, $v(b_j - 1) > r - v(a) - 1/j$, and $f\bar{x} = (\prod_{j=N}^{i-1} \bar{b}_j)\bar{a}\bar{x}$ whenever $v(x) > 1/i$, $i \geqq N$.*

Proof. We have previously shown that P is A-dense if and only if $A = \{x \in V | v(x) \geqq r\}$. This establishes the existence of $r \in \mathbf{R}^+$. By the definition of $v[f]$, for $f \neq 0$, $f \in Q(V/A)$, we note that $r - v[f] > 0$. There is an element $a \in V \backslash A$ such that $v(a) = v[f]$. Since the subgroup H of the value group of V

is not discrete, there is a sequence of elements $\langle x_k \rangle$ in V such that $0 < v(x_k) < 1/k$ and $v(x_{k+1}) < v(x_k)$, where $x_k \notin A$. The integer N is chosen to satisfy $1/N < r - v(a)$ in order that neither x_k nor y_k be elements of A for $k \geqq N$, where $f\bar{x}_k = \bar{y}_k$. Then for $k \geqq N$, $v(y_k) = v(ax_k)$ since $v(y_k) - v(x_k) = v[f] = v(a)$. Hence there is a unit u_k in V such that $y_k = u_k a x_k$.

Let $u = u_N$ and define the unit b_j to be the product $u_{j+1}u_j^{-1}$ for $j \geqq N$. Then $u_k = u \prod_{j=N}^{k-1} b_j$ for $k \geqq N + 1$. Further for $j \geqq N$, $v(b_j - 1) = v(u_{j+1}u_j^{-1} - 1) = v(u_{j+1} - u_j)$ since $v(u_j) = 0$. Since $v(x_{j+1}) < v(x_j)$, there is a z_{j+1} in V such that $x_j = z_{j+1}x_{j+1}$. Hence $\bar{u}_j\bar{a}\bar{x}_j = f\bar{x}_j = f\bar{z}_{j+1}\bar{x}_{j+1} = \bar{u}_{j+1}\bar{a}\bar{x}_j$, and $(u_{j+1} - u_j)ax_j \in A$. Thus

$$\begin{aligned} v(b_j - 1) &= v(u_{j+1} - u_j) \\ &\geqq r - v(a) - v(x_j) \\ &> r - v(a) - 1/j \end{aligned}$$

for $j \geqq N$.

By our choice of N, this latter result shows that $b_j - 1 \in P$ for each $j \geqq N$.

For arbitrary $\bar{x} \in \bar{P} \subseteq \operatorname{dom} f$, pick $i \geqq N$ such that $v(x) > 1/i$. Then $v(x) > 1/i > v(x_i)$, so that there exists $s_i \in V$ such that $x = s_i x_i$. Hence

$$\begin{aligned} f\bar{x} &= f\overline{s_i x_i} \\ &= \bar{s}_i \cdot f\bar{x}_i \\ &= \bar{s}_i \cdot \bar{u}_i\bar{a}\bar{x}_i \\ &= \bar{s}_i \cdot \bar{u}\left(\prod_{j=N}^{i-1} \bar{b}_j\right)\bar{a}\bar{x}_i \\ &= \bar{u}\left(\prod_{j=N}^{i-1} \bar{b}_j\right)\bar{a}\bar{x}. \end{aligned}$$

Without loss of generality we may replace ua by a, since u is a unit in V. Thus $f\bar{x} = (\prod_{j=N}^{i-1} \bar{b}_j)\bar{a}\bar{x}$ is the desired representation.

If f is the zero element in $Q(V/A)$, then $f\bar{x} = \bar{a}\bar{x}$ where $a \in A$.

The converse of Theorem 2.3 is also true.

THEOREM 2.4. *Let V be a valuation ring with valuation v and maximal ideal P and let A be a P-primary ideal of V, where either $A = \{x \in V | v(x) > r\}$ or $A = \{x \in V | v(x) \geqq r\}$. Let an element $a \in V$ and a sequence $\langle b_j \rangle_{j \geqq N}$ of units in V be given such that $v(b_j - 1) > r - v(a) - 1/j$, where $j \geqq N$ and $1/N < r - v(a)$. If $v(x) > 1/i$, $i \in \mathbf{Z}^+$ and $i > N$, define $f\bar{x} = (\prod_{j=N}^{i-1} \bar{b}_j)\bar{a}\bar{x}$. Then $f \in Q(V/A)$.*

Proof. First we show that f is well-defined. For s a positive integer, if $v(x) > 1/i$, then $v(x) > 1/(i + s)$. Hence we must show that

$$\left(\prod_{j=N}^{i-1} \bar{b}_j\right)\bar{a}\bar{x} = \left(\prod_{j=N}^{i+s-1} \bar{b}_j\right)\bar{a}\bar{x}.$$

First

$$1 - \prod_{j=i}^{i+s-1} b_j = (1 - b_i) + \sum_{j=i+1}^{i+s-1} b_i \ldots b_{j-1}(1 - b_j)$$

and since the minimum valuation that any summand can have is greater than $r - v(a) - 1/i$,

$$v\left(1 - \prod_{j=i}^{i+s-1} b_j\right) > r - v(a) - \frac{1}{i}.$$

Thus

$$v\left[\left(\prod_{j=N}^{i-1} b_j\right)ax - \left(\prod_{j=N}^{i+s-1} b_j\right)ax\right]$$
$$= v(a) + v(x) + v\left(\prod_{j=N}^{i-1} b_j\right) + v\left(1 - \prod_{j=i}^{i+s-1} b_j\right)$$
$$> v(a) + \frac{1}{i} + 0 + r - v(a) - \frac{1}{i} = r.$$

Finally $(\prod_{j=N}^{i-1} b_j)ax - (\prod_{j=N}^{i+s-1} b_j)ax \in A$ and the above equality is established.

Next, suppose that $v(x_1) > 1/i_1$ and $v(x_2) > 1/i_2$ where $i_1, i_2 \in \mathbf{Z}^+$. Let $i = \max \{i_1, i_2\}$. Then

$$f(\bar{x}_1 + \bar{x}_2) = \left(\prod_{j=N}^{i-1} \bar{b}_j\right)\bar{a}(\bar{x}_1 + \bar{x}_2)$$
$$= f\bar{x}_1 + f\bar{x}_2.$$

Let $v(x) > 1/i$ where $i \in \mathbf{Z}^+$ and let z be an arbitrary element of V. Then it follows that $f(\bar{z}\bar{x}) = \bar{z}(f\bar{x})$. Therefore $f \in Q(V/A)$.

By Theorem 1.7, $Q(D/A)$ is isomorphic to a subring of $\prod Q^{(P)}$. We now proceed to show that the subring is $\prod Q^{(P)}$ itself when A has an irredundant primary decomposition.

LEMMA 2.5. *Let D be a semilocal Prüfer domain with maximal prime ideals P. Let A be an ideal of D such that $Q_{cl}(D/A) = D/A$. If A has an irredundant primary decomposition, $A = \bigcap A_P$, where each A_P is P-primary, then D/A is isomorphic to $\prod D_P/A^{(P)}$.*

Proof. Let A have an irredundant primary decomposition $\bigcap A_P$ with maximal primes P, as stated above. Then the A_P are pairwise comaximal and, since D is a Prüfer domain, D satisfies the Chinese Remainder Theorem (pp. 307-10, Gilmer [**5**]).

Let $(a_P/b_P + A^{(P)})$ be an element in $\prod D_P/A^{(P)}$ where $a_P \in D$ and $b_P \in D\backslash P$. Applying the Chinese Remainder Theorem, let a be a solution of $x \equiv a_P \pmod{A_P}$ and let b be a solution of $x \equiv b_P \pmod{A_P}$, so that $b \in D\backslash \bigcup P$. Then $a/b \equiv a_P/b_P \pmod{A^{(P)}}$. Thus b is a unit in D and $ab^{-1} \in D$. Hence the natural map $D \to \prod D_P/A^{(P)}$ is a surjection with kernel $\bigcap A^{(P)} = A$; that is, D/A is isomorphic to $\neq D_P/A^{(P)}$.

The monomorphism η described in Theorem 1.7 is an isomorphism in the setting of this section. Namely, if $f \in Q(\bar{D})$, then $\eta(f) = (f^{(P)}) \in \prod Q^{(P)}$, where for $x \in B$, $y \in D$ and $f\bar{x} = \bar{y}$, $f^{(P)}(x + A^{(P)}) = y + A^{(P)}$. To verify that η is onto $\prod Q^{(P)}$, let $f^{(P)} \in Q^{(P)}$ for $P \in \Delta$. Let $x \in B$; then $f^{(P)}(x + A^{(P)}) = a_P/b_P + A^{(P)} \in D_P/A^{(P)}$. But by the above Lemma 2.5 there is a $\bar{y} \in D/A$ such that $\varphi(y) = (a_P/b_P + A^{(P)})$. Now define $f \in Q(D/A)$ by $f\bar{x} = \bar{y}$; f is well defined, for if $y' \equiv y \equiv a_P/b_P \pmod{A^{(P)}}$ for all $P \in \Delta$, then $y' \equiv y$ $(\text{mod} \bigcap A^{(P)} = A)$. Since $f^{(P)}$ is a D homomorphism of $D_P/A^{(P)}$, and $A = \bigcap A^{(P)}$, f is a D homomorphism of D/A and thus also a D/A homomorphism. Thus f is an element in $Q(D/A)$. Further, $\eta(f) = (f^{(P)})$. Thus we have established the following:

THEOREM 2.6. *Let D be a semilocal Prüfer domain with maximal prime ideals P. Let A be an ideal of D such that $Q_{c1}(D/A) = D/A$. If A has an irredundant primary decomposition, $A = \bigcap A_P$, where each A_P is P-primary, then $Q(D/A)$ is isomorphic to $\prod Q^{(P)}$.*

Let $f \in Q(\bar{D})$ and let $P \in \Delta$ be A-dense. Then it follows from Theorem 2.3 that for any $f^{(P)} \in Q^{(P)}$, there is a sequence $\langle a_i/b_i \rangle$, $i \geqq N$, and an element c/d in D_P, where $c \in D$ and $a_i, b_i, d \in D \backslash P$. If $P \in \Delta$ is not A-dense, we let $a_i = b_i = 1$ and $c = z$, $d = x$ as discussed above. Notice that the positive integer N can be chosen the same for all of the elements in Δ. Furthermore, the elements a_i, b_i, c and d need only be defined modulo $A^{(P)} \cap D = A_P$. Since the Chinese Remainder Theorem holds in D and the A_P are pairwise comaximal, the a_i, b_i, c and d can be chosen the same for all the maximal ideals P in D. Hence the elements a_i, b_i and d are units in D, $a_i b_i^{-1}$ is a unit in D and $cd^{-1} \in D$. Replacing $a_i b_i^{-1}$ by a_i and cd^{-1} by c for $f \in Q(\bar{D})$, we can establish the following theorem.

THEOREM 2.7. *Given $f \in Q(\bar{D})$ there is a sequence $\langle a_i \rangle_{i \geqq N}$ of units in D and an element $c \in D$ such that for $x \in B$, $f\bar{x} = \prod_{i=N}^{N'-1} \bar{a}_i \bar{c}\bar{x}$ where $v_P(x) > 1/N'$ for all A-dense ideals P. Furthermore, if P is A-dense, then $A_P = \{x \in D | v_P(x) \geqq r_P\}$ and either*

(i) $v_P(a_i - 1) > r_P - v_P(c) - 1/i$ *for* $r_P \in \mathbf{R}$, *or*

(ii) $c \in A_P$.

If P is not A-dense, then $a_i - 1 \in A_P$.

Proof. From above, for all $P \in \Delta$ and $x \in B$, $f^{(P)}(x + A^{(P)}) = \prod_{i=N}^{N'-1} (a_i cx + A^{(P)})$. Thus if $f\bar{x} = \bar{y}$, then $y - \prod_{i=N}^{N'-1} a_i cx \in A^{(P)}$ for all $P \in \Delta$ and $f\bar{x} = \prod_{i=N}^{N'-1} \bar{a}_i \bar{c}\bar{x}$.

If P is A-dense and $f^{(P)} = 0$ in $Q^{(P)}$, then $c \in A^{(P)} \cap D = A_P$. If P is A-dense and $f^{(P)} \neq 0$, then since our choices of a_i and c are congruent modulo

$A^{(P)}$ to the elements of Theorem 2.3, we have $v_P(a_i - 1) > r_P - v_P(c) - 1/i$. If P is not A-dense, then a_i is congruent to 1 modulo A_P and thus $a_i - 1 \in A_P$.

Finally we show that the converse of Theorem 2.7 is valid, thus completely describing the elements of the complete quotient ring of D/A in terms of certain products.

THEOREM 2.8. *Let D, Δ, A and $r_P \in \mathbf{R}$ be as above. Let $N \in \mathbf{Z}^+$ and $c \in D$ be given. Let $\langle a_i \rangle_{i \geqq N}$ be a sequence of units in D such that if $P \in \Delta$ is A-dense, then either*

(i) $v_P(a_i - 1) > r_P - v_P(c) - 1/i, \quad i > N$, *or*

(ii) $c \in A_P$,

and if $P \in \Delta$ is not A-dense, then $a_i - 1 \in A_P$. If P is A-dense and $x \in B$, where $v_P(x) > 1/N'$, $N' \in \mathbf{Z}^+$ and $N' > N$, define $f\bar{x} = \prod_{i=N}^{N'-1} \bar{a}_i \bar{c} \bar{x}'$. Then $f \in Q(D)$.

The proof is similar to that of Theorem 2.4.

3. Topological closure. Let D be a semilocal Prüfer domain and let D/A be its own classical quotient ring. We construct a topology in D/A and demonstrate that the completion of the topological ring is $Q(D/A)$. Let v_P be the valuation associated with $P \in \Delta$, $v_P : Q_{\mathrm{cl}}(D) \rightarrow G_P$, where we assume the map is onto the ordered group G_P. If P is A-dense and $\xi \in G_P^+$, let

$$U_\xi^{(P)} = \{\bar{x} \in D_P/A^{(P)} | \text{there is a } y \in Q_{\mathrm{cl}}(D) \text{ such that } \bar{y} = \bar{x} \text{ and } v_P(y) > \xi\}$$

and let G_P' be the set of all $\xi \in G_P^+$ for which $U_\xi^{(P)} \neq \{\bar{0}\}$. If P is not A-dense, we define $U_\xi^{(P)}$ as above but let $G_P' = G_P^+$. For P A-dense, we have $\bigcap_{\xi \in G_P'} U_\xi^{(P)} = \{\bar{0}\}$ since $\bar{x} \notin U_{v_P(x)}^{(P)}$ for $x \in D_P \backslash A^{(P)}$. And for P not A-dense, we have $\bigcap U_\xi^{(P)} = \{\bar{0}\}$. By Chapter III, 6.3, Bourbaki [**4**], the set $U^{(P)} = \{U_\xi^{(P)} | \xi \in G_P'\}$ forms a filter basis of neighborhoods of the origin for $D_P/A^{(P)}$. Furthermore this topology is Hausdorff (page 223, ibid.).

Let $\xi \in \mathbf{X}_\Delta G_P' = G'$ and define U_ξ to be

$$\varphi^{-1}(\mathbf{X}_\Delta U_{\xi(P)}^{(P)}) = \{\bar{x} \in D/A | v_P(X) > \xi(P) \text{ for all } P\}$$

(see the definition of φ after Lemma 1.5). Then $\{U_\xi | \xi \in G'\}$ is a filter base of neighborhoods of the origin for a Hausdorff topology of D/A and φ is continuous.

By Proposition 6, page 278, ibid., there is a complete Hausdorff ring T such that D/A is a dense subring of T. We now proceed to show that T is isomorphic to $Q(D/A)$.

Let $\{\bar{z}_\alpha\}_{\mathscr{A}}$ be a Cauchy net in D/A. For $x \in B$, we define $\xi \in G'$ as follows:

(i) if P is A-dense then, since xD_P is not $A^{(P)}$-dense, there exists $w_P \in (A^{(P)} : xD_P) \backslash A^{(P)}$; then let $\xi(P) = v_P(w_P)$.

(ii) if P is not A-dense, let $\xi(P) \in v_P^{-1}(A^{(P)})$.

Since $\{\bar{z}_\alpha\}_{\mathscr{A}}$ is a Cauchy net, there is an $\alpha \in \mathscr{A}$ such that if $\beta, \gamma > \alpha$, then $v_P(z_\beta - z_\gamma) > \xi(P)$ for all $P \in \Delta$. From above, $z_\beta x - z_\gamma x \in A^{(P)}$ for all $P \in \Delta$. Hence $\overline{z_\beta x} = \overline{z_\gamma x}$ in D/A. That is, for each $x \in B$, $\{\overline{z_\alpha x}\}$ is a Cauchy net in D/A which is eventually constant and converges to an element in D/A. By defining $f\bar{x}$ to be this limit, it follows that f is a well-defined map from $\bar{B}$ to $\bar{B}$. Since f is a D/A homomorphism, f is an element of $Q(D/A)$.

We now define μ: $T \to Q$ by considering $z \in T$. Then z is the limit of a Cauchy net $\{\bar{z}_\alpha\}_{\mathscr{A}}$ in D/A and corresponds to an element $f \in Q$ as above. The element f is seen to be independent of the choice of Cauchy nets. Thus μ is a function from T to Q. Since sums and products in T correspond to sums and products in Q, μ is a homomorphism.

Let $z \in T$ and suppose that $z \in \ker \mu$; then z is the limit of a Cauchy net $\{\bar{z}_\alpha\}_{\mathscr{A}}$ in D/A. Let $\xi \in G'$. For fixed $P \in \Delta$, where P is A-dense, there exists $x \in P\backslash A$ such that $v_P(x) = \xi(P)$. Since $xP \not\subseteq A$, there is a $y \in B\backslash A$ such that $xy \in P\backslash A$. Since $z \in \ker \mu$ and since $\{\overline{z_\alpha y}\}$ is eventually constant, there is an $\alpha_P \in \mathscr{A}$ such that if $\beta > \alpha_P$, then $z_\beta y \in A$. Thus $v_P(z_\beta y) > v_P(yx)$ and

$$v_P(z_\beta) = v_P(z_\beta y) - v_P(yx) + v_P(x) > v_P(x) = \xi(P).$$

Now for fixed $P \in \Delta$, where P is not A-dense, by Theorem 1.1 there is a $y \in B$ such that $v_P(y) = 0$. Again, since $z \in \ker \mu$, there is an $\alpha_P \in \mathscr{A}$ such that for $\beta > \alpha_P$ we have $z_\beta y \in A \subseteq A^{(P)}$. Thus, since $\xi(P) \in G_P' = G_P^+$, $v_P(z_\beta) = v_P(z_\beta y) > \xi(P)$. Since Δ is finite, we can find $\alpha \in \mathscr{A}$ independent of P such that if $\beta > \alpha$, then $v_P(z_\beta) > \xi(P)$ for all $P \in \Delta$. Thus $\{\bar{z}_\alpha\}_{\mathscr{A}}$ converges to $\bar{0} \in D/A \subseteq T$ and μ is a monomorphism.

In the following, we show that μ is onto. We now let the index set $\mathscr{A}$ be the collection of elements in B which

(i) are not in P if P is not A-dense,

(ii) are not in $P^* \cup A$ if P is A-dense,

where P^* is defined in Theorem 1.1. By Theorem 1.1, $\mathscr{A}$ is not empty. We order the elements in $\mathscr{A}$ by defining, for each pair x, y in $\mathscr{A}$, $x \leqq y$ if and only if $v_P(x) \geqq v_P(y)$ for all $P \in \Delta$. Note that if P is not A-dense, then $v_P(x) = 0$. The inequality "$\leqq$" on $\mathscr{A}$ is a partial ordering and $\mathscr{A}$ becomes a directed set under "$\leqq$" by Theorem 1.1.

Let $x \in \mathscr{A}$ and let $f\bar{x} = \bar{y}$. We now show that $y/x \in D$. Assume the opposite. Then $y/x \notin D_P$ for some $P \in \Delta$. But if P is not A-dense, then x is a unit in D_P and $y/x \in D_P$. If P is A-dense and $y/x \notin D_P$, then $b = x/y \in PD_P$ and $\bar{b} \in \operatorname{dom} f^{(P)}$, since D_P is a valuation domain. Thus in $D_P/A^{(P)}$,

$$\bar{y} = f^{(P)}\bar{x} = f^{(P)}\bar{b}\bar{y} = \bar{y}f^{(P)}\bar{b}.$$

Let $f^{(P)}\bar{b} = \bar{c}$, $c \in D_P$; then $y(1 - c) \in A^{(P)}$. If c is not a unit, then $1 - c$ is a unit and $y \in A^{(P)}$, contrary to $y/x \notin D_P$. If c is a unit, then without loss of generality there is a $d \in PD_P$, where $f^{(P)}\bar{d} = \bar{1}$. Since $P^2 = P$, there exist u,

$v \in PD_P$ such that $d = uv$ and $\bar{u}f^{(P)}\bar{v} = f^{(P)}\bar{d} = \bar{1}$. Thus u is a unit in D_P, contrary to $u \in PD_P$. Therefore $y/x \in D_P$ for all $P \in \Delta$ and $y/x \in D$.

For convenience we write the elements of the index set $\mathscr{A}$ as Greek letters, $\alpha \in \mathscr{A}$, and let $x_\alpha \in D$ be α itself. For $\alpha \in \mathscr{A}$ let $f\bar{x}_\alpha = \bar{y}_\alpha$ and let $z_\alpha = y_\alpha/x_\alpha \in D$. Next we show that $\{\bar{z}_\alpha\}_{\mathscr{A}}$ is a Cauchy net in D/A and thus converges to an element z in T. By the definition of μ: $T \to Q(D/A)$, $\mu(z) = f$ and consequently μ is onto.

Let $f \in Q(D/A)$ and $f\bar{x}_\alpha = \bar{y}_\alpha$ for $\alpha \in \mathscr{A}$. Suppose $\xi \in G'$ and $P \in \Delta$. We must consider two cases. If P is A-dense, let $w \in PD_P \backslash A^{(P)}$, where $v_P(w) = \xi(P)$. Then there is a $u \in PD_P$ such that $uw \in wPD_P\backslash A^{(P)}$ by Lemma 1.4 (since $A^{(P)}$ is a proper subset of wPD_P). By Theorem 1.1, let γ be such that $v_P(x_\gamma) \leqq v_P(u)$. For α, $\beta > \gamma$, we assume without loss of generality that $v_P(x_\alpha) \leqq v_P(x_\beta) \leqq v_P(u)$. Then there exists $b \in D_P$ such that $bx_\alpha = x_\beta$ and $b = c/d$, where $c \in D$ and $d \in D\backslash P$. Hence

$$\bar{c}\bar{y}_\alpha = f\bar{c}\bar{x}_\alpha = f\bar{d}\bar{x}_\beta = \bar{d}\bar{y}_\beta$$

or

$$a = by_\alpha - y_\beta \in A^{(P)}.$$

And

$$\begin{aligned} v_P(y_\alpha/x_\alpha - y_\beta/x_\beta) &= v_P(a/x_\beta) \\ &\geqq v_P(a/u) \\ &= v_P(aw/uw) = v_P(a/uw) + v_P(w) \\ &> v_P(w) = \xi(P). \end{aligned}$$

Thus $y_\alpha/x_\alpha - y_\beta/x_\beta \in U_{\xi(P)}{}^{(P)}$ for $\alpha, \beta > \gamma$.

On the other hand, if P is not A-dense, then for any $\alpha, \beta \in \mathscr{A}$, $x_\alpha, x_\beta \in B\backslash P$. We can again find $b \in D_P$ and $a \in A^{(P)}$, where $bx_\alpha = x_\beta$ and $a = by_\alpha - y_\beta \in A^{(P)}$. Hence

$$v_P(y_\alpha/x_\alpha - y_\beta/x_\beta) = v_P(a/x_\beta) = v_P(a)$$

and $y_\alpha/x_\alpha - y_\beta/x_\beta \in A^{(P)} \subseteq U_{\xi(P)}{}^{(P)}$.

Thus we can find a γ independent of P such that $y_\alpha/x_\alpha - y_\beta/x_\beta \in U_\xi$ for all $\alpha, \beta > \gamma$. Therefore $\{\overline{y_\alpha/x_\alpha}\}_{\mathscr{A}}$ is a Cauchy net in D/A and hence converges to an element $z \in T$, and z maps onto $f \in Q(D/A)$.

The above establishes the following:

THEOREM 3.1. *Let D be a semilocal Prüfer domain with ideal A and let D/A be its own classical quotient ring. Then the valuations on D define a Hausdorff topology on D/A, the completion of which is the complete quotient ring of D/A.*

4. An example. We now give an example of a homomorphic image of a Prüfer domain R which has a non-trivial complete quotient ring. The following is adapted from Example (6), pp. 390–1, Bourbaki [**3**]:

https://doi.org/10.4153/CJM-1977-092-x Published online by Cambridge University Press

Let Γ be the real numbers $\mathbf{R}$ and let F be an arbitrary field. Let $\Gamma_+ = \mathbf{R}^+ \cup \{0\}$ and let C be the semigroup algebra of Γ_+ over F. Then C is a domain and an F-algebra with basis $\langle x_\alpha \rangle_{\alpha \in \Gamma_+}$ and multiplication given by $x_\alpha x_\beta = x_{\alpha+\beta}$. Note that $x_0 = 1$. Let K be the quotient field of C. Define v: $K \rightarrow \Gamma \cup \{\infty\}$ by

$$v\left(\sum_\alpha a_\alpha x_\alpha \Big/ \sum_\beta b_\beta x_\beta\right) = \min_{a_\alpha \neq 0} (\alpha) - \min_{b_\beta \neq 0} (\beta),$$

where $v(0) = \infty$. Let $R = \{p | p \in K \text{ and } v(p) \geqq 0\}$, R being the valuation ring of v. Then R is a local Prüfer domain with maximal ideal $M = \{p \in R | v(p) > 0\}$. Let $A = \{p \in R | v(p) \geqq 1\}$; A is an M-primary ideal of R and $\bar{M}$ is the only dense ideal in R/A. Since units are the only regular elements in R/A, R/A is its own classical quotient ring.

In Theorem 2.4, let $a = 1$, $N = 2$ and choose the sequence of units $\langle b_j \rangle_{j=2}^\infty$ in R where $b_j - 1 = x_{1-1/(j+1)} = c_j$. Note that $v(c_j) = 1 - 1/(j+1)$. By Theorem 2.4, an element $f \neq \bar{0}$ in $Q(R/A)$ is determined such that

$$f\bar{x} = \prod_{j=2}^{n-1} \overline{(1 + c_j)}\bar{x} = \bar{y}$$

for $x \in \operatorname{dom} f = \bar{M}$ and $v(x) > 1/n$. Now

$$\prod_{j=2}^{n-1} \overline{1 + c_j} = \bar{1} + \sum_{j=2}^{n-1} \bar{c}_j = \bar{1} + \sum_{j-2}^{n-1} \bar{x}_{j/(j+1)},$$

since the valuation of a product of two or more of the c_j's is greater than 1. Hence for $\bar{x} \in \operatorname{dom} f$, where $v(x) > 1/n$, we obtain

$$f\bar{x} = \left(\bar{1} + \sum_{j=2}^{n-1} \bar{x}_{j/(j+1)}\right)\bar{x} = \left(\bar{1} + \sum_{j=2}^{n-1} \bar{c}_j\right)\bar{x}.$$

Suppose that f is in the classical quotient ring of R/A; that is, for some $e \in R$, $f = \bar{e} \in R/A$. Thus for $x \in R$ with $v(x) > 1/n$,

$$f\bar{x} = \overline{ex} \quad \text{or} \quad \left(\bar{1} + \sum_{j=2}^{n-1} \bar{c}_j - \bar{e}\right)\bar{x} = \bar{0}.$$

By definition of the valuation v, we can choose a positive integer L such that if $k \geqq L$ then

$$t = v\left(1 + \sum_{j=2}^{k-1} c_j - e\right) = v\left(1 + \sum_{j=2}^{L-1} c_j - e\right).$$

If $t < 1$, let s be a positive integer such that $s > L$ and $t < 1 - 1/s$. For $x = x_{1/(s+1)}$, $f\bar{x} = \overline{ex}$ and

$$1 \leqq v\left(\left(1 + \sum_{j=2}^{L-1} c_j - e\right)x\right) = t + 1/(s+1) < 1 - 1/s + 1/(s+1) < 1,$$

a contradiction. If $t \geqq 1$, then $\bar{e} = \bar{1} + \sum_{j=2}^{L-1} \bar{c}_j$. For $x = x_{1/(L+2)}$, we obtain for $n = L + 1$,

$$\left(\bar{1} + \sum_{j=2}^{L-1} \bar{c}_j\right)\bar{x} = \bar{e}\bar{x} = f\bar{x} = \left(\bar{1} + \sum_{j=2}^{L} \bar{c}_j\right)\bar{x}.$$

Thus

$$\bar{0} = \left(\bar{1} + \sum_{j=2}^{L} \bar{c}_j\right)\bar{x} - \left(\bar{1} + \sum_{j=2}^{L-1} \bar{c}_j\right)\bar{x}$$

$$= \bar{x}_{L/(L+1)} \cdot \bar{x}_{1/(L+2)}$$

$$= \bar{x}_{1-1/(L+1)(L+2)} = \bar{c}_{(L+1)(L+2)} \neq 0,$$

a contradiction.

Hence the assumption that f is in R/A leads to a contradiction and R/A is not its own classical quotient ring.

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Eastern Montana College,
Billings, Montana;
Montana State University,
Bozeman, Montana