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# Canonical transformation of polynomial hamiltonians

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This thesis is an investigation of time-dependent canonical transformations as applied to polynomial hamiltonians. The motivation came from two sources. The first was the work of Lewis [*14*, *15*, *16*, *17*] on an exact invariant for the time-dependent harmonic oscillator. The second was the construction by Fradkin ([*2*]; see also [*4*]) of an invariant tensor for the time-independent three dimensional isotropic oscillator. This tensor, together with the angular momentum, was used as a basis for the generators of the dynamical symmetry group of that system, namely SU(3) .

In Part A, the theory and application of time-dependent linear canonical transformation of quadratic hamiltonians is developed. The Lewis invariant is derived from a time-independent hamiltonian. Similar invariants are shown to exist for all time-dependent quadratic hamiltonians, with particular instances being the damped and forced harmonic oscillator, the harmonic oscillator whose source of potential is undergoing translation and the time-dependent anisotropic oscillator.

The equivalence of all quadratic hamiltonians to the same-dimensioned isotropic oscillator leads to a generalization of the Jauch-Hill-Fradkin tensor to all such hamiltonians. (The generalization to simple time-dependent oscillators was undertaken with Günther [*3*].) Every quadratic hamiltonian has an associated invariant with which the group SU(3) may be associated. In general this is not an invariance symmetry group of the hamiltonian.

The corresponding Schrödinger equation for general oscillator systems

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may be solved using time-dependent linear canonical transformations. The Schrödinger wave function is found by means of an integral transform whose kernel is essentially the complex exponentiation of the appropriate generating function of the transformation. In some cases this reduces to a simple multiplier function. The eigenvalues may be obtained directly by the transform method. This method is valid only when the signatures of the two quadratic hamiltonians are the same.

In Part B some work is done on the application of non-linear time-dependent transformations to non-linear systems. This work was motivated by a suggestion of Lewis (private communication) in the hope that it would be of some help in current problems in plasma physics. The work is formal in approach and shows how to construct invariant series for time-dependent systems, in particular the time-dependent oscillator with cubic anharmonicity. It is possible that such series are ultimately divergent. This part concludes with the derivation of the action-angle transformation for the time-independent harmonic oscillator using a Laurent series.

Much of the material in the thesis has either been published [*1*, 5, *6*, *7*, *8*] or has been accepted for publication [*9*, *10*, *11*]. Although the results in Part B were largely inconclusive, subsequent work has shown that the considerations there are of relevance in the search for a "third integral" in the Hénon-Heiles problem [*12*]. Further work has also been undertaken on the classification of invariants for quadratic hamiltonians on the basis of their utility in the determination of invariance symmetry groups [*13*].

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