The Modified Bessel Function $K_n(z)$.

By Dr T. M. MacRobert.

(*Read and Received 13th February 1920*).

Gray and Mathews, in their treatise on Bessel Functions, define the function $K_n(z)$ to be

$$\frac{2^n \sqrt{\pi}\, z^{-n}}{\Gamma(\frac{1}{2}-n)} \int_0^\infty \frac{\cos(z \sinh\phi)\, d\phi}{\cosh^{2n}\phi}.$$

We shall denote this function by $V_n(z)$. This definition only holds when z is real, and $R(n) \gtreqless 0$. The asymptotic expansion of the function is also given; but the proof, which is said to be troublesome and not very satisfactory, is omitted. Basset (*Proc. Camb. Phil. Soc.*, Vol. 6) gives a similar definition of the function.

The simplest definition of $K_n(z)$ is *

$$K_n(z) = \frac{\pi}{2 \sin n\pi} \{I_{-n}(z) - I_n(z)\} = i^n G_n(iz).$$

This paper contains two alternative proofs of the relation

$$V_n(z) = \cos n\pi\, K_n(z),$$

and two corresponding methods of establishing the asymptotic expansion of $K_n(z)$. A short discussion of the relation between this function and Whittaker's Function $W_{k,m}(z)$ is also given.

First Method.

If $0 < R(z) < 1$, then

$$\int_0^\infty \cos t \,.\, t^{z-1}\, dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right).$$

Hence

$$\int_{C_1} \cos\zeta \,.\, \zeta^{z-1}\, d\zeta = \left(e^{2\pi i z} - 1\right) \Gamma(z) \cos\left(\frac{\pi z}{2}\right), \quad \text{..........(1)}$$

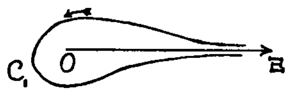

Fig. 1

* *Cf.* Macdonald, *Proc. London Math. Soc.*, XXX.

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where C_1 is the contour of Fig. 1, and amp $\zeta = 0$ initially. This relation holds if $R(z) < 1$.

Let $n < 0$, and let x be real; then if C is the contour of Fig. 2, deformed, if necessary, so that $|\zeta| > 1$ for all points on the contour,

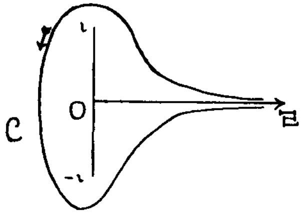

Fig. 2

$$I \equiv x^n \int_C \cos(x\zeta)(\zeta^2+1)^{n-\frac{1}{2}}\, d\zeta = x^{n-1}\int_C \cos\zeta\left(\frac{\zeta^2}{x^2}+1\right)^{n-\frac{1}{2}} d\zeta.$$

In the latter integral $|x/\zeta| < 1$ for all points on the contour. It can be shown (*Bromwich*, *Inf. Ser.*, §176, B) that it is permissible to expand the binomial in descending powers of ζ and then integrate term by term.

It follows, by means of equation (1), that

$$I = \left(e^{4\pi i n} - 1\right)\cos n\pi\, \Gamma(2n)\, 2^{-n}\, \Gamma(-n+1)\, I_{-n}(x)$$

$$= \left(e^{4\pi i n} - 1\right)\cot n\pi\, \sqrt{\pi}\, 2^{n-1}\, \Gamma(n+\tfrac{1}{2})\, I_{-n}(x).$$

Now, let $n > -\frac{1}{2}$, and let C be deformed into the contour of Fig. 3; then the integrals round the circles vanish with the circles, so that

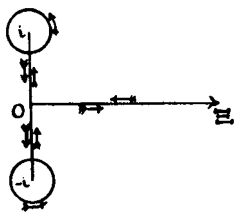

Fig. 3

$$I = \left(e^{4\pi i n} - 1\right) x^n \int_0^\infty \cos(x\xi)(\xi^2+1)^{n-\frac{1}{2}}\, d\xi$$

$$+ \left(e^{2\pi i n} + 1\right)^2 i x^n \int_0^1 \cosh(x\eta)(1-\eta^2)^{n-\frac{1}{2}}\, d\eta.$$

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Now divide by $\left(e^{4\pi in}-1\right)$; then

$$\cot n\pi \ \sqrt{\pi}\, 2^{n-1}\,\Gamma(n+\tfrac{1}{2})\, I_{-n}(x)$$
$$= x^n \int_0^\infty \cos(x\xi)\,(\xi^2+1)^{n-\frac{1}{2}}\, d\xi + \cot n\pi \ \sqrt{\pi}\, 2^{n-1}\,\Gamma(n+\tfrac{1}{2})\, I_n(x).$$

Again,* $I_n(x)$ is holomorphic in n if $x \neq 0$; hence, if $n \not\equiv 0$,

$$K_n(x) = \frac{\pi}{2\sin n\pi}\{I_{-n}(z) - I_n(z)\}$$
$$= \frac{\sqrt{\pi}}{\cos n\pi\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \int_0^\infty \cos(x\xi)\,(\xi^2+1)^{n-\frac{1}{2}}\, d\xi.$$

Therefore, if $n \not\equiv 0$,

$$\cos n\pi\, K_n(x) = \cos n\pi\, K_{-n}(x)$$
$$= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)}\left(\frac{x}{2}\right)^{-n} \int_0^\infty \frac{\cos(x\xi)}{(\xi^2+1)^{n+\frac{1}{2}}}\, d\xi = V_n(x).$$

There is a factor $\cos n\pi$ in the definitions of $K_n(z)$ given by Gray and Mathews and by Whittaker and Watson. This factor is better omitted, as it makes the function vanish when n is half an odd integer.

The Asymptotic Expansion.† — Assume that x is real and positive, and let

$$I \equiv x^n \int_{C_2} e^{-x\zeta}(\zeta^2-1)^{n-\frac{1}{2}}\, d\zeta = x^{-n}\int_{C_2} e^{-\zeta}\,\zeta^{2n-1}\left(1-\frac{x^2}{\zeta^2}\right)^{n-\frac{1}{2}} d\zeta,$$

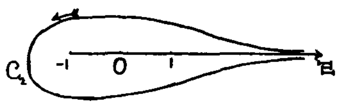

Fig. 4

where amp $(\zeta^2-1)=0$ initially, and C_2 is the contour of Fig. 4, chosen so that $|\zeta|>1$. Now expand the binomial expression in descending powers of ζ, and integrate term by term. Thus

$$I = \left(e^{4\pi in}-1\right)\Gamma(2n)\Gamma(1-n)\,2^{-n}\, I_{-n}(x)$$
$$= i\left(e^{3\pi in}+e^{\pi in}\right)2^n\,\sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})\, I_{-n}(x).$$

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* *Cf.* MacRobert's ***Functions of a Complex Variable***, p. 239.

† *Cf.* **Whittaker and Watson**, ***Analysis***, p. 376.

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Now let C_2 be deformed into the contour of Fig. 5; then, if $R(n+\frac{1}{2}) > 0$,

Fig. 5

$$I = \left(e^{4\pi in} - 1\right) x^n \int_1^\infty e^{-x\xi} (\xi^2 - 1)^{n-\frac{1}{2}} d\xi$$

$$+ i\left(e^{3\pi in} + e^{\pi in}\right) 2x^n \int_0^1 \cosh(x\xi)(1-\xi^2)^{n-\frac{1}{2}} d\xi$$

$$= \left(e^{4\pi in} - 1\right) x^n \int_1^\infty e^{-x\xi} (\xi^2 - 1)^{n-} \, d\xi$$

$$+ i\left(e^{3\pi in} + e^{\pi in}\right) 2^n \sqrt{\pi}\, \Gamma(n+\tfrac{1}{2}) I_n(x).$$

Hence

$$K_n(x) = \frac{\sqrt{\pi}}{2^n} \frac{1}{\Gamma(n+\frac{1}{2})} x^n e^{-x} \int_0^\infty e^{-x\xi} \xi^{n-\frac{1}{2}} (2+\xi)^{n-\frac{1}{2}} d\xi$$

$$= \sqrt{\left(\frac{\pi}{2x}\right)} \frac{1}{\Gamma(n+\frac{1}{2})} e^{-x} \int_0^\infty e^{-\xi} \xi^{n-\frac{1}{2}} \left(1 + \frac{\xi}{2x}\right)^{n-\frac{1}{2}} d\xi.$$

Since both sides of the equation are holomorphic for

$$-\pi < \operatorname{amp} x < \pi, \; x \neq 0,$$

this formula for $K_n(x)$ holds at all points in that region.

Let the binomial expression be expanded in descending powers of x; then

$$K_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)} e^{-z} \left\{ 1 + \frac{4n^2-1}{\lfloor 1\,(8z)} + \frac{(4n^2-1)(4n^2-3^2)}{\lfloor 2\,(8z)^2} + \ldots\ldots + R_s \right\}.$$

In a paper read at a recent meeting of the Society, Professor Gibson has shown, by a very simple method, that if $s + \frac{1}{2} > R = (n)$

$$|R_s| < \frac{1}{k} \left| \frac{\Gamma(n+\frac{1}{2}+s)}{\Gamma(s+1)\,\Gamma(n+\frac{1}{2}-s)} \right| \frac{1}{|2z|^s},$$

where

$k = 1$ if $-\pi/2 \leqq \operatorname{amp} z \leqq \pi/2$,

$$k = \left|\sin\phi\right|^{s+\frac{1}{2}-\alpha} e^{-\pi|\beta|} \text{ if } \begin{cases} \pi/2 \leqq \phi = \operatorname{amp} z < \pi \\ -\pi/2 \geqq \phi \qquad\qquad > -\pi, \end{cases}$$

$(n = \alpha + i\beta)$.

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Since $K_{-n}(z)=K_n(z)$, the expansion also holds when n is negative.

The asymptotic expansions of $G_n(z)$, $J_n(z)$, and $I_n(z)$ follow as corollaries from that of $K_n(z)$.

Second Method.

The equation

$$z w'' + (n+1) w' + w = 0$$

has solutions

$$C_n(z) \text{ and } z^{-n} C_{-n}(z),$$

where

$$C_n(z) = z^{-\frac{1}{2}n} J_n(2\sqrt{z}).$$

The function* $C_n(z)$ has recently been discussed by Sir George Greenhill (*Phil. Mag.*, XXXVIII.). Some of its properties are simpler than those of the Bessel Functions; for example, for all values of n,

$$C'_n(z) = -C_{n+1}(z).$$

The equation is a particular case of Laplace's Linear Differential Equation; so that, by applying the method for obtaining the definite integral solutions of that equation, it can be shown that

$$w = \int e^{z/\zeta} e^{-\zeta} \zeta^{-n-1} d\zeta,$$

where $\theta(\zeta) \equiv e^{z/\zeta} e^{-\zeta} \zeta^{-n-1}$ has the same value at both ends of the contour of integration.

By expanding $e^{z/\zeta}$ in powers of z it can be shown that

$$\int_{C_1} e^{z/\zeta} e^{-\zeta} \zeta^{-n-1} d\zeta = 2\pi i \; e^{-\pi i n} C_n(z),$$

where C_1 is the contour of Fig. 1 and amp $\zeta = 0$ initially.

It follows that

$$J_n(z) = \left(\frac{z}{2}\right)^n C_n\left(\frac{z^2}{4}\right) = \frac{1}{2\pi i} e^{\pi i n} z^n \int_{C_1} e^{-\frac{1}{2}(\zeta - z^2/\zeta)} \zeta^{-n-1} d\zeta,$$

a well known form for $J_n(z)$.

* *Cf.* Fourier, *Théorie Analytique de la Chaleur*, Ch. VI., Gray & Mathews, Ch. V., and Clifford, *Mathematical Papers*, p. 346.

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Hence

$$I_n(z) = i^{-n} J_n(iz) = \frac{1}{2\pi i} e^{\pi i n} z^n \int_{C_1} e^{-\frac{1}{2}(\zeta + z^2/\zeta)} \zeta^{-n-1}\, d\zeta .$$

Here let x be real and positive, and replace ζ by $x\zeta$; then

$$I_n(x) = \frac{1}{2\pi i} e^{\pi i n} \int_{C_1} e^{-\frac{x}{2}\left(\zeta + \frac{1}{\zeta}\right)} \zeta^{-n-1}\, d\zeta .$$

In this equation put $1/\zeta$ for ζ, and reverse the direction of integration: in order to make amp $\zeta = 0$ initially the factor $e^{-2\pi i n}$ is taken out of the integral; thus

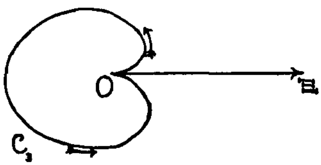

Fig 6

$$I_n(x) = \frac{1}{2\pi i} e^{-\pi i n} \int_{C_3} e^{-\frac{x}{2}\left(\zeta + \frac{1}{\zeta}\right)} \zeta^{n-1}\, d\zeta ,$$

where C_3 is the contour of Fig. 6.

Now the contour C_1 can be replaced by the ξ-axis from $+\infty$ to 0, followed by C_3, followed by the ξ-axis from 0 to ∞; hence

$$I_{-n}(x) = \frac{1}{2\pi i} e^{-\pi i n} \int_{C_1} e^{-\frac{x}{2}\left(\zeta + \frac{1}{\zeta}\right)} \zeta^{n-1}\, d\zeta$$

$$= \frac{1}{2\pi i} \left(e^{\pi i n} - e^{-\pi i n}\right) \int_0^\infty e^{-\frac{x}{2}\left(\xi + \frac{1}{\xi}\right)} \xi^{n-1}\, d\xi + I_n(x) .$$

Thus *

$$K_n(x) = \tfrac{1}{2} \int_0^\infty e^{-\frac{x}{2}\left(\xi + \frac{1}{\xi}\right)} \xi^{n-1}\, d\xi .$$

* *Cf.* Macdonald, *Proc. London Math. Soc.*, XXX.

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In this equation let $\xi = e^t$; then

$$K_n(x) = \int_0^\infty e^{-x\cosh t} \cosh nt \, dt.$$

These two equations hold if $R(x) > 0$; J. W. Nicholson defines $K_n(x)$ by means of the latter equation. (*Quarterly Journal*, XLII.).

It follows that, if $I(z) > 0$,

$$G_n(z) = (-i)^n K_n(-iz) = (-i)^n \int_0^\infty e^{iz\cosh t} \cosh nt \, dt.$$

This is the function that Heine * denoted by $K_n(z+0i)$; for points below the real axis he used

$$K_n(-z-0i) = (-1)^n K_n(z+0i) = (-1)^n G_n(z).$$

In Professor Gibson's *Calculus*, p. 477, it is shown that, by changing the order of integration of a certain double integral, the equation

$$\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} (2x)^n \int_0^\infty \frac{\cos \xi \, d\xi}{(\xi^2+x^2)^{n+\frac{1}{2}}} = (2x)^n \int_0^\infty e^{-\left(x^2\xi^2 + \frac{1}{4\xi^2}\right)} \xi^{2n-1} \, d\xi \ldots (2)$$

can be established for x real and positive and $R(n) \geqq 0$; the substitution $\xi = x \sinh \phi$ reduces the left hand side of the equation to the form $V_n(x)/\cos(n\pi)$: in the second integral put $\xi/(2x)$ for ξ^2; then *

$$\frac{V_n(x)}{\cos n\pi} = \frac{1}{2} \int_0^\infty e^{-\frac{x}{2}(\xi + 1/\xi)} \xi^{n-1} \, d\xi$$

$$= K_n(x).$$

The integrals in equation (2) have been studied by Glaisher (*Phil. Trans.*, 1881), J. J. Thomson (*Quarterly Journal, XVIII.*), and others. Glaisher says that he delayed the publication of his paper for three years in order to investigate the connection between the integrals and the Bessel Functions. His difficulties were probably due to the fact that he was dealing with $J_n(z)$ and not with $K_n(z)$.

Again (Gibson's *Calculus*, p. 478),

$$\int_0^\infty e^{-\left(x^2\xi^2 + \frac{1}{4\xi^2}\right)} \xi^{2n-1} \, d\xi = \frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} e^{-x} \int_0^\infty e^{-2x\xi} (1+\xi)^{n-\frac{1}{2}} \xi^{n-\frac{1}{2}} \, d\xi.$$

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* *Kugelfunctionen*, p. 237.

* *Cf.* Macdonald, *Proc. London Math. Soc.*, XXX., p. 170, and Hardy, *Quarterly Journal*, XXXIX.

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Hence

$$K_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} \frac{1}{\Gamma(n+\frac{1}{2})} e^{-x} \int_0^\infty e^{-\xi} \left(1 + \frac{\xi}{2x}\right)^{n-\frac{1}{2}} \xi^{n-\frac{1}{2}} \, d\xi,$$

and the asymptotic expansion follows as before.

Note on the function $W_{k,m}(z)$.* The methods used in this paper can be employed to obtain the expression for $W_{k,m}(z)$ in terms of $M_{k,m}(z)$ and $M_{k,-m}(z)$.

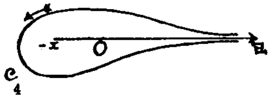

Fig. 7

Let

$$I \equiv x^k e^{-\frac{1}{2}x} \int_{C_4} e^{-\zeta} \zeta^{-k-\frac{1}{2}+m} \left(1 + \frac{\zeta}{x}\right)^{k-\frac{1}{2}+m} d\zeta,$$

where x is real and positive, amp $\zeta = 0$ initially, and C_4 is the contour of Fig. 7. Then

$$I = x^{\frac{1}{2}-m} e^{-\frac{1}{2}x} \int_{C_4} e^{-\zeta} \zeta^{2m-1} \left(1 + \frac{x}{\zeta}\right)^{k-\frac{1}{2}+m} d\zeta.$$

Now expand in ascending powers of x, and integrate term by term; thus

$$I = \left(e^{4\pi i m} - 1\right) \Gamma(2m) \, M_{k,-m}(x).$$

Again, if $R(\pm k + \frac{1}{2} + m) > 0$, the contour C_4 can be deformed into the contour of Fig. 8, and

$$I = x^k e^{-\frac{1}{2}x} \left(e^{4\pi i m} - 1\right) \int_0^\infty e^{-\zeta} \zeta^{-k-\frac{1}{2}+m} \left(1 + \frac{\zeta}{x}\right)^{k-\frac{1}{2}+m} d\zeta$$

$$- x^k e^{-\frac{1}{2}x} e^{\pi i (m+\frac{1}{2}-k)} \left\{ e^{2\pi i (m-\frac{1}{2}+k)} - 1 \right\}$$

$$\times \int_0^x e^{\zeta} \zeta^{-k-\frac{1}{2}+m} \left(1 - \frac{\zeta}{x}\right)^{k-\frac{1}{2}+m} d\zeta.$$

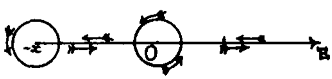

Fig. 8

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* *Cf.* Whittaker and Watson, *Analysis*, Chapter XVI.

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The first integral can be expressed in terms of $W_{k,m}(x)$; in the second integral let $\zeta = xt$; then

$$I = \left(e^{4\pi i m} - 1\right)\Gamma\left(\tfrac{1}{2}+m-k\right)W_{k,m}(x)$$
$$- x^{m+\frac{1}{2}} e^{-\frac{1}{2}x} e^{\pi i (m-\frac{1}{2}-k)}\left\{e^{2\pi i(k+m)}+1\right\}$$
$$\times \int_0^1 e^{xt}\, t^{m-\frac{1}{2}-k}\left(1-t\right)^{m-\frac{1}{2}+k}\,dt.$$

Now expand e^{xt} in powers of x, and integrate term by term; then

$$I = \left(e^{4\pi i m} - 1\right)\Gamma\left(-\tfrac{1}{2}+m-k\right)\, W_{k,m}(x)$$
$$- e^{\pi i(m-\frac{1}{2}-k)}\left\{e^{2\pi i(k+m)}+1\right\} B\left(\tfrac{1}{2}+m+k, \tfrac{1}{2}+m-k\right) M_{k,m}(x)$$
$$= \left(e^{4\pi i m} - 1\right)\Gamma\left(\tfrac{1}{2}+m-k\right) W_{k,m}(x)$$
$$- \left(e^{4\pi i m} - 1\right)\frac{\Gamma\left(\tfrac{1}{2}+m-k\right)\Gamma\left(-2m\right)}{\Gamma\left(\tfrac{1}{2}-m-k\right)} M_{k,m}(x).$$

Again, in the expression

$$x^{m+\frac{1}{2}} e^{-\frac{1}{2}x}\int^{(1+,\,0+,\,1-,\,0-)} e^{xt}\, t^{m-\frac{1}{2}-k}\left(1-t\right)^{m-\frac{1}{2}+k}\,dt,$$

where the initial point lies on the real axis between 0 and 1, and amp t and amp $(1-t)$ are initially zero, expand e^{xt} and integrate term by term; then for all values of m and k, the expression is equal to

$$\left\{1 - e^{2(m+\frac{1}{2}-k)\pi i}\right\}\left\{1 - e^{2(m+\frac{1}{2}+k)\pi i}\right\}$$
$$\times B\left(m+\tfrac{1}{2}-k, m+\tfrac{1}{2}+k\right) M_{k,m}(x).$$

Hence $M_{k,m}(x)$ is holomorphic in k and m, except at isolated singularities.

Thus the restrictions on k and m can be removed; also both sides of the equation are holomorphic for $-\pi < \operatorname{amp} x < \pi$, so that, in that region

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$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} M_{k,-m}(x).$$

Since $M_{0,n}(2x) = 2^{2n}\Gamma(n+1)\sqrt{2x}\, I_n(x)$, it follows that

$$K_n(x) = \sqrt{\frac{\pi}{2\pi}}\, W_{0,n}(2x).$$

The asymptotic expansion of $K_n(x)$ can thus be derived from that of $W_{k,n}(x)$; the latter can be easily established by the method employed for the Bessel Function by Professor Gibson.

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