THE SPIN REPRESENTATION OF THE SYMMETRIC GROUP

A. O. MORRIS

1. Let Γ_n be the *representation group* or *spin group* (**9**; **4**) of the symmetric group S_n. Then the irreducible representations of Γ_n can be allocated into two classes which we shall call (i) *ordinary* representations, which are the irreducible representations of the symmetric group, and (ii) *spin* or *projective* representations.

As is well known (**3**; **5**), there is an ordinary irreducible representation $[\lambda]$ corresponding to every partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n with

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0.$$

Of fundamental importance in the study of the ordinary representation is the concept of a *hook graph* (**2**; **6**; **8**). Our aim in this note is to develop a similar concept for the spin representations of Γ_n.

There is an irreducible spin representation $\langle\lambda\rangle$ of Γ_n corresponding to every partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n with $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$. In the following, we shall say that a partition satisfies *Condition* A if it has no equal parts, where the parts are not necessarily in descending order.

Any partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ can be associated with a *graph* consisting of rows of symbols called *nodes*. λ_1 in the first row, λ_2 in the second row, ..., λ_m in the last row. The node in the ith row and jth column of $\langle\lambda\rangle$ is called its (i,j)-node.

Definition 1. An $(i,j)_{(k)}$-bar $(k = i, i+1, \ldots, m)$ of $\langle\lambda\rangle$ will consist of the (i,j)-node together with the remaining $\lambda_i - j$ nodes to the right of it and

(i) the $\lambda_k - j + 1$ nodes from the kth row of $\langle\lambda\rangle$ if $k > i$,

(ii) no further nodes if $k = i$,

so that the resulting graph on deleting these nodes satisfies Condition A.

From this definition, it is clear that there is more than one $(i,j)_{(k)}$-bar attached to the (i,j)-node. In fact, if r_{ij} denotes the number of $(i,j)_{(k)}$-bars attached to the (i,j)-node, then $0 \leqslant r_{ij} \leqslant m - i + 1$. Further, since the resulting graphs on deleting the nodes have to satisfy Condition A, we have that

$$r_{i1} = m - i + 1 \qquad (i = 1, 2, \ldots, m),$$
$$r_{ij} = \begin{cases} 1, & \text{if } j > 1 \text{ and } j - 1 \neq \lambda_k \text{ for any } k, \\ 0, & \text{if } j > 1 \text{ and } j - 1 = \lambda_k \text{ for some } k. \end{cases}$$

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Definition 2. The *length* of an $(i,j)_{(k)}$-bar is

$$S_{ij}^{(k)} = \begin{cases} \lambda_i - j + 1 & \text{if } k = i, \\ \lambda_i + \lambda_k - 2j + 2 & \text{if } k > i. \end{cases}$$

In an $(i,j)_{(k)}$-bar $(k \geqslant i)$, the number $q = \lambda_k - j + 1$ is called the *arm length* of the $(i,j)_{(k)}$-bar. An $(i,j)_{(k)}$-bar with arm length q will be called a *q-bar*. An $(i,j)_{(i)}$-bar will be called an *O*-bar.

We easily see that

$$S_{i1}^{(k)} = \begin{cases} \lambda_i & \text{if } k = i, \\ \lambda_i + \lambda_k & \text{if } k > i. \end{cases}$$

Let

$$S_{i1} = \prod_{k=i}^{m} s_{i1}^{(k)},$$

and

$$S_{ij} = \begin{cases} \lambda_i - j + 1 & \text{if } j > 1 \text{ and } j - 1 \neq \lambda_k \text{ for any } k, \\ 1 & \text{if } j > 1 \text{ and } j - 1 = \lambda_k \text{ for some } k. \end{cases}$$

Definition 3. The graph obtained by replacing the (i,j)-node of $\langle\lambda\rangle$ by S_{ij} is called the *bar graph* $S\langle\lambda\rangle$. The *bar product* S^λ is the product of all the S_{ij}'s in $S\langle\lambda\rangle$, that is,

$$S^\lambda = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} S_{ij}.$$

2. The degree of an irreducible spin representation $\langle\lambda\rangle$ of Γ_n. Schur **(9)** has shown that the degree f^λ of $\langle\lambda\rangle$ is given by the formula

$$f^\lambda = 2^{[\frac{1}{2}(n-m)]} \frac{n!}{\lambda_1!\,\lambda_2!\,\ldots\,\lambda_m!} \prod_{1 \leqslant r < s \leqslant m} \frac{\lambda_r - \lambda_s}{\lambda_r + \lambda_s}, \tag{1}$$

where $[x]$ denotes the greatest integer less than or equal to x. We now prove the following.

THEOREM 1. *Let S^λ denote the bar product of an irreducible spin representation $\langle\lambda\rangle$ of Γ_n. Then the degree f^λ of $\langle\lambda\rangle$ is given by the formula*

$$f^\lambda = \frac{2^{[\frac{1}{2}(n-m)]}\, n!}{S^\lambda}.$$

The proof follows closely the proof of the corresponding theorem on the degree of the ordinary representations of Γ_n in terms of the hook product **(2)**.

From § 1, we have

$$\prod_{j=1}^{\lambda_i} S_{ij} = \lambda_i \prod_{k=i+1}^{m} (\lambda_i + \lambda_k) \prod_{j} (\lambda_i - j + 1),$$

where $j = 2, \ldots, \lambda_m, \lambda_m + 2, \ldots, \lambda_{m-1}, \lambda_{m-1} + 2, \ldots, \lambda_{i-1}, \lambda_{i-1} + 2, \ldots, \lambda_i$. Thus, we see that

$$\prod_{j=1}^{\lambda_i} S_{ij} = \lambda_i \prod_{k=i+1}^{m} (\lambda_i + \lambda_k) \prod_{j=2}^{\lambda_i} (\lambda_i - j + 1) \Big/ \prod_{k=i+1}^{m} (\lambda_i - \lambda_k)$$
$$= \lambda_i! \prod_{k=i+1}^{m} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}.$$

Hence, it follows that

$$S^\lambda = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} S_{ij} = \lambda_1!\, \lambda_2! \ldots \lambda_m! \prod_{1 \leqslant i < k \leqslant m} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k},$$

and from (1)

$$f^\lambda = \frac{2^{[\frac{1}{2}(n-m)]} n!}{S^\lambda}.$$

Example. We find the degree of the irreducible spin representation $\langle\lambda\rangle = \langle 863\rangle$ of Γ_{17}. $\langle\lambda\rangle$ has the graph

$$\langle\lambda\rangle \qquad \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & & & & \end{matrix}$$

and the bar graph

$$\begin{matrix} S^\lambda & 14 \times 11 \times 8 & 7 & 6 & 1 & 4 & 3 & 1 & 1 \\ & 9 \times 6 & 5 & 4 & 1 & 2 & 1 & & \\ & 3 & 2 & 1 & & & & & \end{matrix}$$

Hence,

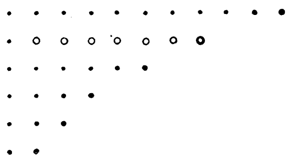
$$f^\lambda = \frac{2^7 \times 17!}{14 \times 11 \times 9 \times 8 \times 7 \times 6 \times 4 \times 3 \times 6 \times 5 \times 4 \times 2 \times 3 \times 2 \times 1}$$
$$= 5{,}657{,}600.$$

3. The removal of a bar from a graph. First, we consider the effect of removing an O-bar from the graph $\langle\lambda\rangle$ corresponding to the partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n. Suppose that an O-bar is removed from the (i, j)-node of this graph, that is, the $\lambda_i - j + 1$ nodes from the ith row of this graph to the right of and including the (i, j)-node, assuming that $j - 1 \neq \lambda_k$ for any k. Then $j - 1 > \lambda_k$ and $j - 1 < \lambda_{k-1}$ for some value of k, $i \leqslant k \leqslant m$. Move up the nodes which are below the empty spaces to obtain a graph with parts in descending order, that is, the graph of the partition

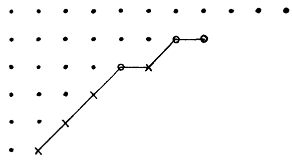
$$(\lambda)' = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{k-1}, j - 1, \lambda_k, \ldots, \lambda_m)$$

of $n - \lambda_i + j - 1$. Sometimes, it is more convenient to consider removing the *equivalent rim* of the graph $\langle\lambda\rangle$, that is, $\langle\lambda\rangle'$ is obtained from $\langle\lambda\rangle$ by removing $\lambda_i - \lambda_{i+1}$ nodes from the ith row, $\lambda_{i+1} - \lambda_{i+2}$ nodes from the $(i+1)$th row, ..., $\lambda_{k-1} - j + 1$ nodes from the kth row.

For example, we remove an O-bar from the $(2, 2)$-node of the graph $\langle 11, 8, 6, 4, 3, 2\rangle$.



is equivalent to



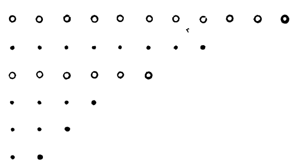
We shall call the equivalent rim removed a *skew* O-bar. An (i, j)-node on the skew O-bar is called a *leg node* of the skew O-bar if the $(i-1, j+1)$-node also lies on the skew O-bar. The total number of leg nodes is called the *leg length* of the skew O-bar. The leg length is clearly $k - i - 1$. In the above example, the nodes indicated by $\times$ are leg nodes and thus the leg length of the skew O-bar is 4.

Next, we consider the effect of removing a q-bar from the graph $\langle\lambda\rangle$, for some value of q. Suppose that we remove an $(i, 1)_k$-bar $(k > i)$ from $\langle\lambda\rangle$, that is, $q = \lambda_k$. The resulting graph is

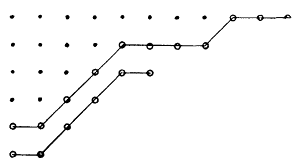
$$\langle\lambda\rangle' = \langle\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m\rangle$$

of $n - \lambda_i - \lambda_k$. Thus, $\langle\lambda\rangle'$ is obtained from $\langle\lambda\rangle$ by removing $\lambda_i - \lambda_{i+1}$ nodes from the ith row, $\lambda_{i+1} - \lambda_{i+2}$ nodes from the $(i+1)$th row, ..., $\lambda_{k-1} - \lambda_{k+1}$ nodes from the $(k-1)$th row, $\lambda_k - \lambda_{k+2}$ nodes from the kth row, $\lambda_{k+1} + \lambda_{k+3}$ nodes from the $(k+1)$th row, ..., $\lambda_{m-2} - \lambda_m$ nodes from the $(m-2)$th row, λ_{m-1} nodes from the $(m-1)$th row, and λ_m nodes from the mth row. Hence, this is equivalent to removing two skew O-bars from $\langle\lambda\rangle$; first remove a skew O-bar of length λ_k and then remove a skew O-bar of length λ_i. This equivalent rim removed is called a *skew* λ_k-*bar*. The total number of leg nodes in both parts of the skew λ_k-bar is called the *leg length* of the skew λ_k-bar. In a skew λ_k-bar, λ_k is the *arm length* defined in § 1.

This is now illustrated by an example. Suppose that we remove a $(1, 1)_{(3)}$-bar from $\langle\lambda\rangle = \langle 11, 8, 6, 4, 3, 2\rangle$. That is



or equivalently



The leg length of this skew 6-bar is $4 + 3 = 7$, and the arm length is 6.

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4. The spin characters of Γ_n. Using the terminology of § 3, we can now prove the following theorem, which corresponds to the well-known Murnaghan–Nakayama formula **(8)** for the ordinary characters of Γ_n.

THEOREM 2. *Let T denote the graph corresponding to the partition (λ) of n and $T - S_i$ the graph obtained by removing a bar S_i of length i from T. Let $\zeta_\pi(T)$ denote the irreducible spin character of the positive class $(\pi) = (1^{\alpha_1}3^{\alpha_3}5^{\alpha_5}\ldots)$ of Γ_n corresponding to the graph T, then*

$$\zeta_\pi(T) = \sum_i (-1)^{h_i+k_i} 2^{[\frac{1}{2}(\epsilon'-\epsilon+1)]}\zeta_{\pi'}(T - S_i),$$

where the summation runs over all possible bars S_i of length i which can be removed from T; h_i and k_i denote the arm length and leg length respectively of S_i; π' is obtained from π by deleting a cycle of length i; ϵ and ϵ' are 0 or 1 according as $\zeta_\pi(T)$ and $\zeta_{\pi'}(T - S_i)$ are double or associate spin characters.

Schur **(9)** has shown that the irreducible spin characters of Γ_n are generated by a certain class of symmetric functions Q_λ, known as Q-functions **(5)**, with the property that

$$Q_\lambda = \sum_\pi \frac{h_\pi}{2.n!} 2^{\frac{1}{2}(p+m+\epsilon)} \zeta_\pi^{\lambda} S_\pi,$$

where $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $(\pi) = (1^{\alpha_1}3^{\alpha_3}5^{\alpha_5}\ldots)$, $p = \alpha_1 + \alpha_3 + \alpha_5 + \ldots$, and $\epsilon = 0$ or 1 according as ζ_π^λ is a double spin character or an associate spin character.

Remark. If ζ_ρ^λ is a spin character, then $\zeta_\rho^{\lambda'} = (-1)^\alpha \zeta_\rho^\lambda$, where $\alpha = 0$ or 1 according as (ρ) is a positive or negative class, is a second spin character. ζ_ρ^λ and $\zeta_\rho^{\lambda'}$ are known as associate spin characters. If $\zeta_\rho^\lambda = \zeta_\rho^{\lambda'}$ for all classes (ρ), then ζ_ρ^λ is a double spin character.

In **(4)**, certain rules have been proved whereby a Q_λ corresponding to a partition (λ) with some parts possibly negative or zero and not necessarily in descending order is written in terms of a $Q_{(\lambda)}$ with positive parts in descending order. These rules are

(i) if any two parts of Q_λ are equal, then $Q_\lambda \equiv 0$;

(ii) $Q_{(\lambda_1\lambda_2\ldots\lambda_r\lambda_{r+1}\ldots\lambda_m)} = -Q_{(\lambda_1\lambda_2\ldots\lambda_{r+1}\lambda_r\ldots\lambda_m)}$,

(iii) $Q_{(\lambda_1\lambda_2\ldots\lambda_m)} \equiv 0$ if $\lambda_i < 0$ for any $1 \leqslant i \leqslant m$ and $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|$ are all different,

(iv) $Q_{(\lambda_1,\lambda_2,\ldots,-\lambda_r,\lambda_r,\ldots,\lambda_m)} = 2(-1)^{\lambda_r} Q_{(\lambda_1\lambda_2\ldots\lambda_{r-1}\lambda_{r+1}\ldots\lambda_m)}$ and

$$Q_{(\lambda_1\lambda_2\ldots\lambda_r,-\lambda_r,\ldots,\lambda_m)} \equiv 0.$$

In **(5)**, it has been proved that

$$\zeta_\pi^\lambda = \sum_{(\mu)} k_{(\mu)} 2^{\frac{1}{2}(m'-m+\epsilon'-\epsilon+1)}\zeta_{\pi'}^{\mu},$$

where the summation is taken over all partitions $(\mu) = (\mu_1, \mu_2, \ldots, \mu_{m'})$ corresponding to the Q_μ obtained from $Q_{(\lambda_{12},\lambda,\ldots,\lambda_{j-i},\ldots,\lambda_m)}$ $(j = 1, 2, \ldots, m)$

by means of rules (i)–(iv) above, $k_{(\mu)}$ is the coefficient of Q_μ and $(\pi)'$ is the class obtained from (π) by deleting a cycle of length i. Thus, in order to prove this theorem, we must show that

$$k_{(\mu)} 2^{\frac{1}{2}(m'-m+\epsilon'-\epsilon+1)} = (-1)^{h_i+k_i} 2^{\frac{1}{2}(\epsilon'-\epsilon+1)}. \tag{2}$$

From the rules (i)–(iv) above, $k_{(\mu)} \neq 0$ if and only if

(a) $\lambda_j - i \neq \lambda_k$ for any $k > j$,

(b) $\lambda_j - i = 0$,

(c) $\lambda_j - i + \lambda_k = 0$, for some $k > j$.

Case (a) is equivalent to removing an O-bar of length i from $\langle\lambda\rangle$. If $\lambda_j - 1 \neq 0$, $m' = m$, and if k_i denotes the arm length of the O-bar, $k_i = 0$. Use (i) repeatedly until the partition has parts in descending order to give the value of k_μ. Clearly, by the definition of the leg length h_i of the skew O-bar, $k_\mu = (-1)^{h_i}$, and thus (2) follows in this case.

In case (b), $m' = m - 1$ and thus $m' - m + 1 + \epsilon' - \epsilon = \epsilon' - \epsilon = 0$ and by the same argument as for case (a), $k_{(\mu)} = (-1)^{h_i}$, and thus (2) follows in this case.

Case (c) is equivalent to removing a λ_k-bar of length i from $\langle\lambda\rangle$. Now $m' - m = -2$, and thus

$$k_\mu 2^{\frac{1}{2}(m'-m+1+\epsilon'-\epsilon)} = \tfrac{1}{2} k_\mu \, 2^{\frac{1}{2}(1+\epsilon'-\epsilon)}$$

and by rules (i) and (iv) above, $k_\mu = (-1)^{h_i+k_i}$, where h_i is the length of the skew λ_k-bar and $k_i = \lambda_k$ is the arm length of the skew λ_k-bar. Thus (b) follows in this case again, and the proof of the theorem is complete.

It would be of interest to obtain a direct proof of this theorem of the same type as the proof of the Murnaghan–Nakayama recursion formula given by Robinson **(7)**. In order that this may be done, a theory corresponding to the theory of Young diagrams must be developed for the graph $\langle\lambda\rangle$.

5. In a future publication, the theory of bar graphs will be applied to the study of the modular representations of the group Γ_n. It will be shown that bar graphs play a similar role for the modular representations of Γ_n as hook graphs play for the modular representation of the group S_n. For instance, we shall prove a result corresponding to the well-known Nakayama conjecture **(6)** first proved by Brauer and Robinson **(1)**. If we define the p-core of a graph $\langle\lambda\rangle$ to be the graph obtained after removing all possible bars of length p from $\langle\lambda\rangle$, then we shall prove that irreducible spin representations of Γ_n belong to the same p-block if and only if their corresponding graphs have the same p-core.

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University College of Wales,
Aberystwyth, Wales