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THE CUSPIDAL MODULES OF THE FINITE GENERAL LINEAR GROUPS

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Abstract

In this paper we prove a conjecture due to R. Carter [2], concerning the action of the finite general linear group $GL_n(q)$ on a cuspidal module. As an application of this result, we work out the case $GL_4(q)$.

It is well known that the cuspidal characters of the finite groups of Lie type play a particularly important role in the complex representation theory of such groups. For the general linear group $\mathrm{GL}_n(q)$ over the field $\mathbf{F}_q$ of q elements, these characters were determined by J. A. Green [5]. In the cases $n = 2$ and $n = 3$, the action of $\mathrm{GL}_n(q)$ on modules affording cuspidal characters has been studied in [9] and [8] respectively. In 1992 R. Carter [2] considered these cases using a different method from that used in [9] and [8]. The purpose of the present paper is to extend Carter's method to the general case, and to prove a conjecture stated in the introduction of [2]. This concerns the form of the entries of the matrices representing the action of $\mathrm{GL}_n(q)$ on a module affording a given cuspidal character.

This paper is organized as follows. In Section 1, we set up the basic definitions and notation from the representation theory needed here, and we determine a cuspidal module affording a given cuspidal character of $\mathrm{GL}_n(q)$. In Section 2, we do the work to obtain the action of $\mathrm{GL}_n(q)$ on such modules, and we discuss Carter's conjecture. In Section 3 we apply the general theory to the group $\mathrm{GL}_4(q)$. A considerable amount of detailed work was involved in the compilation of the results needed here to obtain the cuspidal matrices representing the generators of $\mathrm{GL}_4(q)$. This part of the work has not been included in the paper, since for our calculations we have followed more or less the same recipe as that used for the group $\mathrm{GL}_3(q)$ in [2].

1. *The cuspidal modules*

Throughout this paper, G will denote the group $\mathrm{GL}_n(q)$, T the maximal torus of diagonal matrices in G, and W the Weyl group with respect to T, which we shall identify with the subgroup of G of permutation matrices. Then TW is the subgroup N of all monomial matrices in G.

We consider a maximal Coxeter torus T_n of G. Then the elements of T_n can be viewed as the diagonal matrices $\mathrm{diag}\,(t, t^q, \ldots, t^{q^{n-1}})$, $t \in \mathbf{F}^*_{q^n}$ where $\mathbf{F}^*_{q^n}$ denotes the multiplicative group of $\mathbf{F}_{q^n}$. Thus $T_n \cong \mathbf{F}^*_{q^n}$. It is well known (see [10]) that all the cuspidal characters have degree

$$(q-1)(q^2-1)\ldots(q^{n-1}-1)$$

and they are the Deligne–Lusztig irreducible characters $(-1)^{n-1}R_{T_n,\theta}$, where θ is a regular character of T_n (see [1, Theorem 9.3.2]).

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Thus, counting the regular characters of T_n, it turns out from [7] that the group G has

$$\frac{1}{n}\sum_{d|n}\mu(d)(q^{n/d}-1)$$

cuspidal characters in number, where μ is the Möbius function on the natural numbers.

Now, given a cuspidal character χ of G, we want to determine a module affording χ. Such a module can be found inside the Gelfand–Graev module which is constructed as follows. We consider the root system Φ of G with respect to T, and we fix a fundamental basis $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ in Φ. Then a positive root α has the form

$$\alpha = \sum_{i\leqslant k<j}\alpha_k,\ 1\leqslant i<j\leqslant n.$$

Let X_α be the root subgroup corresponding to the positive root α. Then X_α consists of all upper unitriangular matrices $x_\alpha(\lambda)$, $\lambda \in \mathbf{F}_q$, where the entry in the (i, j)-position is equal to λ and all the other non-diagonal entries are equal to zero. The product

$$\prod_{\alpha\in\Phi^+} X_\alpha$$

is the upper unitriangular subgroup of G, where Φ^+ is the set of positive roots. Let ψ be a non-trivial complex linear character of the additive group $\mathbf{F}_q^+$ of $\mathbf{F}_q$. Since each X_α is isomorphic to $\mathbf{F}_q^+$, ψ induces the linear character of U (denoted also by ψ), given by

$$\psi\left(\prod_{\alpha\in\Phi^+} x_\alpha(\lambda_\alpha)\right) = \prod_{\alpha\in\Delta}\psi(\lambda_\alpha).$$

The induced character ψ^G is the so-called Gelfand–Graev character of G which is afforded by the module $\mathbb{G}e$, where e is the central primitive idempotent

$$|U|^{-1}\sum_{u\in U}\psi(u^{-1})u$$

in the group algebra $\mathbb{C}U$ of U over the field $\mathbb{C}$ of the complex numbers. One of the main properties of ψ^G is that each of its irreducible components occurs with multiplicity one, and their number is $q^{n-1}(q-1)$. Moreover, every Deligne–Lusztig generalized character meets ψ^G (see [1, Theorem 8.4.5]). In particular, every cuspidal character of G occurs in ψ^G.

We recall that the Gelfand–Graev module is isomorphic to the G-module $\mathcal{F}$ of all $\mathbb{C}$-valued functions f on G defined by

$$f(gu) = \psi(u^{-1})f(g),\quad u \in U,\quad g \in G,$$

where the G-action is given by

$$(gf)(x) = f(g^{-1}x),\quad g, x \in G,\quad f \in \mathcal{F}.$$

For $x \in G$ we define the function $f_x : G \to \mathbb{C}$ by

$$f_x(xu) = \psi(u^{-1}),\quad \text{for } u \in U$$

and

$$f_x(y) = 0,\quad \text{for } y \notin xU.$$

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Then these functions have the following properties:

Lemma 1. *Let $x \in G$. Then (i) $f_x \in \mathcal{F}$, (ii) $gf_x = f_{gx}$, for all $g \in G$, and (iii) $f_{xu} = \psi(u)f_x$, for all $u \in U$.*

Proof. (i) For $g \in xU$ and $u \in U$ we have, $f_x(gu) = f_x(xu^{'}u) = \psi(u^{-1})\psi(u^{'-1}) = \psi(u^{-1})f_x(g)$ where $g = xu^{'}, u^{'} \in U$. If $g \notin xU$ and $u \in U$ we have $f_x(g) = f_x(gu) = 0$ as $gu \notin xU$. Thus $f_x(gu) = \psi(u^{-1})f_x(g) = 0$. (ii) For $y = gxu$, $u \in U$, we have $(gf_x)(y) = f_x(g^{-1}y) = \psi(u^{-1}) = f_{gx}(y)$, while for $y \notin gxU$ we have $f_{gx}(y) = 0$ as well as $gf_x(y) = f_x(g^{-1}y) = 0$, since $g^{-1}y \notin xU$. (iii) Let $y = xu^{'}$, $u^{'} \in U$. Then $f_{xu}(y) = f_{xu}(xuu^{-1}u^{'}) = \psi(u^{'-1}u) = \psi(u)\psi(u^{'-1}) = \psi(u)f_x(y)$. If $y \notin xU$, then $\psi(u)f_x(y) = 0$ and $f_{xu}(y) = 0$ as $y \notin xuU$.

□

Corollary 2. $ef_1 = f_1$.

Proof. We have $uf_1 = f_{1u} = \psi(u)f_1$, for all $u \in U$. Thus

$$ef_1 = |U|^{-1}\sum_{u\in U}\psi(u^{-1})uf_1 = |U|^{-1}\sum_{u\in U}\psi(u^{-1})\psi(u)f_1 = f_1.$$

□

We next consider an irreducible component χ of ψ^G and let ε_χ be the corresponding central primitive idempotent of $\mathbb{C}G$. Then the G-module $\varepsilon_\chi\mathcal{F}$ is the unique irreducible G-submodule of $\mathcal{F}$ affording χ and so $\varepsilon_\chi\mathcal{F}$, as U-module, affords the character $\chi|_U$. By the Frobenius reciprocity theorem, we have

$$(\chi|_U, \psi) = (\chi, \psi^G) = 1.$$

Therefore the U-module $\varepsilon_\chi\mathcal{F}$ contains a unique 1-dimensional submodule affording ψ. Since $u(\varepsilon_\chi f_1) = \varepsilon_\chi(uf_1) = \psi(u)\varepsilon_\chi f_1$, $u \in U$, the subspace $\mathbb{C}\varepsilon_\chi f_1$ of $\varepsilon_\chi\mathcal{F}$ is the required one, and we have $\varepsilon_\chi\mathcal{F} = \mathbb{C}G\varepsilon_\chi f_1$. The non-zero multiples of the vector $\varepsilon_\chi f_1$ are called Bessel vectors for the character χ.

Let χ be a cuspidal character. Following [2], to determine the action of G on the Bessel vector $\varepsilon_\chi f_1$, we write $\varepsilon_\chi f_1$ in terms of a basis of $\mathcal{F}$ which is obtained by considering the left cosets of U in G. These are determined by the Bruhat decomposition

$$G = \dot{\bigcup}_{w\in W} U_{w^{-1}}TwU$$

where $U_{w^{-1}} = U \cap ww_0Uw_0w^{-1}$, w_0 being the element of maximal length in W. Thus the set $X = \dot{\bigcup}_{w\in W} U_{w^{-1}}Tw$ is a complete set of left coset representatives of U in G, and every element $f \in \mathcal{F}$ can be expressed uniquely in the form $f = \sum_{x\in X} f(x)f_x$. In other words, the set $\{f_x \mid x \in X\}$ is a basis of $\mathcal{F}$, and we call this basis the "Bruhat basis" of $\mathcal{F}$. In order to express $\varepsilon_\chi f_1$ as a linear combination of the basis elements f_x, $x \in X$, we need to recall [3] a basic fact concerning the relation of the central primitive idempotents of $\mathbb{C}G$ and those of the Hecke algebra $\mathcal{H} = e\mathbb{C}Ge$ of ψ^G. We put ${}^gU = gUg^{-1}$, $U_g^+ = U \cap {}^{g^{-1}}U$, for $g \in G$, and we let

$$N_\psi = \left\{n \in N \mid {}^n\psi(u) = \psi(u), \;\; u \in U_{n^{-1}}^+\right\}$$

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where ${}^n\psi$ is the character of ${}^nU$ given by ${}^n\psi(nun^{-1}) = \psi(u)$, $u \in U$. Then the elements $a_n = |U : U^+_{n^{-1}}|ene$, $n \in N_\psi$, form a basis for $\mathcal{H}$. Since ψ^G is multiplicity-free, $\mathcal{H}$ is a commutative algebra, and hence a direct sum of 1-dimensional subalgebras. The primitive idempotents of $\mathcal{H}$ are precisely the elements $e_\chi = \varepsilon_\chi e$ for all central primitive idempotents ε_χ of $\mathbb{C}G$ with $(\chi, \psi^G) = 1$, and we have

$$e_\chi = \frac{\chi(1)}{|G : U|} \sum_{n \in N_\psi} |U : U^+_{n^{-1}}|^{-1} \chi(|U : U^+_{n^{-1}}|en^{-1}e)|U : U^+_{n^{-1}}|ene$$

(see [3], §11D). Thus Corollary 2 and Lemma 1 (ii) give

$$e_\chi f_1 = \frac{\chi(1)}{|G : U|} \sum_{n \in N_\psi} |U : U^+_{n^{-1}}| \chi(en^{-1}e)ef_n.$$

Therefore we have to express ef_n, $n \in N_\psi$, in terms of the elements f_x, $x \in X$.

Proposition 3. *Let $n \in N_\psi$. Then*

$$ef_n = \frac{1}{q^{\ell(w)}} \sum_{u \in U_{w^{-1}}} \psi(u^{-1}) f_{un}$$

where $n = tw$, $t \in T$, $w \in W$, and $\ell(w)$ denotes the length of w.

Proof. We recall that if $w \in W$, then $U = U_w U_w^+ = U_w^+ U_w$ and $U_w \cap U_w^+ = 1$. Thus for $u \in U$ we may write $u = u_1 u_2$, $u_1 \in U_{w^{-1}}$ and $u_2 \in U^+_{w^{-1}}$, and we have

$$\begin{aligned}
ef_n &= |U|^{-1} \sum_{u \in U} \psi(u^{-1}) f_{un} \\
&= |U|^{-1} \sum_{u_1 \in U_{w^{-1}}} \sum_{u_2 \in U^+_{w^{-1}}} \psi(u_1^{-1})\psi(u_2^{-1}) f_{u_1 u_2 n} \\
&= |U|^{-1} \sum_{u_1 \in U_{w^{-1}}} \sum_{u_2 \in U^+_{w^{-1}}} \psi(u_1^{-1})\psi(u_2^{-1}) f_{u_1 n(n^{-1} u_2 n)} \\
&= |U|^{-1} \sum_{u_1 \in U_{w^{-1}}} \sum_{u_2 \in U^+_{w^{-1}}} \psi(u_1^{-1})\psi(u_2^{-1})\psi(n^{-1} u_2 n) f_{u_1 n}
\end{aligned}$$

by Lemma 1 (iii) as $n^{-1}u_2 n \in U \cap {}^{n^{-1}}U$. Since $n \in N_\psi$, we have $\psi(n^{-1}u_2 n) = {}^n\psi(u_2) = \psi(u_2)$. Thus

$$\begin{aligned}
ef_n &= |U|^{-1} \sum_{u_1 \in U_{w^{-1}}} \sum_{u_2 \in U^+_{w^{-1}}} \psi(u_1^{-1})\psi(u_2^{-1})\psi(u_2) f_{u_1 n} \\
&= |U|^{-1} \left(\sum_{u_2 \in U^+_{w^{-1}}} \psi(u_2^{-1})\psi(u_2) \right) \sum_{u_1 \in U_{w^{-1}}} \psi(u_1^{-1}) f_{u_1 n} \\
&= |U|^{-1} |U^+_{w^{-1}}| \sum_{u_1 \in U_{w^{-1}}} \psi(u_1^{-1}) f_{u_1 n} \\
&= q^{-\ell(w)} \sum_{u_1 \in U_{w^{-1}}} \psi(u_1^{-1}) f_{u_1 n},
\end{aligned}$$

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as $|U| = q^{\ell(w_0)}$ and $|U^+_{w^{-1}}| = q^{\ell(w_0)-\ell(w)}$. □

Corollary 4. $e_\chi f_1 = \dfrac{\chi(1)}{|G:U|} \sum_{n\in N_\psi} \sum_{u\in U_{w^{-1}}} \chi(en^{-1}e)\psi(u^{-1}) f_{un},$
where in the second sum the element w of W is such that $n \in Tw$, for $n \in N_\psi$.

2. *The G-action on the cuspidal modules*

Let χ be a cuspidal character of G. From now on we shall take as a Bessel vector the vector $b = |G : U|\chi(1)^{-1} e_\chi f_1 = (q^n - 1)e_\chi f_1$. Since $U_1 = 1$ and if $n = 1$ then $w = 1$, we have

$$b = f_1 + \sum_{\substack{n\in N_\psi \\ n\neq 1}} \sum_{u\in U_{w^{-1}}} \chi(en^{-1}e)\psi(u^{-1}) f_{un}$$

as $\chi(e) = 1$. This is the unique Bessel vector that has the coefficient of f_1 equal to 1.

Now to determine the action of G on the cuspidal module $\mathbb{C}Gb$, we consider as in [2] the affine subgroup A of G consisting of all matrices (a_{ij}) in G with $a_{nj} = 0$ for $j = 1, 2, \dots, n-1$ and $a_{nn} = 1$. Then G is generated by A and the permutation matrix

$$s := s_{n-1} = \begin{pmatrix} 1 & & & & O \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & 1 \\ O & & & 1 & 0 \end{pmatrix}.$$

We also consider the subgroup G_0 of A consisting of all matrices $(a_{ij}) \in A$ with $a_{in} = 0$, $i = 1, 2, \dots, n-1$.

Lemma 5. *If Y is a set of left coset representatives of the subgroup $U_0 = G_0 \cap U$ in G_0, then Y is also a set of left coset representatives of U in A.*

Proof. We have $A = G_0 U^+_\sigma$ and $U = U_0 U^+_\sigma$ where

$$\sigma = \begin{pmatrix} 0 & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ 1 & & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \text{ and so } U^+_\sigma = \begin{pmatrix} 1 & & 0 & * \\ & \ddots & & \vdots \\ 0 & & 1 & * \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $G_0 \cap U^+_\sigma = 1$, we have $A = \dot{\bigcup}_{y\in Y} yU_0U^+_\sigma = \dot{\bigcup}_{y\in Y} yU$ as required. □

Let $W_0 = \{w \in W \mid w \in G_0\}$ and $T_0 = T \cap A$. Then we have the Bruhat decomposition of G_0

$$G_0 = \dot{\bigcup}_{w\in W_0} U_{w^{-1}} T_0 w U_0$$

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and so by Lemma 5, we may take the set $Y = \dot{\bigcup}_{w \in W_0} U_{w^{-1}} T_0 w$ as a set of left coset representatives of U in A. We show now that the vectors yb, $y \in Y$, form a basis of the cuspidal module $\mathbb{C}Gb$. For this we need first to determine the matrix form of the elements in N_ψ.

Lemma 6. *The set N_ψ consists of the monomial matrices of the form*

$$\begin{pmatrix} 0 & & t_1 I_{n_1} \\ & \cdot^{\cdot^{\cdot}} & \\ t_k I_{n_k} & & 0 \end{pmatrix}, \quad t_1, \ldots, t_k \in \mathbf{F}_q^*,$$

where $(n_1, \ldots, n_k)$ is an unordered partition of n and I_{n_i} denotes the identity matrix of degree n_i.

Proof. Let $n = tw$, $t \in T$, $w \in W$. We show first that $n \in N_\psi$ if and only if $w(\Phi^+) \cap \Delta = w(\Delta) \cap \Phi^+$ and $w(\alpha)(t) = 1$ for all $\alpha \in w^{-1}(\Delta) \cap \Phi^+$. We have

$$U \cap {}^nU = \prod_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^+}} X_{w(\alpha)}.$$

Let $u = \prod_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^+}} x_{w(\alpha)}(\lambda_\alpha) \in U \cap {}^nU$. Then $\psi(u) = \prod_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Delta}} \psi(\lambda_\alpha)$ and ${}^n\psi(u) = \psi(u^n) = \prod_{\substack{\alpha \in \Delta \\ w(\alpha) \in \Phi^+}} \psi\left(w(\alpha)(t^{-1})\lambda_\alpha\right)$. Thus $n \in N_\psi$ if and only if

$$\prod_{\alpha \in \Phi^+ \cap w^{-1}(\Delta)} \psi(\lambda_\alpha) = \prod_{\alpha \in \Delta \cap w^{-1}(\Phi^+)} \psi\left(w(\alpha)(t^{-1})\lambda_\alpha\right) \tag{1}$$

for any choice of $\lambda_\alpha \in \mathbf{F}_q^+, \alpha \in \Phi^+$. If there were $\alpha \in \Phi^+ \cap w^{-1}(\Delta)$ with $\alpha \notin w^{-1}(\Phi^+) \cap \Delta$, then taking $\lambda_\alpha \in \mathbf{F}_q^+$ with $\psi(\lambda_\alpha) \neq 1$ and $\lambda_\beta = 0$, $\beta \neq \alpha$, $\beta \in (\Phi^+ \cap w^{-1}(\Delta)) \cup (\Delta \cap w^{-1}(\Phi^+))$, we would have $\psi(\lambda_\alpha) = 1$ by (1), a contradiction. Similarly, if $\alpha \in w^{-1}(\Phi^+) \cap \Delta$ then $\alpha \in \Phi^+ \cap w^{-1}(\Delta)$. Thus $\Phi^+ \cap w^{-1}(\Delta) = w^{-1}(\Phi^+) \cap \Delta$. That is, $w(\Phi^+) \cap \Delta = \Phi^+ \cap w(\Delta)$. Moreover, if $w(\alpha)(t^{-1}) \neq 1$ for some $\alpha \in \Phi^+ \cap w^{-1}(\Delta)$, taking $\lambda_\alpha \in \mathbf{F}_q^+$ with $\psi((w(\alpha)(t^{-1}) - 1)\lambda_\alpha) \neq 1$ and $\lambda_\beta = 0$, $\beta \neq \alpha$, $\beta \in \Phi^+ \cap w^{-1}(\Delta)$, we have by (1), $\psi(\lambda_\alpha) = \psi(w(\alpha)(t^{-1})\lambda_\alpha)$ or $\psi((w(\alpha)(t^{-1}) - 1)\lambda_\alpha) = 1$, a contradiction. This shows that $w(\alpha)(t^{-1}) = 1$, and so $w(\alpha)(t) = (w(\alpha)(t^{-1}))^{-1} = 1$. The converse is obvious.

We identify now Φ with the root system $\{e_i - e_j \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$ in $\mathbb{R}^n$ where $\{e_1, \ldots, e_n\}$ is the normal basis in $\mathbb{R}^n$. Let $\Delta = \{e_i - e_{i+1} \mid 1 \leqslant i \leqslant n-1\}$. Suppose that t and w have the form

$$t = \begin{pmatrix} t_1 I_{n_1} & & 0 \\ & \ddots & \\ 0 & & t_k I_{n_k} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & & I_{n_1} \\ & \ddots & \\ I_{n_k} & & 0 \end{pmatrix}.$$

Then it is straightforward to see that $w(\Phi^+) \cap \Delta = w(\Delta) \cap \Phi^+$ and $w(\alpha)(t) = 1$ for all $\alpha \in w^{-1}(\Delta) \cap \Phi^+$. Conversely, suppose that w and t satisfy this condition and let π be the

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permutation that corresponds to w. We have $w(\Phi^+) \cap \Delta = \{e_{\pi(i)} - e_{\pi(j)} \mid 1 \leqslant i < j \leqslant n$ and $\pi(j) = \pi(i)+1\}$ and $w(\Delta) \cap \Phi^+ = \{e_{\pi(i)} - e_{\pi(i+1)} \mid 1 \leqslant \pi(i) < \pi(i+1) \leqslant n\}$. Since $w(\Phi^+) \cap \Delta = \Phi^+ \cap w(\Delta)$, π must satisfy the condition: $\pi(i+1) = \pi(i) + 1$ whenever $\pi(i) < \pi(i+1)$, $1 \leqslant i \leqslant n-1$. If $\delta_{\rho,k}$ denotes the Kronecker delta, then $w = (\delta_{i,\pi(j)})$. Let $\pi(j_1) = 1$. As $\pi(j_1) < \pi(j_1+1)$, we have $\pi(j_1+1) = \pi(j_1)+1 = 2$ and inductively $\pi(j_1+k) = k+1$, $0 \leqslant k \leqslant n-j_1$. That is, $\delta_{i,\pi(n-n_1+i)} = 1$, $1 \leqslant i \leqslant n_1$, where $n_1 = n - j_1 + 1$. In the same way, if $\pi(j_2) = n - j_1 + 2 = n_1 + 1$, then $\pi(j_2) < \pi(j_2+1)$ and so $\pi(j_2+1) = \pi(j_2) + 1 = n_1 + 2$ and inductively $\pi(j_2+k) = n_1 + 1 + k$, $0 \leqslant k \leqslant j_1 - j_2 - 1$. That is, $\delta_{n_1+i,\pi(n-n_1-n_2+i)} = 1$, $1 \leqslant i \leqslant n_2$, where $n_2 = j_1 - j_2$. Therefore, an easy induction argument gives that w has the required form.

Now as $w(\alpha)(t) = 1$, $\alpha \in w^{-1}(\Phi^+) \cap \Delta$, we have $\alpha(t) = 1$, $\alpha \in \Phi^+ \cap w(\Delta)$. But $\alpha(t) = \lambda_i \lambda_j^{-1}$, where $t = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $\alpha = e_i - e_j$, $1 \leqslant i, j \leqslant n$. In our case α is one of the roots $e_{\pi(i)} - e_{\pi(i+1)}$, where π is the corresponding permutation. Thus $t_{\pi(i)} = t_{\pi(i+1)} = t_{\pi(i)+1}$ whenever $\pi(i) < \pi(i+1)$. That is, $\lambda_1 = \ldots = \lambda_{n_1}, \lambda_{n_1+1} = \ldots = \lambda_{n_1+n_2}, \ldots, \lambda_{n_1+\ldots+n_{k-1}+1} = \ldots = \lambda_n$. This shows that t must have the required form, where $t_1 = \lambda_1$, $t_2 = \lambda_{n_1+1}, \ldots, t_k = \lambda_{n_1+\ldots+n_{k-1}+1}$.

□

From this Lemma one can obtain the dimension of the Hecke algebra $\mathcal{H}$.

Corollary 7. $|N_\psi| = q^{n-1}(q-1)$.

Proof. We consider the elements

$$\begin{pmatrix} 0 & t_1 I_k \\ t_2 I_{n-k} & 0 \end{pmatrix}, \quad 1 \leqslant k \leqslant n-1$$

in N_ψ. Replacing for a given k the block $t_2 I_{n-k}$ by any element of the corresponding set in $\mathrm{GL}_{n-k}(q)$, we obtain by induction $q^{n-k-1}(q-1)^2$ elements in N_ψ. Doing this for any k, $1 \leqslant k \leqslant n-1$, we obtain $(q-1)^2 \sum_{k=1}^{n-1} q^{n-k-1} = (q-1)(q^{n-1}-1)$ elements in N_ψ. The only elements of N_ψ that have not been considered are those in $Z(G)$. Thus $|N_\psi| = (q-1)(q^{n-1}-1) + (q-1) = q^{n-1}(q-1)$.

□

Corollary 8. $N_\psi \cap A = 1$.

Proof. The proof is obvious. □

Proposition 9. *The set $\{yb \mid y \in Y\}$ is a basis of the cuspidal module $\mathbb{C}Gb$.*

Proof. The set $N_0 = \dot{\bigcup}_{w \in W_0} T_0 w$ is a set of double coset representatives of U in A. We consider the induced character ψ^A of A, where ψ is the character of U considered before. Applying Mackey's theorem on ψ^A we have

$$(\psi^A, \psi^A) = \sum_{n \in N_0} (\psi^n, \psi)_{U^+_{n-1}} = 1$$

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as $\psi \neq {}^n\psi$ on U^+_{n-1}, $n \in N_0$ unless $n = 1$ by Corollary 8. That is, the character ψ^A is irreducible. Let χ be any cuspidal character of G, and let $\chi|_A$ be its restriction on A. By Frobenius reciprocity we have

$$(\chi|_A, \psi^A)_A = (\chi, \psi^G)_G = 1.$$

Since ψ^A is an irreducible character of A and $\deg \chi|_A = \deg \psi^A$, we must have $\psi^A = \chi|_A$ for any cuspidal character χ of G. Therefore, if χ is a given cuspidal character of G, the G-module $\mathbb{C}Gb$ affording χ remains irreducible as an A-module, since this affords $\chi|_A$. On the other hand, the A-module $\mathbb{C}Ab$ is a non-zero A-submodule of the irreducible A-module $\mathbb{C}Gb$. Thus $\mathbb{C}Ab = \mathbb{C}Gb$. Now if $a \in A$, then $a = yu$ for some $y \in Y$ and $u \in U$. Thus $ab = \psi(u)yb$. This means that the elements yb, $y \in Y$, generate the space $\mathbb{C}Gb$. But $|Y| = \dim \mathbb{C}Gb$, and so the set $\{yb \mid y \in Y\}$ is a basis of $\mathbb{C}Gb$ as required.

□

To investigate the action of G on $\mathbb{C}Gb$ it is sufficient to obtain the action of A and that of the element $s = s_{n-1}$ on the basis $\{yb \mid y \in Y\}$. But if $a \in A$, $y \in Y$, then by the Bruhat decomposition, we have $a = y^{'}u$ for some $y^{'} \in Y$ and $u \in U$. Therefore we have $a(yb) = y^{'}ub = \psi(u)y^{'}b$. In other words, the action of A is monomial with respect to the above basis. Thus it remains to investigate the action of the element s on the vectors yb, $y \in Y$.

Let $X_\psi = \{un \mid n \in N_\psi,\ u \in U_{w^{-1}}, \text{where } n \in Tw\}$.
We put $\xi_a = \chi(en^{-1}e)\psi(u^{-1})$, where $a = un \in X_\psi$. We then have

$$b = \sum_{a \in X_\psi} \xi_a f_a.$$

The coefficients ξ_a are usually called "Bessel coefficients". Thus for $y \in Y$, we have by Lemma 1 (ii)

$$yb = \sum_{a \in X_\psi} \xi_a f_{ya} = \sum_{z \in Z} \xi_z f_{yz} + \sum_{a \in X_\psi \setminus Z} \xi_a f_{ya}$$

where $Z = Z(G)$. For $a \in X_\psi$, $y \in Y$, we let $sya = x_{sya}v_{sya}$ be the Bruhat decomposition of sya, where $x_{sya} \in X$ and $v_{sya} \in U$. With this notation we have by Lemma 1 (iii)

$$s(yb) = \sum_{z \in Z} \xi_z \psi(v_{syz}) f_{x_{syz}} + \sum_{a \in X_\psi \setminus Z} \xi_a \psi(v_{sya}) f_{x_{sya}}. \tag{2}$$

We notice that in (2) if $a, a^{'} \in X_\psi \setminus Z$, then $x_{sya} = x_{sya^{'}}$ implies $a = a^{'}$. For, if $a = un$ and $a^{'} = u^{'}n^{'}$, then $unv_{sya}^{-1} = u^{'}n^{'}v_{sya^{'}}^{-1}$ and from the uniqueness of the Bruhat decomposition we get $a = a^{'}$.

Next, we express the vector $s(yb)$ as a linear combination of the basis elements $y^{'}b$, $y^{'} \in Y$. We have

$$s(yb) = \sum_{y^{'} \in Y} \lambda_{y,y^{'}}(y^{'}b) = \sum_{z \in Z} \sum_{y^{'} \in Y} \lambda_{y,y^{'}} \xi_z f_{y^{'}z} + \sum_{a \in X_\psi \setminus Z} \sum_{y^{'} \in Y} \lambda_{y,y^{'}} \xi_a f_{y^{'}a}. \tag{3}$$

For this expression (3) of $s(yb)$ we notice that the vectors $f_{y^{'}z}$, $y^{'} \in Y$, $z \in Z$, belong to the Bruhat basis of the G-module $\mathcal{F}$, and $y^{'}z = y^{''}z^{'}$, $y^{'}, y^{''} \in Y$, $z, z^{'} \in Z$ if and only if $y^{'} = y^{''}$ and $z = z^{'}$. Also, since $A \cap N_\psi = 1$ the vectors $f_{y^{'}a}$, $a \in X_\psi \setminus Z$, $y^{'} \in Y$, are multiples of the Bruhat basis elements f_g, $g \in X$, with $g \notin YZ$.

Now we compare the two expressions (2) and (3) of $s(yb)$. Since, for any $z^{'} \in Z$ the elements $syz^{'}$ are not in AZ (and so $x_{syz} \notin YZ$), we see that there is no $z^{'} \in Z$ for which $f_{y'z} = f_{x_{syz'}}$ for some $z \in Z$ and $y^{'} \in Y$. Therefore we must have

$$\sum_{z\in Z}\sum_{y'\in Y}\lambda_{y,y'}\xi_z f_{y'z} = \sum_{a\in X_\psi\setminus Z}\xi_a\psi(v_{sya})f_{x_{sya}}.$$

Now if we have an $a \in Z_\psi$ such that $sya \in A$, then $x_{sya}z = x_{sy(az)}$ and $v_{sya} = v_{sy(az)}$ for all $z \in Z$. Thus we obtain

$$\sum_{z\in Z}\sum_{y'\in Y}\lambda_{y,y'}\xi_z f_{y'z} = \sum_{z\in Z}\sum_{\substack{a\in X_\psi\setminus Z\\ x_{sya}\in Y}}\xi_{az}\psi(v_{sya})f_{x_{sy(az)}}. \tag{4}$$

The equation 4 says that if we put $Y^{'} = \{y^{'} \in Y \mid y^{'} = x_{sya}$ for some $a \in X_\psi \setminus Z\}$, then for $y^{'} \in Y^{'}$ we have $\lambda_{y,y'} = \xi_a\psi(v_{sya}) = \xi_z^{-1}\xi_{az}\psi(u_{sya})$, for all $z \in Z$, where a is the element of $X_\psi \setminus Z$ such that $y^{'} = x_{sya}$. Otherwise, if $y^{'} \in Y \setminus Y^{'}$, then $\lambda_{y,y'} = 0$. Thus, given $y, y^{'} \in Y$, we consider the element $y^{-1}sy^{'}$ and its Bruhat decomposition $y^{-1}sy^{'} = utwv^{-1}$, $w \in W$, $t \in T$, $u \in U_{w^{-1}}$, $v \in U$. Then if $utw \in X_\psi \setminus Z$, we have

$$\lambda_{y,y'} = \xi_{utw}\psi(v).$$

Otherwise $\lambda_{y,y'} = 0$.

Now if $\lambda_{y,y'} = \xi_{utw}\psi(v)$, then $w \neq 1$; otherwise, u would be 1 and so by Lemma 6, $t \in Z$, a contradiction, since $utw \notin Z$. Thus $w_0w^{-1} \neq w_0$, and so there is a simple root $\alpha \in \Delta$ such that $w_0w^{-1}(\alpha) = \beta \in \Phi^+$. Therefore we have ${}^{ww_0}X_\beta = X_{ww_0(\beta)} = X_\alpha \subseteq U \cap^{ww_0} U = U_{w^{-1}}$. Now we may choose an element $u_1 \in X_\alpha$ such that $\psi(v) = \psi(u_1)$. This allows us to write $\xi_a\psi(v) = \xi_a\psi(u_1) = \chi(en^{-1}e)\psi(u^{-1})\psi(u_1) = \xi_{u_1^{-1}un}$, where $a = un$, $n = tw$, $sya = y^{'}v$. In other words, the coefficient $\lambda_{y,y'}$ is one of the Bessel coefficients, namely the coefficient $\xi_{u_1^{-1}un}$ (which is equal to $\xi_{u'^{-1}un}$ for any other choice $u^{'} \in U_{w^{-1}}$ such that $\psi(u^{'}) = \psi(v)$). This proves Carter's conjecture [2], which we state now as a theorem.

Theorem 10. *Let R be the matrix representation corresponding to the cuspidal module $\mathbb{C}Gb$ with respect to the basis $\{yb \mid y \in Y\}$. Then every non-zero entry of the matrix $R(s) = (\lambda_{y,y'})_{y,y'\in Y}$ is a Bessel coefficient.*

More precisely, if $y^{-1}sy^{'} = utwv$ is the Bruhat decomposition of the element $y^{-1}sy^{'}$ for $y, y^{'} \in Y$, then $\lambda_{y,y'} = \xi_a$ for any a of the form $u_1utw \in X_\psi \setminus Z$ where $u_1 \in U_{w^{-1}}$ such that $\psi(u_1) = \psi(v)$. Otherwise $\lambda_{y,y'} = 0$. Moreover

$$\lambda_{y,y'} = \xi_z^{-1}\xi_{az} \ \ \textit{(for every } z \in Z).$$

□

As a result of the above Theorem 10, one can prove Proposition 24 in [2].

Corollary 11. *Let $y, y^{'} \in Y$. Then $\lambda_{y,y'} = \overline{\lambda_{y',y}}$. In other words, the matrix S representing the action of s on $\mathbb{C}Gb$ is Hermitian.*

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Proof. We may suppose that $\lambda_{y,y'} \neq 0$. Then there exist $a \in X_\psi \setminus Z$ such that $sya = y'zv \in AZ$, $z \in Z$, $v \in U$. Without loss of generality, we may take $z = 1$. Then we have $\lambda_{y,y'} = \xi_a\psi(v)$. Let $a = un$, $n \in N_\psi \setminus Z$, $u \in U_{w^{-1}}$, where $n = tw$, $t \in T$, $w \in W$. Therefore we have

$$\lambda_{y,y'}\xi_n\psi(u^{-1})\psi(v).$$

By writing $v = v_1v_2$, $v_1 \in U_w$, $v_2 \in U_w^+$, we have $yu = sy'v_1v_2n^{-1} = sy'v_1n^{-1}(nv_2n^{-1})$. But $v_1n^{-1} \in X_\psi \setminus Z$, as $v_1 \in U_{(w^{-1})^{-1}}$, and $nv_2^{-1}n^{-1} \in {}^wU \cap U \subseteq U$. Thus we have $sy'a' = yv' \in A$, where $a' = v_1n^{-1} \in X_\psi \setminus Z$ and $v' = unv_2^{-1}n^{-1} \in U$.

Therefore $\lambda_{y',y} = \xi_{a'}\psi(v') = \xi_{n^{-1}}\psi(v_1^{-1})\psi(u)\psi(nv_2^{-1}n^{-1}) =$ (as $nv_2^{-1}n^{-1} \in U \cap {}^nU$ and $\psi = {}^n\psi$ on $U \cap {}^nU$) $= \xi_{n^{-1}}\psi(u)\psi(v_1^{-1})\,{}^n\psi(nv_2^{-1}n^{-1}) = \overline{\xi}_n\overline{\psi(u^{-1})}\,\overline{\psi(v_1)}\,\overline{\psi(v_2)} = \overline{\xi_n\psi(u^{-1})\psi(v)} = \overline{\lambda_{y',y}}$ since $\xi_{n^{-1}} = \overline{\xi}_n$.

□

3. *The group* $\mathrm{GL}_4(q)$

Here we determine explicitly the entries $\lambda_{y,y'}$, of the representing matrix S of the element $s = s_3$ for the group $\mathrm{GL}_4(q)$.

According to Theorem 10 we have to determine for each $y \in Y$ the Bruhat decomposition of the elements $y^{-1}sy'$, $y' \in Y$. Instead of considering these elements, we may equivalently consider for each $y \in Y$ the elements sya, $a \in X_\psi \setminus Z$, and determine whether such an element belongs to AZ or not.

Lemma 12. *Let $a = utw \in X_\psi \setminus Z$, $w \in W$, $t \in T$, $u \in U_{w^{-1}}$ and $y = u't_0w'$, $w' \in W_0$, $t_0 \in T_0$, $u' \in U_{w'^{-1}}$. Then $sya \in AZ$ if and only if $sw'w \in W_0$ and $u \in \prod_{\alpha \in \Phi^+ \cap w(\Phi_0^-)} X_\alpha$, where Φ_0 is the subsystem of Φ generated by the simple roots α_i, $i = 1, \ldots, n-2$.*

Proof. We have $sya = u'^s t_0^s sw'wu^w t^w$. Since $w' \in W_0$ and $u' \in U_{w'^{-1}} = \prod_{\alpha \in \Phi^+ \cap w'(\Phi^-)} X_\alpha$, we see that u' is of the form

$$\begin{pmatrix} B & O \\ O & 1 \end{pmatrix},$$

where B is an upper unitriangular matrix of degree $n-1$. Thus $u'^s \in A$. Also $t_0^s, t^w \in AZ$. Suppose that $sya \in AZ$; that is, $sw'wu^w \in AZ$. Since $u \in U_{w^{-1}}$, $u^w \in {}^{w_0}U \cap {}^{w^{-1}}U = \prod_{\alpha \in w^{-1}(\Phi^+) \cap \Phi^-} X_\alpha$; that is, u^w is a lower unitriangular matrix. Now, the left multiplication action of $sw'w$ on u^w changes the rows of u^w. Since $sw'wu^w \in AZ$, $sw'w$ must leave the last row of u^w fixed. Thus the last row of $sw'wu^w$ must be that of u^w, and so $u^w \in A \cap \prod_{\alpha \in w^{-1}(\Phi^+) \cap \Phi^-} X_\alpha = \prod_{\alpha \in w^{-1}(\Phi^+) \cap \Phi_0^-} X_\alpha$. Therefore $sw'w \in W_0$ and $u \in \prod_{\alpha \in \Phi^+ \cap w(\Phi_0^-)} X_\alpha$. The converse is obvious. □

Thus for a given $w' \in W_0$ we have to determine the elements $w \in N_\psi \cap W$ that satisfy the condition that $sw'w \in W_0$. For this, we let σ_k be the permutation matrix that corresponds

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to the permutation $(k, n-1)$, $k = 1, \ldots, n-1$. Then the set $\{\sigma_k \mid k = 1, \ldots, n-1\}$ is a set of right coset representatives of the Weyl subgroup W_{00} of W_0 which is generated by the reflections defined by the simple roots $\alpha_1, \ldots, \alpha_{n-3}$.

Corollary 13. *Let a and y be the elements considered in Lemma 12. Suppose $w' \in W_{00}\sigma_k$ for some $k = 1, \ldots, n-1$. Then $sya \in A$ if and only if $t_0^s t^w \in T_0$, $w(e_n) = e_k$ and $u \in \prod_{\alpha \in \Phi^+ \cap w(\Phi_0^- \setminus \Phi_k^-)} X_\alpha$, where $\Phi_k^- = \varnothing$ if $k = 1$; otherwise Φ_k^- is the set of negative roots of the subsystem Φ_k of Φ which is generated by the roots $\alpha_{n-1}, \ldots, \alpha_{n-(k-2)}, \alpha_{n-(k-1)}$.*

Proof. Let $w' \in W_{00}\sigma_k$, $k \in \{1, \ldots, n-1\}$. Suppose that $sya \in A$. Then by Lemma 12 we have $sw'w \in W_0$ and $u \in \prod_{\alpha \in \Phi^+ \cap w(\Phi_0^-)} X_\alpha$. Let $w(e_n) = e_{k'}$. Then $k' \neq n$, since the only element w in $N_\psi \cap W$ that fixes e_n is the identity, and in this case we should have $sw' \in W_0$, which does not hold. Suppose that $k' \neq k$. Then if $w' = w''\sigma_k$, $w'' \in W_{00}$, we have $sw''\sigma_k w(e_n) = sw''\sigma_k(e_{k'}) = sw'(e_i)$, where i can be either k (if $k' = n-1$) or k' (if $k' \neq n-1$) and so $i \in \{1, \ldots, n-2\}$. Thus $sw''(e_i) = e_j$, $j \in \{1, 2, \ldots, n-2\}$. This means that $sw'w \notin W_0$ as $sw'w(e_n) = e_j \neq e_n$. Therefore we must have $w(e_n) = e_k$ and so, by Lemma 6, w must be a permutation matrix of the form

$$\begin{pmatrix} O & & I_k \\ & \cdot^{\cdot^{\cdot}} & \\ I_\nu & & O \end{pmatrix}.$$

Thus, if $k > 1$, we have $w(e_{n-\lambda} - e_{n-\lambda'}) = e_{k-\lambda} - e_{k-\lambda'}$ for λ, λ', $0 \leqslant \lambda' < \lambda \leqslant k-1$. This means that $w(\Phi_k^+) \subseteq \Phi^+$. Therefore we have $\Phi^+ \cap w(\Phi_0^-) = \Phi^+ \cap w(\Phi_0^- \setminus \Phi_k^-)$ as required. Finally, we notice that $sw'wu^w \in AZ$ if and only if $sw'wu^w \in A$, and so $t_0^s sw'wu^w(t_0^s)^{-1} \in A$. Thus if $sya \in A$ then $t_0^s t^w \in A \cap T = T_0$. The converse is obvious. □

Now we return to the case $G = \mathrm{GL}_4(q)$. Here the root system is the set $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1+\alpha_2), \pm(\alpha_2+\alpha_3), \pm(\alpha_1+\alpha_2+\alpha_3)\}$. For the root subgroups X_{α_i}, $i = 1, 2, 3$, we shall write X_i, and X_{12}, X_{23}, X_{123} for the subgroups $X_{\alpha_1+\alpha_2}$, $X_{\alpha_2+\alpha_3}$, $X_{\alpha_1+\alpha_2+\alpha_3}$, respectively. A typical element of the subgroup X_i, $i = 1, 2, 3$, will be denoted by x_i or x_i', etc., and similar notation will be used for the elements of the groups X_{12}, X_{23}, X_{123}.

Let s, s_1 and s_2 be the permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively.

We denote by T_i, $i = 1, 2, 3, 4, 5, 6$, the subgroups of T that consist respectively of the

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diagonal matrices of the form

$$\begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \nu & \\ & & & \nu \end{pmatrix}, \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \mu & \\ & & & \nu \end{pmatrix}, \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \rho & \\ & & & \rho \end{pmatrix},$$

$$\begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \nu & \\ & & & \rho \end{pmatrix}, \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \nu & \\ & & & \rho \end{pmatrix} \text{ and } \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \nu & \\ & & & \nu \end{pmatrix}.$$

By Lemma 6 the subset N_ψ, which consists of $q^3(q-1)$ monomial matrices, is the set $\{t_i w_i \mid t_i \in T_i, i = 1, \ldots, 6\} \cup T w_0 \cup Z$, where $w_0 = s_1 s_2 s_1 s s_2 s_1$, $w_1 = s_1 s_2 s$, $w_2 = s s_2 s_1$, $w_3 = s_1 s_2 s_1 s s_2$, $w_4 = s_2 s_1 s s_2 s_1$, $w_5 = s_1 s_2 s s_2 s_1$ and $w_6 = s_2 s_1 s s_2$, Z being the center of G. Using Corollary 13 we consider the right coset representatives σ_k of W_{00} in W_0. We have $W_0 = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$, $W_{00} = \{1, s_1\}$ and $\sigma_1 = s_1 s_2 s_1$, $\sigma_2 = s_2$, $\sigma_3 = 1$. The subgroups $U_{w^{-1}}$, $w \in W_0$, of U are the following:

$U_{1^{-1}} = 1,\ U_{s_1^{-1}} = X_1,\ U_{s_2^{-1}} = X_2,\ U_{(s_1 s_2)^{-1}} = X_1 X_{12},\ U_{(s_2 s_1)^{-1}} = X_2 X_{12}$ and $U_{(s_1 s_2 s_1)^{-1}} = X_1 X_2 X_{12}$.

Thus the set Y of the left coset representatives of the subgroup U in the affine subgroup A of G is

$$Y = \{t_0, x_1 t_0 s_1, x_2 t_0 s_2, x_1 x_{12} t_0 s_1 s_2, x_2 x_{12} t_0 s_2 s_1, x_1 x_2 x_{12} t_0 s_1 s_2 s_1 \mid t_0 \in T_0,\ x_1 \in X_1,\ x_2 \in X_2,\ x_{12} \in X_{12}\}.$$

Now we classify the elements of Y into three classes according to the right coset representatives σ_1, σ_2 and σ_3.
σ_1 gives rise to the class

$$\Sigma_1 = \{x_1 x_2 x_{12} t_0 s_1 s_2 s_1,\ x_2 x_{12} t_0 s_2 s_1 \mid x_1 \in X_1,\ x_2 \in X_2,\ x_{12} \in X_{12},\ t_0 \in T_0\},$$

σ_2 gives rise to the class

$$\Sigma_2 = \{x_2 t_0 s_2,\ x_1 x_{12} t_0 s_1 s_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ x_{12} \in X_{12},\ t_0 \in T_0\}, \text{ and}$$

σ_3 gives rise to the class

$$\Sigma_3 = \{t_0,\ x_1 t_0 s_1 \mid x_1 \in X_1,\ t_0 \in T_0\}.$$

Let $t_0 = \mathrm{diag}\,(\kappa_1, \kappa_2, \kappa_3, 1)$ be an element of T_0. Then by simple calculations it turns out that if $y \in \Sigma_1$ then the elements $a \in X_\psi \setminus Z$ satisfying the condition $sya \in A$ are of the form $t_1 w_1$, $x_2'' x_{23}'' t_3 w_3$, $x_3'' x_{23}'' t_5 w_5$ and $x_2'' x_3'' x_{23}'' t w_0$, where $t_1 = \mathrm{diag}\,(\kappa_3^{-1}, \nu, \nu, \nu) \in T_1$, $t_3 = \mathrm{diag}\,(\kappa_3^{-1}, \nu, \rho, \rho) \in T_3$, $t_5 = \mathrm{diag}\,(\kappa_3^{-1}, \nu, \nu, \rho) \in T_5$, $t = \mathrm{diag}\,(\kappa_3^{-1}, \nu, \rho, \tau) \in T$, $x_2'' \in X_2$, $x_3'' \in X_3$ and $x_{23}'' \in X_{23}$. If $y \in \Sigma_2$, then the corresponding $a \in X_\psi \setminus Z$ are of the form $x_3'' x_{12}'' x_{123}'' t_4 w_4$ or of the form $x_{12}'' x_{123}'' t_6 w_6$, where $t_4 = \mathrm{diag}\,(\kappa_3^{-1}, \kappa_3^{-1}, \nu, \rho) \in T_4$, $t_6 = \mathrm{diag}\,(\kappa_3^{-1}, \kappa_3^{-1}, \nu, \nu) \in T_6$, $x_3'' \in X_3$, $x_{12}'' \in X_{12}$ and $x_{123}'' \in X_{123}$. Finally for the

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elements of Σ_3 the corresponding $a \in X_\psi \setminus Z$ are elements of the form $x''_{23}x''_{123}t_2w_2$, where $t_2 = \mathrm{diag}\,(\kappa_3^{-1}, \kappa_3^{-1}, \kappa_3^{-1}, \nu) \in T_2$, $x''_{23} \in X_{23}$ and $x''_{123} \in X_{123}$.

To describe the Bruhat decomposition of sya we put $y_1 = t_0$, $y_2 = x_1t_0s_1$, $y_3 = x_2t_0s_2$, $y_4 = x_1x_{12}t_0s_1s_2$, $y_5 = x_2x_{12}t_0s_2s_1$, $y_6 = x_1x_2x_{12}t_0s_1s_2s_1$ and $a_0 = x''_2x''_3x''_{23}tw_0$, $a_1 = t_1w_1$, $a_2 = x''_{23}x''_{123}t_2w_2$, $a_3 = x''_2x''_{23}t_3w_3$, $a_4 = x''_3x''_{12}x''_{123}t_4w_4$, $a_5 = x''_3x''_{23}t_5w_5$ and $a_6 = x''_{12}x''_{123}t_6w_6$. For $i = 1, \ldots, 6$ and $j = 0, 1, \ldots, 6$ let y'_{ij} be the element of Y and $v_{ij} \in U$ such that $y'_{ij}v_{ij}$ is the Bruhat decomposition of sy_ia_j.

Tables 1 and 2 give all possible y'_{ij} and v_{ij} such that $sy_ia_j \in A$. In these tables we have put

$$d_1 = (t_0t_2)^s,\ d_2 = (t_0t_1^{s_1})^s,\ d_3 = (t_0t_6^{s_2})^s,\ d_4 = (t_0t_4^{s_2})^s,\ d_5 = (t_0t_6^{s_2s_1})^s,$$
$$d_6 = (t_0t_4^{s_2s_1})^s,\ d_7 = (t_0t_1^{s_1s_2})^s,\ d_8 = (t_0t_5^{s_1s_2})^s,\ d_9 = (t_0t_3^{s_1s_2})^s,\ d_{10} = (t_0t^{s_1s_2})^s,$$
$$d_{11} = (t_0t_1^{s_1s_2s_1})^s,\ d_{12} = (t_0t_5^{s_1s_2s_1})^s,\ d_{13} = (t_0t_3^{s_1s_2s_1})^s,\ d_{14} = (t_0t^{s_1s_2s_1})^s.$$

Moreover, the elements β_1, γ_1, d and f appearing in these tables are determined as follows. For the element $x''_{12} \neq 1$ in X_{12} in the expression of a_4 and a_6 we have $x_{12}^{''s_2s_1t_0^{-1}s} \in X_{-1}$, where X_{-1} denotes the subgroup X_{-a_1}. As one can see, the Bruhat decomposition of the non-trivial elements in X_{-1} has the form $\beta_1ds_1\beta_1$, where $\beta_1 \in X_1$, $d \in T$. Also, for the element $x''_2 \neq 1$ of X_2 in the expression of a_0 and a_3, we have $x_2^{''s_1s_2s_1t_0^{-1}s} \in X_{-1}$, and so its Bruhat decomposition should be of the form $\gamma_1fs_1\gamma_1$, $\gamma_1 \in X_1$, $f \in T$. m , In Table 1 we have written $y'_{ij} = x'_1x'_2x'_{12}t'_0w'$, where $x'_1 \in X_1$, $x'_2 \in X_2$, $x'_{12} \in X_{12}$, $t'_0 \in T_0$ and $w' \in W_0$. Thus, for example, if $i = 3$, $j = 6$, we have $x'_1 = x_{12}^{''s_2t_0^{-1}}$, $x'_2 = 1$, $x'_{12} = x_{123}^{''s_2t_0^{-1}s}$, $t'_0 = d_3 = (t_0t_6^{s_2})^s$ and $w' = s_1s_2$.

Now, using Tables 1 and 2, we can determine all pairs $(y, y') \in Y \times Y$ such that the Bruhat decompositions of the elements $y^{-1}sy'$ are of the form av, where a is one of the elements in $X_\psi \setminus Z$ and $v \in U$.

By Theorem 10 the entry $\lambda_{y,y'}$ of the matrix S corresponding to the pair (y, y') is equal to $\xi_a\psi(v^{-1})$. Below, we give these entries in terms of y and y'. For example, let us take $y = t_0$ and $a = a_2 = x''_{23}x''_{123}t_2ss_2s_1$ where $x''_{23} \in X_{23}$, $x''_{123} \in X_{123}$ and $t_2 \in T_2$. The Bruhat decomposition of sya is $sya = y'v$, where $y' = x'_2x'_{12}t'_0s_2s_1$, $x'_2 = x_{23}^{''t_0^{-1}s}$, $x'_{12} = x_{123}^{''t_0^{-1}s}$, $t'_0 = d_1 = (t_0t_2)^s$ and $v = 1$ by Tables 1 and 2.

Thus we have $\lambda_{y,y'} = \xi_{x''_{23}x''_{123}t_2ss_2s_1} = \xi_{t_2ss_2s_1}$ (*), as $\psi(x_{23}^{''-1}x_{123}^{''-1}) = 1$. Eliminating t_2 in (*) we get $\lambda_{y,y'} = \xi_{t_0^{-1}st'_0s_2s_1}$ if $t_0^{-1}t_0^{'s} \in T_2$, otherwise $\lambda_{y,y'} = 0$. Thus the function that gives the entry $\lambda_{y,y'}$ where $y = t_0$ and $y' = x'_2x'_{12}t'_0s_2s_1$ is of the form

$$\Xi_{(t_0;t'_0)} = \begin{cases} \xi_{t_0^{-1}st'_0s_2s_1} & \text{if } t_0^{-1}t_0^{'s} \in T_2; \\ 0 & \text{otherwise.} \end{cases}$$

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Table 1: The form of elements a.

$\mathbf{y}'_{\mathbf{ij}}$	$\mathbf{x}'_{\mathbf{1}}$	$\mathbf{x}'_{\mathbf{2}}$	$\mathbf{x}'_{\mathbf{12}}$	$\mathbf{t}'_{\mathbf{0}}$	$\mathbf{w}'$
y'_{12}		$x_{23}^{\prime\prime t_0^{-1}s}$	$x_{123}^{\prime\prime t_0^{-1}s}$	d_1	s_2s_1
y'_{22}	x_1	$x_{123}^{\prime\prime s_1t_0^{-1}s}$	$x_{23}^{\prime\prime s_1t_0^{-1}s}$	d_2	$s_1s_2s_1$
y'_{36}	$x_{12}^{\prime\prime s_2t_0^{-1}}$		$x_{123}^{\prime\prime s_2t_0^{-1}s}$	d_3	s_1s_2
y'_{34}	$x_{12}^{\prime\prime s_2t_0^{-1}}$	$x_3^{\prime\prime s_2t_0^{-1}s}$	$([x_3^{\prime\prime-1}, x_{12}^{\prime\prime-1}]x''_{123})^{s_2t_0^{-1}s}$	d_4	$s_1s_2s_1$
y'_{46} $(x''_{12}=1)$		$x_{123}^{\prime\prime s_2s_1t_0^{-1}s}$		d_5	s_2
y'_{46} $(x''_{12}\neq 1)$	$x_1\beta_1$		$[(x_{123}^{\prime\prime s_2s_1t_0^{-1}s})^{-1}, \beta_1^{-1}]$	$dd_5^{s_1}$	s_1s_2
y'_{44} $(x''_{12}=1)$		$x_{123}^{\prime\prime s_2s_1t_0^{-1}s}$	$x_3^{\prime\prime s_2s_1t_0^{-1}s}[x_1^{-1}, (x_{123}^{\prime\prime s_2s_1t_0^{-1}s})^{-1}]$	d_6	s_2s_1
y'_{44} $(x''_{12}\neq 1)$	$x_1\beta_1$	$x_{123}^{\prime\prime s_2s_1t_0^{-1}s}$	$[(x_{123}^{\prime\prime s_2s_1t_0^{-1}s})^{-1}, \beta_1^{-1}]x_3^{\prime\prime s_2s_1t_0^{-1}s}$	$dd_6^{s_1}$	$s_1s_2s_1$
y'_{51}				d_7	1
y'_{55}		$x_3^{\prime\prime s_2t_0^{-1}s}$	$x_{23}^{\prime\prime s_1s_2t_0^{-1}s}$	d_8	s_2s_1
y'_{53}	$x_2^{\prime\prime s_1s_2t_0^{-1}}$		$x_{23}^{\prime\prime s_1s_2t_0^{-1}s}$	d_9	s_1s_2
y'_{50}	$x_2^{\prime\prime s_1s_2t_0^{-1}}$	$x_3^{\prime\prime s_2t_0^{-1}s}$	$x_{23}^{\prime\prime s_1s_2t_0^{-1}s}$	d_{10}	$s_1s_2s_1$
y'_{61}	x_1			d_{11}	s_1
y'_{65}	x_1	$x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}s}$	$x_3^{\prime\prime s_2s_1t_0^{-1}s}$	d_{12}	$s_1s_2s_1$
y'_{63} $(x''_2=1)$		$x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}s}$		d_{13}	s_2
y'_{63} $(x''_2\neq 1)$	$x_1\gamma_1$		$[(x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}s})^{-1}, \gamma_1^{-1}]$	$fd_{13}^{s_1}$	s_1s_2
y'_{60} $(x''_2=1)$		$x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}s}$	$[x_1^{-1}, (x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}s})^{-1}]x_3^{\prime\prime s_2s_1t_0^{-1}s}$	d_{14}	s_2s_1
y'_{60} $(x''_2\neq 1)$	$x_1\gamma_1$	$([x_2^{\prime\prime-1}, x_3^{\prime\prime-1}]x''_{23})^{s_1s_2s_1t_0^{-1}s}$	$[(([x_2^{\prime\prime-1}, x_3^{\prime\prime-1}]x''_{23})^{s_1s_2s_1t_0^{-1}s})^{-1}, \gamma_1^{-1}]x_3^{\prime\prime s_2s_1t_0^{-1}s}$	$fd_{14}^{s_1}$	$s_1s_2s_1$

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Table 2: The form of the elements v.

$$v_{12} = 1$$

$$v_{22} = 1$$

$$v_{36} = x_2^{sd_3s_1s_2}[x_2^{-1}, (x_{12}^{\prime\prime s_2t_0^{-1}})^{-1}]^{sd_3s_1s_2} \in X_{123}X_3$$

$$v_{34} = x_2^{sd_4s_1s_2s_1}[x_2^{-1}, (x_{12}^{\prime\prime s_2t_0^{-1}})^{-1}]^{sd_4s_1s_2s_1} \in X_{23}X_3$$

$$v_{46} = x_1^{d_5s_2}[x_1^{-1}, (x_{123}^{\prime\prime s_2s_1t_0^{-1}})^{-1}]^{sd_5s_2}x_{12}^{sd_5s_2} \in X_{12}X_1X_{123}, \quad \text{if } x_{12}^{\prime\prime} = 1$$

$$v_{46} = x_{123}^{\prime\prime s_2s_1t_0^{-1}sds_1d_5s_2}x_{12}^{sds_1d_5s_2}\beta_1^{d_5s_2} \in X_1X_3X_{12}, \quad \text{if } x_{12}^{\prime\prime} \neq 1$$

$$v_{44} = x_1^{d_6s_2s_1}x_{12}^{sd_6s_2s_1} \in X_2X_{23}, \quad \text{if } x_{12}^{\prime\prime} = 1$$

$$v_{44} = x_{12}^{sds_1d_6s_2}\beta_1^{d_6s_2s_1} \in X_3X_2, \quad \text{if } x_{12}^{\prime\prime} \neq 1$$

$$v_{51} = x_2^{sd_7}x_{12}^{sd_7} \in X_{23}X_{123}$$

$$v_{55} = x_2^{sd_8s_2}x_{12}^{sd_8s_2s_1} \in X_3X_{23}$$

$$v_{53} = x_2^{sd_9s_1s_2}([x_2^{-1}, (x_2^{\prime\prime s_1s_2t_0^{-1}})^{-1}]x_{12})^{sd_9s_1s_2} \in X_{123}X_3$$

$$v_{50} = x_2^{sd_{10}s_1s_2s_1}([x_2^{-1}, (x_2^{\prime\prime s_1s_2t_0^{-1}})^{-1}]x_{12})^{sd_{10}s_1s_2} \in X_{23}X_3$$

$$v_{61} = x_2^{sd_{11}s_1}x_{12}^{sd_{11}s_1} \in X_{123}X_{23}$$

$$v_{65} = x_2^{sd_{12}s_1s_2s_1}x_{12}^{sd_{12}s_1s_2} \in X_{23}X_3$$

$$v_{63} = x_1^{d_{13}s_2}[x_1^{-1}, (x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}})^{-1}]^{sd_{13}s_2}x_2^{sd_{13}s_2}x_{12}^{sd_{13}s_2} \in X_{12}X_1X_3X_{123} \text{ if } x_2^{\prime\prime} = 1$$

$$v_{63} = x_2^{sfs_1d_{13}s_2}([x_2^{-1}, \gamma_1^{-1}]x_{12})^{sfs_1d_{13}s_2}x_{23}^{\prime\prime s_1s_2s_1t_0^{-1}sfs_1d_{13}s_2}\gamma_1^{d_{13}s_2} \in X_{123}X_3X_1X_{12}$$

if $x_2^{\prime\prime} \neq 1$

$$v_{60} = x_1^{d_{14}s_2s_1}x_2^{sd_{14}s_2}x_{12}^{sd_{14}s_2s_1} \in X_2X_3X_{23}$$

if $x_2^{\prime\prime} = 1$

$$v_{60} = x_2^{sfs_1d_{14}s_2s_1}([x_2^{-1}, \gamma_1^{-1}]x_{12})^{sfs_1d_{14}s_2}\gamma_1^{d_{14}s_2s_1} \in X_{23}X_3X_2 \text{ if } x_2^{\prime\prime} \neq 1$$

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Our calculations have shown that the matrix S with respect to the ordered basis $\{t_0 b,\ x_1 t_0 s_1 b,\ x_2 t_0 s_2 b,\ x_1 x_{12} t_0 s_1 s_2 b,\ x_2 x_{12} t_0 s_2 s_1 b,\ x_1 x_2 x_{12} t_0 s_1 s_2 s_1 b;\ t_0 \in T_0,\ x_1 \in X_1,\ x_2 \in X_2,\ x_{12} \in X_{12}\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \Xi_{(t_0;t_0')} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{x_1,x_1'}\Xi'_{(t_0;t_0')} \\ 0 & 0 & 0 & \Xi_{(t_0,x_2;t_0',x_1')} & 0 & \Xi_{(t_0,x_2;t_0',x_1',x_2')} \\ 0 & 0 & \Xi_{(t_0,x_1;t_0',x_2')} & \Xi_{(t_0,x_1,x_{12};t_0',x_1',x_{12}')} & \Xi_{(t_0,x_1;t_0',x_2',x_{12}')} & \Xi_{(t_0,x_1,x_{12};t_0',x_1',x_2',x_{12}')} \\ \Xi''_{(t_0;t_0')} & 0 & 0 & \Xi_{(t_0,x_2,x_{12};t_0',x_1')} & \Xi_{(t_0,x_2;t_0',x_2')} & \Xi_{(t_0,x_2,x_{12};t_0',x_1',x_2')} \\ 0 & \delta_{x_1,x_1'}\Xi'''_{(t_0;t_0')} & \Xi_{(t_0,x_1,x_2;t_0',x_2')} & \Xi_{(t_0,x_1,x_2,x_{12};t_0',x_1',x_{12}')} & \Xi_{(t_0,x_1,x_2;t_0',x_2',x_{12}')} & \Xi_{(t_0,x_1,x_2,x_{12};t_0',x_1',x_2',x_{12}')} \end{pmatrix}$$

where

$$\Xi_{(t_0;t_0')} = \begin{cases} \xi_{t_0^{-1} s t_0' s_2 s_1} & \text{if } t_0^{-1} t_0^{\prime s} \in T_2; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi'_{(t_0;t_0')} = \begin{cases} \xi_{s_1 t_0^{-1} s t_0' s_1 s_2 s_1} & \text{if } (t_0^{-1} t_0^{\prime s})^{s_1} \in T_2; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_2;t_0',x_1')} = \begin{cases} \xi_{s_2 t_0^{-1} s t_0' s_1 s_2} \psi([x_2^{-1}, x_1^{\prime -1}]^{s t_0' s_1 s_2}) & \text{if } (t_0^{-1} t_0^{\prime s})^{s_2} \in T_6; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_2;t_0',x_1',x_2')} = \begin{cases} \xi_{s_2 t_0^{-1} s t_0' s_1 s_2 s_1} \psi((x_2^{\prime -1})^{s t_0 s_2} [x_2^{-1}, x_1^{\prime -1}]^{s t_0' s_1 s_2}) & \text{if } (t_0^{-1} t_0^{\prime s})^{s_2} \in T_4; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1;t_0',x_2')} = \begin{cases} \xi_{s_2 s_1 t_0^{-1} s t_0' s_2} \psi([x_1^{-1}, x_2^{\prime -1}]^{t_0' s_2}) & \text{if } (t_0^{-1} t_0^{\prime s})^{s_1 s_2} \in T_6; \\ 0 & \text{otherwise.} \end{cases}$$

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$$\Xi_{(t_0,x_1,x_{12};t_0',x_1',x_{12}')} = \begin{cases} \xi_{s_2s_1t_0^{-1}s_1sd^{-1}t_0's_1s_2}\psi((\omega_2x_{12}^s)^{t_0's_1s_2}) \text{ if } x_1' \neq x_1 \\ \text{and } (t_0^{-1})^{s_1s_2}(d^{-1}t_0')^{ss_2} \in T_6, \text{ where } d \in T \text{ such that} \\ (x_1^{-1}x_1')ds_1(x_1^{-1}x_1') \in X_{-1} \setminus \{1\} \text{ and} \\ \omega_2 \in X_2 \text{ such that } [\omega_2^{-1}, x_1x_1'^{-1}] = x_{12}'; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1;t_0',x_2',x_{12}')} = \begin{cases} \xi_{s_2s_1t_0^{-1}st_0's_2s_1}\psi(([x_1^{-1},x_2'^{-1}]x_{12}'^{-1})^{st_0s_1s_2}x_1^{t_0's_2s_1}) \text{ if } (t_0^{-1}t_0'^{s})^{s_1s_2} \in T_4; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1,x_{12};t_0',x_1',x_2',x_{12}')} = \begin{cases} \xi_{s_2s_1t_0^{-1}s_1sd^{-1}t_0's_1s_2s_1}\psi((x_{12}'^{-1}[x_2'^{-1},x_1'^{-1}x_1])^{st_0s_1s_2} \\ x_{12}^{st_0's_1s_2}(x_1^{-1}x_1')^{s_1d^{-1}t_0's_1s_2s_1}) \\ \text{if } x_1' \neq x_1 \text{ and } (t_0^{-1})^{s_1s_2}(d^{-1}t_0')^{ss_2} \in T_4, \\ \text{where } d \in T \text{ such that } (x_1^{-1}x_1')ds_1(x_1^{-1}x_1') \in X_{-1} \setminus \{1\}; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi''_{(t_0;t_0')} = \begin{cases} \xi_{s_1s_2t_0^{-1}st_0'} \text{ if } (t_0^{-1}t_0'^{s})^{s_2s_1} \in T_1; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_2;t_0',x_2')} = \begin{cases} \xi_{s_1s_2t_0^{-1}st_0's_2s_1}\psi((x_2'^{-1})^{st_0s_2}x_2^{st_0's_2}) \text{ if } (t_0^{-1}t_0'^{s})^{s_2s_1} \in T_5; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_2,x_{12};t_0',x_1')} = \begin{cases} \xi_{s_1s_2t_0^{-1}st_0's_1s_2}\psi((x_1'^{-1})^{t_0s_2s_1}([x_2^{-1},x_1'^{-1}]x_{12})^{st_0's_1s_2}) \\ \text{ if } (t_0^{-1}t_0'^{s})^{s_2s_1} \in T_3; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_2,x_{12};t_0',x_1',x_2')} = \xi_{s_1s_2t_0^{-1}st_0's_1s_2s_1}\psi((x_1'^{-1}x_2'^{-1})^{st_0s_2s_1}([x_2^{-1},x_1'^{-1}]x_{12})^{st_0's_1s_2}).$$

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$$\Xi'''_{(t_0;t_0')} = \begin{cases} \xi_{s_1s_2s_1t_0^{-1}st_0's_1} & \text{if } (t_0^{-1}t_0'^{s})^{s_1s_2s_1} \in T_1; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1,x_2;t_0',x_2')} = \begin{cases} \xi_{s_1s_2s_1t_0^{-1}st_0's_2}\,\psi(([x_1^{-1},x_2'^{-1}]x_2^{s})^{t_0's_2}) & \text{if } (t_0^{-1}t_0'^{s})^{s_1s_2s_1} \in T_3; \\ 0 & \text{otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1,x_2,x_{12};t_0',x_1',x_{12}')} = \begin{cases} \xi_{s_1s_2s_1t_0^{-1}s_1sf^{-1}t_0's_1s_2}\,\psi(((x_1'^{-1}x_1)s_1f^{-1}(x_1'^{-1}x_1))^{t_0s_1s_2s_1} \\ ([x_2^{-1},x_1x_1'^{-1}]x_{12})^{st_0's_1s_2}\omega_2^{t_0's_1s_2}) \\ \text{if } x_1' \neq x_1 \text{ and } (t_0^{-1})^{s_1s_2s_1}(f^{-1}t_0')^{ss_2s_1} \in T_3, \\ \text{where } f \in T \text{ such that } (x_1^{-1}x_1')fs_1(x_1^{-1}x_1') \in X_{-1}\setminus\{1\} \\ \text{and } \omega_2 \in X_2 \text{ such that } [\omega_2^{-1},x_1x_1'^{-1}] = x_{12}'; \\ 0 \text{ otherwise.} \end{cases}$$

$$\Xi_{(t_0,x_1,x_2;t_0',x_2',x_{12}')} = \xi_{s_1s_2s_1t_0^{-1}st_0's_2s_1}\,\psi((x_{12}'^{-1}[x_1^{-1},x_2'^{-1}])^{st_0s_1s_2}(x_1x_2)^{st_0's_2s_1}).$$

$$\Xi_{(t_0,x_1,x_2,x_{12};t_0',x_1',x_2',x_{12}')} = \begin{cases} \xi_{s_1s_2s_1t_0^{-1}st_0's_1s_2s_1}\,\psi(((x_{12}'^{-1})^{st_0}x_{12}^{st_0'})^{s_1s_2}) \\ \text{if } x_1' = x_1 \text{ and } (t_0^{-1}t_0'^{s})^{s_1s_2s_1} \in T_5; \\ \xi_{s_1s_2s_1t_0^{-1}s_1sf^{-1}t_0's_1s_2s_1}\,\psi((((x_1'x_1^{-1})s_1f^{-1}(x_1'x_1^{-1})x_{12}'^{-1} \\ [x_2'^{-1},x_1'x_1^{-1}])^{st_0}([x_2^{-1},x_1x_1'^{-1}]x_{12})^{st_0'}(x_1^{-1}x_1'^{-1})^{s_1f^{-1}t_0'})^{s_1s_2s_1}) \\ \text{if } x_1' \neq x_1 \text{ where } f \in T \\ \text{such that } (x_1^{-1}x_1')fs_1(x_1^{-1}x_1') \in X_{-1}\setminus\{1\}; \\ 0 \text{ otherwise.} \end{cases}$$

We now determine explicitly the coefficient ξ_a, $a \in X_\psi$ of the Bessel vector b. If $a = un$, $n \in N_\psi$, $u \in U_{n-1}$, then we know that $\xi_a = \xi_n\psi(u^{-1})$. Thus it is enough to consider only the coefficients ξ_n, $n \in N_\psi$.

If χ is the cuspidal character that gives rise to the Bessel vector b, then we have

$$\xi_{n^{-1}} = \chi(ene) = |U|^{-1}\sum_{u\in U}\psi(u^{-1})\chi(utw)$$

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where $n = tw$, $t \in T$, $w \in W \cap N_\psi$. (We notice that $\xi_{n^{-1}} = \overline{\xi}_n$.)

To calculate the value $\chi(utw)$ we must know which conjugacy class of $\mathrm{GL}_4(q)$ contains the element utw.

The conjugacy classes of $\mathrm{GL}_4(q)$ are represented by 22 different types of Jordan forms of the elements of $\mathrm{GL}_4(K)$, where K is the algebraic closure of the field $\mathbf{F}_q$. These types are given in Table 3. Let us note that in this table some of the types of class are empty for small values of q. For example, the conjugacy classes of $\mathrm{GL}_4(2)$ are classified into the types $C_1, C_2, C_3, C_4, C_5, C_{15}, C_{16}, C_{18}, C_{19}, C_{21}$ and C_{22}, the other types being empty.

Table 3: The conjugacy classes of $GL_4(q)$.

Type of class	Jordan form	Number of classes
C_1	$\begin{pmatrix} \kappa &&& \\ & \kappa && \\ && \kappa & \\ &&& \kappa \end{pmatrix}, \ \kappa \in \mathbf{F}_q^*$	$q-1$
C_2	$\begin{pmatrix} \kappa & 1 && \\ & \kappa && \\ && \kappa & \\ &&& \kappa \end{pmatrix}, \ \kappa \in \mathbf{F}_q^*$	$q-1$
C_3	$\begin{pmatrix} \kappa & 1 && \\ & \kappa && \\ && \kappa & 1 \\ &&& \kappa \end{pmatrix}, \ \kappa \in \mathbf{F}_q^*$	$q-1$
C_4	$\begin{pmatrix} \kappa & 1 && \\ & \kappa & 1 & \\ && \kappa & \\ &&& \kappa \end{pmatrix}, \ \kappa \in \mathbf{F}_q^*$	$q-1$
C_5	$\begin{pmatrix} \kappa & 1 && \\ & \kappa & 1 & \\ && \kappa & 1 \\ &&& \kappa \end{pmatrix}, \ \kappa \in \mathbf{F}_q^*$	$q-1$
C_6	$\begin{pmatrix} \kappa &&& \\ & \kappa && \\ && \kappa & \\ &&& \lambda \end{pmatrix}, \ \kappa, \lambda \in \mathbf{F}_q^*, \ \kappa \neq \lambda$	$(q-1)(q-2)$

Table 3 continued on next page

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Table 3 continued from previous page

Type of class	Jordan form		Number of classes
C_7	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & & \\ & & \kappa & \\ & & & \lambda \end{pmatrix}$,	$\kappa, \lambda \in \mathbf{F}_q^*$ $\kappa \neq \lambda$	$(q-1)(q-2)$
C_8	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & 1 & \\ & & \kappa & \\ & & & \lambda \end{pmatrix}$,	$\kappa, \lambda \in \mathbf{F}_q^*$ $\kappa \neq \lambda$	$(q-1)(q-2)$
C_9	$\begin{pmatrix} \kappa & & & \\ & \kappa & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$,	$\kappa, \lambda \in \mathbf{F}_q^*$ $\kappa \neq \lambda$	$\frac{1}{2}(q-1)(q-2)$
C_{10}	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$,	$\kappa, \lambda \in \mathbf{F}_q^*$ $\kappa \neq \lambda$	$(q-1)(q-2)$
C_{11}	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$,	$\kappa, \lambda \in \mathbf{F}_q^*$ $\kappa \neq \lambda$	$\frac{1}{2}(q-1)(q-2)$
C_{12}	$\begin{pmatrix} \kappa & & & \\ & \kappa & & \\ & & \lambda & \\ & & & \mu \end{pmatrix}$,	$\kappa, \lambda, \mu \in \mathbf{F}_q^*$ κ, λ, μ pairwise different	$\frac{1}{2}(q-1)(q-2)(q-3)$
C_{13}	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & & \\ & & \lambda & \\ & & & \mu \end{pmatrix}$,	$\kappa, \lambda, \mu \in \mathbf{F}_q^*$ κ, λ, μ pairwise different	$\frac{1}{2}(q-1)(q-2)(q-3)$
C_{14}	$\begin{pmatrix} \kappa & & & \\ & \lambda & & \\ & & \mu & \\ & & & \nu \end{pmatrix}$,	$\kappa, \lambda, \mu, \nu \in \mathbf{F}_q^*$ $\kappa, \lambda, \mu, \nu$ pairwise different	$\frac{1}{24}(q-1)(q-2)$ $(q-3)(q-4)$

Table 3 continued on next page

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Table 3 continued from previous page

Type of class	Jordan form	Number of classes
C_{15}	$\begin{pmatrix} \kappa & & & \\ & \kappa & & \\ & & x & \\ & & & x^q \end{pmatrix}$, $\kappa \in \mathbf{F}_q^*$, $x \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$	$\frac{1}{2}q(q-1)^2$
C_{16}	$\begin{pmatrix} \kappa & 1 & & \\ & \kappa & & \\ & & x & \\ & & & x^q \end{pmatrix}$, $\kappa \in \mathbf{F}_q^*$, $x \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$	$\frac{1}{2}q(q-1)^2$
C_{17}	$\begin{pmatrix} \kappa & & & \\ & \lambda & & \\ & & x & \\ & & & x^q \end{pmatrix}$, $\kappa, \lambda \in \mathbf{F}_q^*$, $\kappa \neq \lambda$, $x \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$	$\frac{1}{4}q(q-1)^2(q-2)$
C_{18}	$\begin{pmatrix} x & & & \\ & x & & \\ & & x^q & \\ & & & x^q \end{pmatrix}$, $x \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$	$\frac{1}{2}q(q-1)$
C_{19}	$\begin{pmatrix} x & 1 & & \\ & x & & \\ & & x^q & 1 \\ & & & x^q \end{pmatrix}$, $x \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$	$\frac{1}{2}q(q-1)$
C_{20}	$\begin{pmatrix} x & & & \\ & x^q & & \\ & & y & \\ & & & y^q \end{pmatrix}$, $x, y \in \mathbf{F}_{q^2}^* \setminus \mathbf{F}_q$, $x \neq y, y^q$	$\frac{1}{8}q(q+1)(q-1)(q-2)$
C_{21}	$\begin{pmatrix} \kappa & & & \\ & x & & \\ & & x^q & \\ & & & x^{q^2} \end{pmatrix}$, $\kappa \in \mathbf{F}_q^*$, $x \in \mathbf{F}_{q^3}^* \setminus \mathbf{F}_q$	$\frac{1}{3}q(q+1)(q-1)^2$
C_{22}	$\begin{pmatrix} x & & & \\ & x^q & & \\ & & x^{q^2} & \\ & & & x^{q^3} \end{pmatrix}$, $x \in \mathbf{F}_{q^4} \setminus \mathbf{F}_{q^2}$	$\frac{1}{4}q^2(q+1)(q-1)$

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Table 4: The values of the cuspidal character χ.

Class C	$\chi(C)$
C_1	$(q^3-1)(q^2-1)(q-1)\theta(\kappa)$
C_2	$-(q^2-1)(q-1)\theta(\kappa)$
C_3	$(q-1)\theta(\kappa)$
C_4	$(q-1)\theta(\kappa)$
C_5	$-\theta(\kappa)$
C_{18}	$(q^2-1)(\theta(x)+\theta(x^q))$
C_{19}	$-(\theta(x)+\theta(x^q))$
C_{22}	$-(\theta(x)+\theta(x^q)+\theta(x^{q^2})+\theta(x^{q^3}))$
C_i, $i=6,7,8,9,10,11,12,$ 13,14,15,16,17,20,21	0

The values of χ can be easily obtained using the work of Green [5]. These values are given in Table 4.

Now we have

$$\chi(etwe)=|U|^{-1}\sum_{C\ conj.class}\chi(C)\sum_{\substack{u\in U\\ utw\in C}}\psi(u^{-1}). \tag{5}$$

To calculate, for a class C, the coefficient $\sum_{\substack{u\in U\\ utw\in C}}\psi(u^{-1})$ of $\chi(C)$, we use the same method as in [2]. Thus we have to determine, for a given $n=tw\in N_\psi$, $t\in T$, $w\in N_\psi\cap W$, the elements $u\in U$, such that $utw\in C$.

For this, first we determine all the pairs (t',u) so that $ut'w\in C$. This can be done by comparing the characteristic polynomial of $ut'w$ and that of C. Then the use of the minimal polynomial of C and the rank of the matrix $ut'w-\lambda I$ gives these pairs, where λ is an eigenvalue of the elements in C . Having found these pairs, for the given $w\in N_\psi\cap W$ we consider only those where $t'=t$. These calculations need a considerable amount of work, which cannot be presented here.

Tables 5–11 give for each $w\in N_\psi\cap W$ the number of pairs (t',u) with $ut'w\in C$, for every conjugacy class C. In these tables we have omitted the classes that do not contain elements of the form $ut'w$.

In Tables 9 and 10 we have put C_i^1, $i=8,11,13,14,16,17,20,21,22$, and C_i^2, $i=11,13,14,16,17,20$, to denote respectively the conjugacy classes of type: C_8, with $\lambda=\frac{a^2}{\kappa^3}$, $a\neq 0,\pm\kappa^2$, C_{11} with $\lambda=-\kappa$, C_{13} with $\mu=\frac{\kappa^2}{\lambda}$, $\lambda\neq-\kappa$, C_{14} with $\lambda=-\kappa$, $\nu=-\mu$, $\kappa\neq-\mu$, C_{16} with $\kappa^2=x^{q+1}$, C_{17} with $\kappa\lambda=x^{q+1}$, C_{20} with $x^{q+1}=y^{q+1}$, C_{21} with $-\kappa$ and $-x^{1+q+q^2}$ squares in $\mathbf{F}_q$ or with $-\kappa$ and $-x^{1+q+q^2}$ not squares, C_{22} with

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Table 5: $(w = 1)$

Type of Class	Number of Class	Number of pairs (t', u)
C_1	$q-1$	1
C_2	$q-1$	$(q-1)(3q^2+2q+1)$
C_3	$q-1$	$q(q-1)^2(2q+1)$
C_4	$q-1$	$q^2(q-1)^2(3q+1)$
C_5	$q-1$	$q^3(q-1)^3$

Table 6: $(w = w_1$ or $w_2)$

Type of Class	Number of Class	Number of pairs (t', u)
C_5	$q-1$	$q^3(q-1)$
C_8	$(q-1)(q-2)$	$q^3(q-1)$
C_{11}	$\frac{1}{2}(q-1)(q-2)$	$q^3(q-1)$
C_{13}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q^3(q-1)$
C_{14}	$\frac{1}{24}(q-1)(q-2)(q-3)(q-4)$	$q^3(q-1)$
C_{16}	$\frac{1}{2}q(q-1)^2$	$q^3(q-1)$
C_{17}	$\frac{1}{4}(q-1)^2(q-2)q$	$q^3(q-1)$
C_{19}	$\frac{1}{2}q(q-1)$	$q^3(q-1)$
C_{20}	$\frac{1}{8}(q+1)q(q-1)(q-2)$	$q^3(q-1)$
C_{21}	$\frac{1}{3}(q+1)q(q-1)^2$	$q^3(q-1)$
C_{22}	$\frac{1}{4}(q+1)q^2(q-1)$	$q^3(q-1)$

$x^{\frac{1}{2}(1+q+q^2+q^3)} \in \mathbf{F}_q$ and C_{11} with $\lambda \neq -\kappa$, C_{13} with $\mu = \dfrac{a^2}{\lambda\kappa^2}$, where $a \in \mathbf{F}_q^* \smallsetminus \{\pm\kappa^2, \pm\kappa\lambda\}$ such that $\lambda \neq a^2\kappa^{-3}$, C_{14} with $\nu = \dfrac{\kappa\lambda}{\mu}$, $\lambda \neq -\mu$, $\kappa \neq -\mu$, $\dfrac{\mu^2}{\lambda}$, C_{16} with $\kappa^2 \neq x^{q+1}$, C_{17} with $\kappa\lambda \neq x^{q+1}$, C_{20} with $x^{q+1} \neq y^{q+1}$. Moreover C_{13}^3 and C_{14}^3 denote respectively the classes of type C_{13} that do not belong to C_{13}^1, and of C_{14} that do not belong to either C_{14}^1 or to C_{14}^2. Finally, C_{14}^4 denotes the classes of type C_{14} that do not belong to C_{14}^2.

For each $w \in N_\psi \cap W$, we notice, from the above results, that if we multiply, for each type of conjugacy class, the number of classes by the number of pairs, then the sum of these products is equal to

$$|U| \cdot |Tw \cap N_\psi|,$$

as it ought to be.

Now we give the values $\chi(ene)$, for $n \in N_\psi$. As we have mentioned, the detailed calculations have been worked out by Gotsis [6].

We let $f_r(x) = x^4 + a_3(r)x^3 + a_2(r)x^2 + a_1(r)x + a_0(r)$ be the characteristic polynomial

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Table 7: ($w = w_3$ or w_4)

Type of Class	**Number of Class**	**Number of pairs** $(t^{'}, u)$
C_4	$q-1$	$q(q-1)^2$
C_5	$q-1$	$q^3(q-1)^2$
C_7	$(q-1)(q-2)$	$q(q-1)^2$
C_8	$(q-1)(q-2)$	$q^2(q-1)^2(q+1)$
C_{10}	$(q-1)(q-2)$	$q(q-1)^2$
C_{11}	$\frac{1}{2}(q-1)(q-2)$	$q(q-1)^2(q^2+q-1)$
C_{12}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q(q-1)^2$
C_{13}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q^2(q-1)^2(q+2)$
C_{14}	$\frac{1}{24}(q-1)(q-2)(q-3)(q-4)$	$q(q-1)^2(q^2+3q+1)$
C_{15}	$\frac{1}{2}q(q-1)^2$	$q(q-1)^2$
C_{16}	$\frac{1}{2}q(q-1)^2$	$q^3(q-1)^2$
C_{17}	$\frac{1}{4}q(q-1)^2(q-2)$	$q(q-1)^2(q^2+q+1)$
C_{19}	$\frac{1}{2}q(q-1)$	$q(q-1)^2(q^2-q+1)$
C_{20}	$\frac{1}{8}(q+1)q(q-1)(q-2)$	$q(q-1)^2(q^2-q+1)$
C_{21}	$\frac{1}{3}(q+1)q(q-1)^2$	$q(q-1)^2(q^2+1)$
C_{22}	$\frac{1}{4}(q+1)q^2(q-1)$	$q(q-1)^2(q^2-q+1)$

of an element r of the torus T_4, which is identified with $\mathbf{F}_{q^4}^*$. In what follows we put

$$u = \begin{pmatrix} 1 & a & \delta & \zeta \\ 0 & 1 & \beta & \varepsilon \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $w = w_0$ and $t = \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \rho & \\ & & & \tau \end{pmatrix}$, we consider the function

$$F_0(r,t) = \begin{cases} -F_0^{'}(r,t) + q^{-5} \sum_{a \in \mathbf{F}_q^*} \psi\left(-a + \dfrac{\mu(r^2+\nu\rho)^2}{a\nu r^4} - \dfrac{r^2-\nu\rho}{\rho r}\right) \\ \quad \text{if } \ r \in \mathbf{F}_q^*, \\ -F_0^{'}(r,t) + q^{-4}\psi\left(\nu \dfrac{1+r^{q-1}}{r^q}\right) \ \text{if } \ r \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q \ \text{ and } \ \nu\rho = -r^{q+1}, \\ -F_0^{'}(r,t) \ \text{ otherwise,} \end{cases}$$

where

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Table 8: $(w = w_5)$

Type of Class	Number of Class	Number of pairs $(t^{'}, u)$
C_2	$q-1$	$q-1$
C_3	$q-1$	$(q-1)^2$
C_4	$q-1$	$4q(q-1)^2$
C_5	$q-1$	$q(q-1)^3(q+2)$
C_6	$(q-1)(q-2)$	$q-1$
C_7	$(q-1)(q-2)$	$(q-1)^2(4q+1)$
C_8	$(q-1)(q-2)$	$q(q-1)^2(q^2+3q-2)$
C_{10}	$(q-1)(q-2)$	$q(q-1)(3q-2)$
C_{11}	$\frac{1}{2}(q-1)(q-2)$	$q(q-1)^2(q^2+4q-3)$
C_{12}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q(q-1)(3q-2)$
C_{13}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q(q-1)^2(q^2+5q-1)$
C_{14}	$\frac{1}{24}(q-1)(q-2)(q-3)(q-4)$	$q(q-1)^2(q^2+6q+1)$
C_{15}	$\frac{1}{2}q(q-1)^2$	$q(q-1)(3q-2)$
C_{16}	$\frac{1}{2}q(q-1)^2$	$q(q-1)^2(q^2+q-1)$
C_{17}	$\frac{1}{4}q(q-1)^2(q-2)$	$q(q-1)^2(q+1)^2$
C_{19}	$\frac{1}{2}q(q-1)$	$q(q-1)^4$
C_{20}	$\frac{1}{8}(q+1)q(q-1)(q-2)$	$q(q-1)^4$
C_{21}	$\frac{1}{3}(q+1)q(q-1)^2$	$q(q-1)^2(q^2+1)$
C_{22}	$\frac{1}{4}(q+1)q^2(q-1)$	$q(q-1)^4$

$$F_0^{'}(r,t) = q^{-6} \sum_{\substack{a,\beta,\delta\in\mathbf{F}_q \\ a^2\nu+a\beta\delta\rho-\delta^2\rho\neq 0}} \psi(-a-\beta+\mu(a^2\nu+a\beta\delta\rho-\delta^2\rho)^{-1}a_0(r)^{-1}\nu^{-1}(a_0(r)$$
$$(a\nu+\beta\delta\rho)-a_1(r)\delta\nu\rho+a_2(r)a\nu^2\rho+a_3(r)\nu^2\rho^2(a\beta-\delta)$$
$$+\nu^2\rho^2(a\beta^2\rho+a\nu-\beta\delta\rho))).$$

Then $\chi(etw_0e) = \sum_{\substack{r\in T_4 \\ \det r=\det t}} F_0(r,t)\theta(r)$, where $\det r = a_0(r) = r^{1+q+q^2+q^3}$.

For $w = w_1$ and $t = \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \nu & \\ & & & \nu \end{pmatrix}$ we put $F_1(r,t) = -q^{-3}\psi\left(\frac{a_3(r)}{\nu}\right)$. Then

$$\chi(etw_1e) = \sum_{\substack{r\in T_4 \\ \det r=-\det t}} F_1(r,t)\theta(r).$$

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Table 9: ($w = w_6$, $char\ \mathbf{F}_q \neq 2$)

Type of Class	Number of Class	Number of pairs (t', u)
C_3	$q-1$	$q(q-1)$
C_4	$q-1$	$q(q-1)^2$
C_5	$q-1$	$3q^2(q-1)^2$
C_8^1	$\frac{1}{2}(q-1)(q-3)$	$2q^2(q-1)^2$
C_9	$\frac{1}{2}(q-1)(q-2)$	$q(q-1)$
C_{10}	$(q-1)(q-2)$	$q(q-1)^2$
C_{11}^1	$\frac{1}{2}(q-1)$	$q(q-1)(5q^2-4q+1)$
C_{11}^2	$\frac{1}{2}(q-1)(q-3)$	$q(q-1)^2(3q-1)$
C_{13}^1	$\frac{1}{2}(q-1)(q-3)$	$2q^2(q-1)(2q-1)$
C_{13}^2	$\frac{1}{4}(q-1)(q-3)(q-5)$	$2q^2(q-1)^2$
C_{14}^1	$\frac{1}{8}(q-1)(q-3)$	$2q^2(q-1)(3q-1)$
C_{14}^2	$\frac{1}{8}(q-1)(q-3)(q-5)$	$2q^2(q-1)(2q-1)$
C_{14}^3	$\frac{1}{48}(q-1)(q-3)(q-5)(q-7)$	$2q^2(q-1)^2$
C_{16}^1	$\frac{1}{2}(q-1)^2$	$2q^2(q-1)(2q-1)$
C_{16}^2	$\frac{1}{4}(q-1)^2(q-3)$	$2q^2(q-1)^2$
C_{17}^1	$\frac{1}{4}(q-1)^3$	$2q^2(q-1)(2q-1)$
C_{17}^2	$\frac{1}{8}(q-1)^3(q-3)$	$2q^2(q-1)^2$
C_{18}	$\frac{1}{2}q(q-1)$	$q(q-1)$
C_{19}	$\frac{1}{2}q(q-1)$	$q(q-1)^2(3q+1)$
C_{20}^1	$\frac{1}{8}(q-1)^3$	$2q^2(q-1)(2q-1)$
C_{20}^2	$\frac{1}{16}(q-1)(q-3)(q^2+1)$	$2q^2(q-1)^2$
C_{21}^1	$\frac{1}{6}(q+1)q(q-1)^2$	$2q^2(q-1)^2$
C_{22}^1	$\frac{1}{8}(q+1)^2(q-1)^2$	$2q^2(q-1)^2$

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Table 10: $(w = w_6, \ \ char\ \mathbf{F}_q = 2)$

Type of Class	Number of Class	Number of pairs (t', u)
C_3	$q-1$	$q(q-1)$
C_4	$q-1$	$q(q-1)^2$
C_5	$q-1$	$2q^2(q-1)^2$
C_8	$(q-1)(q-2)$	$q^2(q-1)^2$
C_9	$\frac{1}{2}(q-1)(q-2)$	$q^2(q-1)$
C_{11}	$\frac{1}{2}(q-1)(q-2)$	$2q^2(q-1)^2$
C_{13}^1	$\frac{1}{2}(q-1)(q-2)$	$q^2(q-1)(3q-1)$
C_{13}^3	$\frac{1}{2}(q-1)(q-2)(q-4)$	$q^2(q-1)^2$
C_{14}^2	$\frac{1}{8}(q-1)(q-3)(q-4)$	$q^2(q-1)(3q-1)$
C_{14}^4	$\frac{1}{24}(q-1)(q-2)(q-4)(q-6)$	$q^2(q-1)^2$
C_{16}^1	$\frac{1}{2}q(q-1)$	$q^2(q-1)(3q-1)$
C_{16}^2	$\frac{1}{2}q(q-1)(q-2)$	$q^2(q-1)^2$
C_{17}^1	$\frac{1}{4}q(q-1)(q-2)$	$q^2(q-1)(3q-1)$
C_{17}^2	$\frac{1}{4}q(q-1)(q-2)^2$	$q^2(q-1)^2$
C_{18}	$\frac{1}{2}q(q-1)$	$q(q-1)$
C_{19}	$\frac{1}{2}q(q-1)$	$q(q-1)^2(2q+1)$
C_{20}^1	$\frac{1}{8}q(q-1)(q-2)$	$q^2(q-1)(3q-1)$
C_{20}^2	$\frac{1}{8}q^2(q-1)(q-2)$	$q^2(q-1)^2$
C_{21}	$\frac{1}{3}(q+1)q(q-1)^2$	$q^2(q-1)^2$
C_{22}	$\frac{1}{4}(q+1)q^2(q-1)$	$q^2(q-1)^2$

For $w = w_2$ and $t = \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \mu & \\ & & & \nu \end{pmatrix}$ we have $\chi(etw_2e) = \overline{\chi(et'w_1e)}$, where

$$t' = \begin{pmatrix} \nu^{-1} & & & \\ & \mu^{-1} & & \\ & & \mu^{-1} & \\ & & & \mu^{-1} \end{pmatrix}.$$

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Table 11: $(w = w_0)$

Type of Class	Number of Class	Number of Pairs (t', u)
C_3	$q-1$	$(q-1)^2$
C_4	$q-1$	$q(q-1)^3$
C_5	$q-1$	$q^3(q-1)^3$
C_7	$(q-1)(q-2)$	$(q-1)^2(q^2+1)$
C_8	$(q-1)(q-2)$	$q^2(q-1)^3(q+1)$
C_9	$\frac{1}{2}(q-1)(q-2)$	$(q-1)^2$
C_{10}	$(q-1)(q-2)$	$q(q-1)^3$
C_{11}	$\frac{1}{2}(q-1)(q-2)$	$q(q-1)^2(q^3-q+2)$
C_{12}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$(q-1)^2(q^2+1)$
C_{13}	$\frac{1}{2}(q-1)(q-2)(q-3)$	$q(q-1)^2(q^3+q^2-q+1)$
C_{14}	$\frac{1}{24}(q-1)(q-2)(q-3)(q-4)$	$(q-1)^2(q^4+2q^3+2q+1)$
C_{15}	$\frac{1}{2}q(q-1)^2$	$(q-1)^4$
C_{16}	$\frac{1}{2}q(q-1)^2$	$q(q-1)^2(q^3-q^2+q+1)$
C_{17}	$\frac{1}{4}q(q-1)^2(q-2)$	$(q-1)^2(q^4+1)$
C_{18}	$\frac{1}{2}q(q-1)$	$(q-1)^2$
C_{19}	$\frac{1}{2}q(q-1)$	$q(q-1)^3(q^2-q+2)$
C_{20}	$\frac{1}{8}(q+1)q(q-1)(q-2)$	$(q-1)^2(q^4-2q^3+4q^2-2q+1)$
C_{21}	$\frac{1}{3}(q+1)q(q-1)^2$	$(q-1)^4(q^2+q+1)$
C_{22}	$\frac{1}{4}(q+1)q^2(q-1)$	$(q-1)^4(q^2+1)$

For $w = w_3$ and $t = \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \rho & \\ & & & \rho \end{pmatrix}$ we have $\chi(etw_3e) = \overline{\chi(et'w_4e)}$, where

$$t' = \begin{pmatrix} \rho^{-1} & & & \\ & \rho^{-1} & & \\ & & \nu^{-1} & \\ & & & \mu^{-1} \end{pmatrix}.$$

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For $w = w_4$ and $t = \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \nu & \\ & & & \rho \end{pmatrix}$ we consider the function

$$F_4(r,t) = \begin{cases} -F_4'(r,t) + q^{-4}\psi\left(\frac{-r^3+2\mu\nu^2}{r^2\nu}\right) & \text{if } r \in \mathbf{F}_q^*, \\ -F_4'(r,t) & \text{otherwise}, \end{cases}$$

where

$$F_4'(r,t) = q^{-5} \sum_{\substack{\beta,\delta\in\mathbf{F}_q \\ \beta\delta\nu+\mu\neq 0}} \psi(a_0(r)^{-1}(\beta\delta\nu+\mu)^{-1}(-a_3(r)\beta\mu^2\nu^2 - a_2(r)\mu^2\nu + a_1(r)\delta\mu\nu$$
$$-a_0(r)(\beta^2\delta\nu + \beta\mu + \delta^2\nu) - \beta^2\mu^2\nu^3)).$$

Then $\chi(etw_4e) = \sum_{\substack{r\in T_4 \\ \det r = -\det t}} F_4(r,t)\theta(r)$.

For $w = w_5$ and $t = \begin{pmatrix} \mu & & & \\ & \nu & & \\ & & \nu & \\ & & & \rho \end{pmatrix}$ we consider the function

$$F_5(r,t) = \begin{cases} -(q^{-5}+q^{-4}+q^{-3}) & \text{if } r = \nu, \\ -q^{-5} \sum_{a,\beta\in\mathbf{F}_q^*} \psi\left(-a - \beta - \dfrac{f_r(\nu)\mu}{a_0(r)a\beta}\right) & \text{otherwise}. \end{cases}$$

Then $\chi(etw_5e) = \sum_{\substack{r\in T_4 \\ \det r = -\det t}} F_5(r,t)\theta(r)$.

For $w = w_6$ and $t = \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \nu & \\ & & & \nu \end{pmatrix}$, we put $F_6(r,t)$ for the function

$$F_6(r,t) = -q^{-4}\left(F_6'(r,t) + \sum_{\beta\in\mathbf{F}_q^*} \psi\left(-\beta + \frac{a_1(r) + a_3(r)\mu\nu}{\beta\mu\nu^2}\right)\right),$$

where $F_6'(r,t) = \begin{cases} -q & \text{if } r \in \mathbf{F}_{q^2}\setminus\mathbf{F}_q \text{ and } \mu\nu = -r^{q+1}, \\ 0 & \text{otherwise}. \end{cases}$

Then $\chi(etw_6e) = \sum_{\substack{r\in T_4 \\ \det r = \det t}} F_6(r,t)\theta(r)$.

Finally, let $w = 1$ and $t \in Z$. Easily, we have

$$\chi(ete) = \theta(t).$$

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This is a consequence of Burnside's Theorem, which states that the linear span of the images of the elements of a group G, under an irreducible representation R, is the full matrix algebra of all $k \times k$ complex matrices, where k is the degree of R. Combining Burnside's Theorem with the fact that a cuspidal character of $\mathrm{GL}_n(q)$ has degree $(q^{n-1}-1)\cdots(q-1)$, we get $\chi(ete) = \theta(t)$, for $t \in Z$.

We conclude the paper by giving the action of the affine subgroup A of $\mathrm{GL}_4(q)$ on $\mathbb{C}Gb$. As we have seen this is a monomial action. Subgroup A is generated by the subgroups T_0, X_1, X_2 and X_3, and the Weyl elements s_1 and s_2. Below, we give the action of T_0, X_1, X_2, X_3, s_1 and s_2 on the basis $\{yb \mid y \in Y\}$ of $\mathbb{C}Gb$.

The action of $t_0' \in T_0$

$$t_0'(t_0b) = (t_0't_0)b$$
$$t_0'(x_1t_0s_1b) = x_1^{t_0'^{-1}}(t_0't_0)s_1b$$
$$t_0'(x_2t_0s_2b) = x_2^{t_0'^{-1}}(t_0't_0)s_2b$$
$$t_0'(x_1x_{12}t_0s_1s_2b) = x_1^{t_0'^{-1}}x_{12}^{t_0'^{-1}}(t_0't_0)s_1s_2b$$
$$t_0'(x_2x_{12}t_0s_2s_1b) = x_2^{t_0'^{-1}}x_{12}^{t_0'^{-1}}(t_0't_0)s_2s_1b$$
$$t_0'(x_1x_2x_{12}t_0s_1s_2s_1b) = x_1^{t_0'^{-1}}x_2^{t_0'^{-1}}x_{12}^{t_0'^{-1}}(t_0't_0)s_1s_2s_1b$$

The action of $x_1' \in X_1$

We notice first that we have the following relations:

(i) $x_1'(x_2t_0s_2) = x_2t_0s_2x_1'^{t_0s_2}[x_1'^{-1}, x_2^{-1}]^{t_0s_2}$, with $x_1'^{t_0s_2} \in X_{12}$ and $[x_1'^{-1}, x_2^{-1}]^{t_0s_2} \in X_1$.

(ii) $x_1'(x_2x_{12}t_0s_2s_1) = x_2([x_1'^{-1}, x_2^{-1}]x_{12})t_0s_2s_1x_1'^{t_0s_2s_1}$, with $[x_1'^{-1}, x_2^{-1}]x_{12} \in X_{12}$ and $x_1'^{t_0s_2s_1} \in X_2$.

The action of $x_1' \in X_1$ is given as follows:

$$x_1'(t_0b) = \psi(x_1'^{t_0})(t_0b)$$
$$x_1'(x_1t_0s_1b) = (x_1'x_1)t_0s_1b$$
$$x_1'(x_2t_0s_2b) = \psi([x_1'^{-1}, x_2^{-1}]^{t_0s_2})(x_2t_0s_2b) \text{ by (i)}$$
$$x_1'(x_1x_{12}t_0s_1s_2b) = (x_1'x_1)x_{12}t_0s_1s_2b$$
$$x_1'(x_2x_{12}t_0s_2s_1b) = \psi(x_1'^{t_0s_2s_1})(x_2([x_1'^{-1}, x_2^{-1}]x_{12})t_0s_2s_1b) \text{ by (ii)}$$
$$x_1'(x_1x_2x_{12}t_0s_1s_2s_1b) = (x_1'x_1)x_2x_{12}t_0s_1s_2s_1b.$$

The action of $x_2' \in X_2$

We have the following relations:

(i) $x_2'(x_1t_0s_1) = x_1t_0s_1x_2'^{t_0s_1}[x_2'^{-1}, x_1^{-1}]^{t_0s_1}$, with $x_2'^{t_0s_1} \in X_{12}$ and $[x_2'^{-1}, x_1^{-1}]^{t_0s_1} \in X_2$.

(ii) $x_2'(x_1x_{12}t_0s_1s_2) = x_1([x_2'^{-1}, x_1^{-1}]x_{12})t_0s_1s_2x_2'^{t_0s_1s_2}$, with $[x_2'^{-1}, x_1^{-1}]x_{12} \in X_{12}$ and $x_2'^{t_0s_1s_2} \in X_1$.

(iii) $x_2'(x_1x_2x_{12}t_0s_1s_2s_1) = x_1(x_2'x_2)([x_2'^{-1}, x_1^{-1}]x_{12})t_0s_1s_2s_1$, with $[x_2'^{-1}, x_1^{-1}]x_{12} \in X_{12}$.

The action of $x_2' \in X_2$ is as follows:

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$x_2'(t_0b) = \psi(x_2'^{t_0})(t_0b)$
$x_2'(x_1t_0s_1b) = \psi([x_2'^{-1}, x_1^{-1}]^{t_0s_1})(x_1t_0s_1b)$ by (i)
$x_2'(x_2t_0s_2b) = (x_2'x_2)t_0s_2b$
$x_2'(x_1x_{12}t_0s_1s_2b) = \psi(x_2'^{t_0s_1s_2})(x_1([x_2'^{-1}, x_1^{-1}]x_{12})t_0s_1s_2b)$ by (ii)
$x_2'(x_2x_{12}t_0s_2s_1b) = (x_2'x_2)x_{12}t_0s_2s_1b$
$x_2'(x_1x_2x_{12}t_0s_1s_2s_1b) = x_1(x_2'x_2)([x_2'^{-1}, x_1^{-1}]x_{12})t_0s_1s_2s_1b$ by (iii).

The action of $x_3' \in X_3$

The relations we need here are the following:

(i) $x_3'(x_2t_0s_2) = x_2t_0s_2x_3'^{t_0s_2}[x_3'^{-1}, x_2^{-1}]^{t_0s_2}$, with $x_3'^{t_0s_2} \in X_{23}$ and $[x_3'^{-1}, x_2^{-1}]^{t_0s_2} \in X_3$.

(ii) $x_3'(x_1x_{12}t_0s_1s_2) = x_1x_{12}t_0s_1s_2x_3'^{t_0s_2}[x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2}$, with $x_3'^{t_0s_2} \in X_{23}$ and $[x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2} \in X_3$.

(iii) $x_3'(x_2x_{12}t_0s_2s_1) = x_2x_{12}t_0s_2s_1x_3'^{t_0s_2s_1}[x_3'^{-1}, x_{12}^{-1}]^{t_0s_2s_1}[x_3'^{-1}, x_2^{-1}]^{t_0s_2}$, with $x_3'^{t_0s_2s_1} \in X_{123}$, $[x_3'^{-1}, x_{12}^{-1}]^{t_0s_2s_1} \in X_{23}$ and $[x_3'^{-1}, x_2^{-1}]^{t_0s_2} \in X_3$.

(iv) $x_3'(x_1x_2x_{12}t_0s_1s_2s_1) = x_1x_2x_{12}t_0s_1s_2s_1x_3'^{t_0s_2s_1}[x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2}[x_3'^{-1}, x_2^{-1}]^{t_0}$, with $x_3'^{t_0s_2s_1} \in X_{123}$, $[x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2} \in X_3$ and $[x_3'^{-1}, x_2^{-1}]^{t_0} \in X_{23}$.

The action of $x_3' \in X_3$ is given by

$x_3'(t_0b) = \psi(x_3'^{t_0})(t_0b)$
$x_3'(x_1t_0s_1b) = \psi(x_3'^{t_0})(x_1t_0s_1b)$
$x_3'(x_2t_0s_2b) = \psi([x_3'^{-1}, x_2^{-1}]^{t_0s_2})(x_2t_0s_2b)$ by (i)
$x_3'(x_1x_{12}t_0s_1s_2b) = \psi([x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2})(x_1x_{12}t_0s_1s_2b)$ by (ii)
$x_3'(x_2x_{12}t_0s_2s_1b) = \psi([x_3'^{-1}, x_2^{-1}]^{t_0s_2})(x_2x_{12}t_0s_2s_1b)$ by (iii)
$x_3'(x_1x_2x_{12}t_0s_1s_2s_1b) = \psi([x_3'^{-1}, x_{12}^{-1}]^{t_0s_1s_2})(x_1x_2x_{12}t_0s_1s_2s_1b)$ by (iv).

The action of s_1

We have the following relations:

$$\text{(i)}\quad s_1(x_1t_0s_1) = \begin{cases} t_0^{s_1} \ \text{ if } \ x_1 = 1; \\ y_1(t_0't_0)s_1y_1^{s_1t_0s_1} \ \text{ if } \ x_1 \neq 1 \ \text{ with } \ y_1 \in X_1, \\ t_0' \in T_0 \ \text{ such that } \ x_1^{s_1} = y_1t_0's_1y_1. \end{cases}$$

$$\text{(ii)}\quad s_1(x_1x_{12}t_0s_1s_2) = \begin{cases} x_{12}^{s_1}t_0^{s_1}s_2 \ \text{ if } \ x_1 = 1; \\ y_1x_{12}^{t_0'^{-1}}(t_0't_0)s_1s_2y_1^{s_1t_0s_1s_2}[(y_1^{s_1t_0s_1})^{-1}, (x_{12}^{t_0s_1})^{-1}]^{s_2} \ \text{ if } \ x_1 \neq 1 \\ \text{with } \ y_1 \in X_1, t_0' \in T_0 \ \text{ such that } \ x_1^{s_1} = y_1t_0's_1y_1, \\ y_1^{s_1t_0s_1s_2} \in X_{12} \ \text{ and } \ [(y_1^{s_1t_0s_1})^{-1}, (x_{12}^{t_0s_1})^{-1}]^{s_2} \in X_1. \end{cases}$$

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$$\text{(iii)}\quad s_1(x_1x_2x_{12}t_0s_1s_2s_1) = \begin{cases} x_2^{s_1}x_{12}^{s_1}t_0^{s_1}s_2s_1 \ \text{ if } \ x_1 = 1; \\ y_1(x_2[y_1^{-1}, (x_{12}^{s_1})^{-1}]^{s_1})^{t_0^{'-1}} x_{12}^{t_0^{'-1}} (t_0^{'}t_0)s_1s_2s_1y_1^{s_1t_0s_1s_2s_1} \\ \text{if } \ x_1 \neq 1 \ \text{ with } \ y_1 \in X_1, t_0^{'} \in T_0 \ \text{ such that} \\ x_1^{s_1} = y_1t_0^{'}s_1y_1 \quad \text{ and } \ y_1^{s_1t_0s_1s_2s_1} \in X_2. \end{cases}$$

From the above relations we obtain the action of s_1:

$$s_1(t_0b) = t_0^{s_1}s_1b$$

$$s_1(x_1t_0s_1b) = \begin{cases} t_0^{s_1}b \ \text{ if } \ x_1 = 1; \\ \psi(y_1^{s_1t_0s_1})(y_1(t_0^{'}t_0)s_1b) \ \text{ if } \ x_1 \neq 1 \ \text{ with } \ y_1 \in X_1, \\ t_0^{'} \in T_0 \ \text{ such that } \ x_1^{s_1} = y_1t_0^{'}s_1y_1. \end{cases}$$

$$s_1(x_2t_0s_2b) = x_2^{s_1}t_0^{s_1}s_1s_2b$$

$$s_1(x_1x_{12}t_0s_1s_2b) = \begin{cases} x_{12}^{s_1}t_0^{s_1}s_2b \ \text{ if } \ x_1 = 1; \\ \psi([(y_1^{s_1t_0s_1})^{-1}, (x_{12}^{t_0s_1})^{-1}]^{s_2})(y_1x_{12}^{t_0^{'-1}}(t_0^{'}t_0)s_1s_2b) \ \text{ if } \ x_1 \neq 1 \\ \text{with } \ y_1 \in X_1, t_0^{'} \in T_0 \ \text{ such that } \ x_1^{s_1} = y_1t_0^{'}s_1y_1. \end{cases}$$

$$s_1(x_2x_{12}t_0s_2s_1b) = x_{12}^{s_1}x_2^{s_1}t_0^{s_1}s_1s_2s_1b$$

$$s_1(x_1x_2x_{12}t_0s_1s_2s_1b) = \begin{cases} x_2^{s_1}x_{12}^{s_1}t_0^{s_1}s_2s_1b \ \text{ if } \ x_1 = 1; \\ \psi(y_1^{s_1t_0s_1s_2s_1})(y_1(x_2[y_1^{-1}, (x_{12}^{s_1})^{-1}]^{s_1})^{t_0^{'-1}} x_{12}^{t_0^{'-1}} (t_0^{'}t_0)s_1s_2s_1b) \\ \text{if } \ x_1 \neq 1 \ \text{ with } \ y_1 \in X_1, t_0^{'} \in T_0 \ \text{ such that } \ x_1^{s_1} = y_1t_0^{'}s_1y_1. \end{cases}$$

The action of s_2

For the generator s_2 we need the relations:

$$\text{(i)}\quad s_2(x_2t_0s_2) = \begin{cases} t_0^{s_2} \ \text{ if } \ x_2 = 1; \\ y_2(t_0^{'}t_0)s_2y_2^{s_2t_0s_2} \ \text{ if } \ x_2 \neq 1 \ \text{ with } \ y_2 \in X_2, \\ t_0^{'} \in T_0 \ \text{ such that } \ x_2^{s_2} = y_2t_0^{'}s_2y_2. \end{cases}$$

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$$\text{(ii)}\quad s_2(x_2x_{12}t_0s_2s_1) = \begin{cases} x_{12}^{s_2}t_0^{s_2}s_1 \ \text{ if } \ x_2 = 1; \\ y_2x_{12}^{t_0'^{-1}}(t_0't_0)s_2s_1y_2^{s_2t_0s_2s_1}[(y_2^{s_2t_0s_2})^{-1}, (x_{12}^{t_0s_2})^{-1}]^{s_1} \ \text{ if } \ x_2 \neq 1 \\ \text{with } \ y_2 \in X_2, t_0' \in T_0 \ \text{ such that } \ x_2^{s_2} = y_2t_0's_2y_2, \\ y_2^{s_2t_0s_2s_1} \in X_{12} \ \text{ and } \ [(y_2^{s_2t_0s_2})^{-1}, (x_{12}^{t_0s_2})^{-1}]^{s_1} \in X_2. \end{cases}$$

$$\text{(iii)}\quad s_2(x_1x_2x_{12}t_0s_1s_2s_1) = \begin{cases} x_1^{s_2}x_{12}^{s_2}t_0^{s_2}s_1s_2 \ \text{ if } \ x_2 = 1; \\ [y_2^{-1}, (x_{12}^{s_2})^{-1}]^{s_2t_0'^{-1}}y_2([y_2^{-1}, [(x_{12}^{s_2})^{-1}, y_2^{-1}]^{s_2t_0'^{-1}}]x_1^{s_2}x_{12}^{t_0'^{-1}}) \\ (t_0't_0)s_1s_2s_1y_2^{s_2t_0s_1s_2s_1} \ \text{ if } \ x_2 \neq 1 \ \text{ with } \ y_2 \in X_2, t_0' \in T_0 \\ \text{such that } \ x_2^{s_2} = y_2t_0's_2y_2, [y_2^{-1}, (x_{12}^{s_2})^{-1}]^{s_2t_0'^{-1}} \in X_1, \\ [y_2^{-1}, [(x_{12}^{s_2})^{-1}, y_2^{-1}]^{s_2t_0'^{-1}}]x_1^{s_2}x_{12}^{t_0'^{-1}} \in X_{12} \\ \text{and } \ y_2^{s_2t_0s_1s_2s_1} \in X_1. \end{cases}$$

These relations give the following action:

$$s_2(t_0b) = t_0^{s_2}s_2b$$

$$s_2(x_1t_0s_1b) = x_1^{s_2}t_0^{s_2}s_2s_1b$$

$$s_2(x_2t_0s_2b) = \begin{cases} t_0^{s_2}b \ \text{ if } \ x_2 = 1; \\ \psi(y_2^{s_2t_0s_2})(y_2(t_0't_0)s_2b) \ \text{ if } \ x_2 \neq 1 \ \text{ with } \ y_2 \in X_2, \\ t_0' \in T_0 \ \text{ such that } \ x_2^{s_2} = y_2t_0's_2y_2. \end{cases}$$

$$s_2(x_1x_{12}t_0s_1s_2b) = x_{12}^{s_2}x_1^{s_2}t_0^{s_2}s_1s_2s_1b$$

$$s_2(x_2x_{12}t_0s_2s_1b) = \begin{cases} x_{12}^{s_2}t_0^{s_2}s_1b \ \text{ if } \ x_2 = 1; \\ \psi([(y_2^{s_2t_0s_2})^{-1}, (x_{12}^{t_0s_2})^{-1}]^{s_1})(y_2x_{12}^{t_0'^{-1}}(t_0't_0)s_2s_1b) \ \text{ if } \ x_2 \neq 1 \\ \text{with } \ y_2 \in X_2, t_0' \in T_0 \ \text{ such that } \ x_2^{s_2} = y_2t_0's_2y_2. \end{cases}$$

$$s_2(x_1x_2x_{12}t_0s_1s_2s_1b) = \begin{cases} x_{12}^{s_2}x_1^{s_2}t_0^{s_2}s_1s_2b \ \text{ if } \ x_2 = 1; \\ \psi(y_2^{s_2t_0s_1s_2s_1})([y_2^{-1}, (x_{12}^{s_2})^{-1}]^{s_2t_0'^{-1}}y_2([y_2^{-1}, [(x_{12}^{s_2})^{-1}, y_2^{-1}]^{s_2t_0'^{-1}}] \\ x_1^{s_2}x_{12}^{t_0'^{-1}})(t_0't_0)s_1s_2s_1b) \ \text{ if } \ x_2 \neq 1 \ \text{ with } \ y_2 \in X_2, t_0' \in T_0 \\ \text{such that } \ x_2^{s_2} = y_2t_0's_2y_2. \end{cases}$$

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