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MULTIPLICATION MODULES

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All rings R considered here are commutative with identity and all the modules are unital right modules. As defined by Mehdi [6] a module M_R is said to be a multiplication module if for every pair of submodules K and N of M, $K \subset N$ implies $K = NA$ for some ideal A of R. This concept generalizes the well known concept of a multiplication ring. A module M_R is said to be a generalized multiplication module if for every pair of proper submodules K and N of M, $K \subset N$ implies $K = NA$ for some ideal A of R. The quasi-cyclic group Z_{p^∞} is a generalized multiplication module which is not a multiplication module. Another example is given at the end of this note. The purpose of this note is to find the structure of a faithful generalized multiplication module over a noetherian domain; the desired structure is given in Theorems (2.4) and (3.6).

1. **Preliminaries.** A module is said to be uniserial if it has a unique composition series. Since any artinian principal ideal ring is a direct sum of special primary rings, by Nakayama [8], we have:

LEMMA (1.1). *Any module over an artinian principal ideal ring is a direct sum of uniserial modules.*

Mehdi [6, Theorem 4] showed that any faithful multiplication module M over a quasi-local ring R, is isomorphic to R_R and R is a multiplication ring. Now any artinian ring is the direct sum of finitely many local, artinian rings and any local artinian, multiplication ring, is a special primary ring [2]. Further every special primary ring is self-injective. This gives the following.

LEMMA (1.2). *Any multiplication module over an artinian ring is a direct sum of finitely many uniserial modules. Further if M is a faithful multiplication module over a quasi-local ring R, and if R is not a domain, then M is uniserial and injective.*

Thus any finite length multiplication module over a quasi-local ring, is quasi-injective. For definition and some elementary properties of quasi-injective modules we refer to [4]. For any module M over a ring R, $E_R(M)$ (or simply $E(M)$) will denote the injective hull of M.

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2. **Torsion free generalized multiplication modules.** The following lemma is obvious:

LEMMA (2.1). *Let M be a faithful generalized multiplication module over a noetherian ring R. Then*

(I) *Either M is finitely generated or every proper submodule of M is finitely generated and small in M.*

(II) *If $R=R_1\oplus R_2$, then M is finitely generated.*

LEMMA (2.2). *If M is a generalized multiplication module over a domain D, such that M is not a torsion free module, then M is a torsion module.*

Proof. Let N be the torsion submodule of M. Now $N\neq 0$ and M/N is a torsion free module. So if $M/N\neq 0$, we can find a proper submodule T/N of M/N. Then $N=TA$ for some non-zero ideal A of D. That gives T is a torsion submodule of M and hence $N=T$. This is a contradiction. This proves that M is a torsion module.

LEMMA (2.3). *If M is torsion free generalized multiplication module over a domain D, then D is a Dedekind domain and M is a uniform D-module.*

Proof. As M is torsion free, D_D is embeddable in M. So D_D is a multiplication module, and hence D is a Dedekind domain.

If M is not uniform, we can find two non-zero submodules A and B of M such that $A\cap B=0$ and $A\oplus B<M$, then for some ideal C of D, $A=(A+B)C$; which is not possible. This proves that M is uniform.

THEOREM (2.4). *If M is a torsion free generalized multiplication module over a domain D which is not a field, then either M is a multiplication module isomorphic to an ideal of D, or M is isomorphic to the total quotient field Q of D and D is a discrete valuation ring of rank one.*

Proof. By (2.3), D is a Dedekind domain. Thus, if M is finitely generated, then by (2.3) M being uniform, M is isomorphic to an ideal of D, and M is a multiplication module. So let M be not finitely generated. We can regard $D\subset M\subset Q$.

Let $M\neq Q$. Then M is not divisible as D-module, so for some $a\neq 0$, $Ma\neq M$. This gives Ma is finitely generated. Then $M\cong Ma$ further gives M is finitely generated. This is a contradiction. Hence $M=Q$. Suppose D is not a discrete valuation ring. Consider any prime ideal $P\neq 0$ of D, then $D<D_P<M=Q$. This gives D_P is a finite D-module; this is a contradiction. Hence D is a discrete valuation ring. This proves the theorem.

3. **Torsion generalized multiplication modules.** It was proved in [2] that any indecomposable multiplication ring is either a Dedekind domain or a special primary ring. Its immediate consequence is:

LEMMA (3.1). *Any noetherian multiplication ring is a direct sum of Dedekind domains and special primary rings.*

Henceforth throughout all the lemmas, M is a faithful, torsion, generalized multiplication module over a noetherian domain R. Clearly then M is not finitely generated and is indecomposable.

LEMMA (3.2). *For* $0 \neq x \in M$, $xR = \oplus \Sigma x_i R$ *such that for each* i, $R/\text{ann}(x_i)$ *is either a special primary ring or a Dedekind domain, which is not a field (so in the later case* $\text{ann}(x_i)$ *is a non-maximal prime ideal).*

Proof. $xR \cong R/\text{ann}(x)$ gives $R/\text{ann}(x)$ is a noetherian multiplication ring. The rest now follows from (3.1).

LEMMA (3.3). *N, the set of those elements x in M such that xR is a direct sum of uniserial modules, is a submodule of M.*

Proof. Since every special primary ring S is uniserial as S-module, it follows from (3.2) that $x \in N$ if and only if $R/\text{ann}(x)$ is artinian. So for any $x, y \in N$, $r \in R$, $\text{ann}(x) \cap \text{ann}(y) \subset \text{ann}(x-y)$, $\text{ann}(x) \subset \text{ann}(xr)$ imply $R/\text{ann}(x-y)$ and $R/\text{ann}(xr)$ are artinian, and hence $x-y \in N$, $xr \in N$. This proves that N is a submodule of M.

LEMMA (3.4). *N', the set consisting of 0 and all those $x \in M$ for which $R/\text{ann}(x)$ is a direct sum of Dedekind domains, none of which is a field, is a submodule of M.*

Proof. Let P be a non-maximal prime ideal of R such that for some $x \in M$, $xR \cong R/P$ Let

$$M_{(P)} = \{y \in M : yP = 0\}$$

Then $M_{(P)}$ is a finitely generated multiplication module over R/P, such that $M_{(P)}$ is not a torsion R/P-module. Consequently by (2.2) and (2.3) $M_{(P)}$ is a torsion free uniform R/P-module.

Consider all above types of $M_{(P)}$ and let $T = \Sigma_P M_{(P)}$. We show that this sum is direct and that $N' = T$. Since $M_{(P)}$ is a torsion free R/P-module, for any $y \neq 0$ in $M_{(P)}$, $\text{ann}_R(y) = P$.

Let $M_{(P)} \cap (\sum_{P' \neq P} M_{(P')}) \neq 0$. We can find $y(\neq 0) \in M_{(P)}$ such that $y = y_1 + y_2 + \cdots + y_n$, $y_i \neq 0$ and there exist distinct non-maximal prime ideals $P_1, P_2, \ldots, P_n$ all different from P, with $y_i \in M_{(P_i)}$. Then $P_1 P_2 \cdots P_n \subset P$ gives $P_i \subset P$ for some i. As R/P_i is a Dedekind domain and P/P_i is a non-maximal prime ideal of R/P_i, we get $P = P_i$. This is a contradiction. Thus $T = \oplus \sum_P M_{(P)}$.

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Clearly $N' \subset T < M$. Consider $0 \neq y \in T$. Then $yR = TA$ for some ideal A of R. Therefore $yR = \oplus \sum_P M_{(P)}A$. If for any P, $M_{(P)}A \neq 0$ then it being a homomorphic image of yR, is cyclic. So if $M_{(P)}A = y_P R$, then $\operatorname{ann}(y_P) = P$. Therefore $yR = \oplus \sum_P y_P R$, gives $y \in N'$. This completes the proof.

LEMMA (3.5). *There exists a maximal ideal P of R such that for each* $x \in M$, $xP^n = 0$ for some n.

Proof. In the notations of (3.3) and (3.4), $M = N + N'$, By (2.1) $M = N$ or $M = N'$.

CASE I. $M = N' = \oplus \sum_P M_{(P)}$ gives $M = M_{(P)}$. Hence for some $x \neq 0$ in M, $xR \cong R/P$ and also $MP = 0$. This gives $P = 0$, and that M is a torsion free R-module. This is a contradiction. Hence this case is not possible.

CASE II. $M = N$. Here given $x(\neq 0) \in M$, $xR = \oplus \sum_{i=1}^{t} x_i R$, with $R/\operatorname{ann}(x_i)$ a special primary ring. So there exists a maximal ideal P_i such that $x_i P_i^{n_i} = 0$ for some n_i. Thus if for each maximal ideal P of R, for which, for some $x \in M$, $\operatorname{ann}(x) = P$, we define $M_P = \{x \in M, xP^n = 0 \text{ for some } n\}$, then M_P is a submodule of M and on similar lines as in Case I, $M = \oplus \sum M_P$. This gives $M = M_P$. Hence the result follows.

THEOREM (3.6). *Let M be a faithful torsion generalized multiplication module over a noetherian domain R. Then M has an infinite properly ascending chain of submodules:*

$$0 = x_0 R < x_1 R < \cdots < x_n R \cdots < M$$

such that $x_i R / x_{i-1} R\,(i \geq 1)$ are simple, mutually isomorphic, and $x_i R$ are the only submodules of M different from M. Further more R is embeddable in a complete discrete valuation ring S such that M can be made into an S-module with the property that M is an injective S-module.

Proof. By (3.5) there exists a maximal ideal P of R such that for each $x \in M$, $xP^n = 0$ for some n. Thus given x and $y \in M$, $xR + yR$ is a multiplication module over P/P^n for some n. So by (1.2) $xR + yR$ is uniserial. Hence $xR \subset yR$ or $yR \subset xR$ and each xR is of finite length. Further if $xR + yR = zR \cong R/A$ for some ideal A then R/A is a special primary ring with maximal ideal P/A, hence all composition factor of $xR + yR$ are isomorphic to R/P. This proves the first part.

Consider $E = E_R(M)$. By Matlis [5, Theorem (3.6)] $E = E_R(R/P)$ is an $\hat{R}_P$-module, where $\hat{R}_P$ is the P-adic completion of R_P. Further by Matlis [5, Theorem (3.7)] $\hat{R}_P = \operatorname{Hom}_R(E, E)$. Since each $x_n R$ is quasi-injective by (1.2), using Johnson and Wong [4], we get that each $x_n R$ is an $\hat{R}_P$-submodule of E. Hence M itself is an $\hat{R}_P$-submodule of E. Hence by Johnson and Wong [4], M is a quasi-injective $\hat{R}_P$-module. Consider A, the annihilator of M in $\hat{R}_P$. Then

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$S=\hat{R}_P/A$ is a complete local ring and R is embeddable in S. Further M is a quasi-injective uniform, S-module; each x_nR is an S-module. For each $n\geq 1$, let

$$A_n=\{s\in S: x_ns=0\}$$

Then $x_nA_n=0$. The maximal ideal N of S is $P\hat{R}_P/A$. By 9, Chap. VIII, Theorem 13, $N^2\supset A_n$ for some n. However by (1.2) S/A_n is a special primary ring. Thus S/N^2 is special primary ring and hence N/N^2 is a simple S-module. This implies N is principal, and S is a complete discrete valuation ring. However every infinite length torsion, uniform, module over a Dedekind domain is always injective we get N is injective as an S-module. This proves the result.

REMARK. It follows from the above proof that if R is a complete local domain, admitting a faithful, torsion generalized multiplication module M, then R is a discrete valuation ring and M is an injective R-module. If K is the quotient field of R, then M is isomorphic to K/R. Now any indecomposable module over a complete discrete valuation ring R, is isomorphic to K, R, K/R or $R/(p^n)$, *where* K is the quotient field of R, and (p) is the maximal ideal of R [3, p. 53]. Using this we get the following from (1.2), (2.4), and (3.6).

THEOREM. *Generalized multiplications modules over a complete discrete valuation ring R are precisely the indecomposable modules over R.*

We end this paper by giving an example of a uniform finite length generalized multiplication module over a local ring, which is not uniserial.

EXAMPLE. Let R be any local ring with maximal ideal W such that $W^2=0$ and composition length $l(W)=2$, Then $W=x_1R\oplus x_2R$. Consider $M=(R/x_1R\oplus R/x_2R)/D$ where $D=\{(\bar{x}_2r,-\bar{x}_1r): r\in R$. Then M is a uniform R-module of length 3, its proper submodules are isomorphic to R/x_1R, R/x_2R and R/W. Since each of these modules is uniserial and hence a multiplication module, we get M is a generalized multiplication module. It can be easily verified that M is uniform, but M is not uniserial.

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