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A GENERALIZATION OF FINAL RANK OF PRIMARY ABELIAN GROUPS

DOYLE O. CUTLER AND PAUL F. DUBOIS

Let G be a p-primary Abelian group. Recall that the final rank of G is $\inf_{n\in\omega}\{r(p^nG)\}$, where $r(p^nG)$ is the rank of p^nG and ω is the first limit ordinal. Alternately, if Γ is the set of all basic subgroups of G, we may define the final rank of G by $\sup_{B\in\Gamma}\{r(G/B)\}$. In fact, it is known that there exists a basic subgroup B of G such that $r(G/B)$ is equal to the final rank of G. Since the final rank of G is equal to the final rank of a high subgroup of G plus the rank of $p^\omega G$, one could obtain the same information if the definition of final rank were restricted to the class of p-primary Abelian groups of length ω.

In this paper we show the existence of appropriate generalizations of these two definitions of final rank; and, when the length of G is an accessible limit ordinal (the limit of a countable increasing sequence of lesser ordinals), we show that the two resulting cardinals are indeed the same. The notation is pretty close to that of [**1**] or [**3**]. We use $\langle\ldots\rangle$ for "subgroup generated by . . .", and ordinals are in the sense of von Neumann.

Let G be a reduced p-primary Abelian group. Let

$$pG = \{x \in G \mid x = py \text{ for some } y \in G\}.$$

Inductively we define

$$p^{\beta+1}G = p(p^\beta G) \qquad \text{and} \qquad p^\alpha G = \bigcap_{\beta\in\alpha} p^\beta G$$

for α a limit ordinal. A subgroup H of G is called p^α-pure in G if

$$0 \to H \to G \to G/H \to 0$$

represents an element of $p^\alpha\,\mathrm{Ext}(G/H, H)$. For α a limit ordinal let

$$\Gamma_\alpha = \{H \mid H \text{ is a } p^\alpha\text{-pure subgroup of } G \text{ and } G/H \text{ is divisible}\}.$$

Then the following two generalizations of final rank can be defined:

$$r_\alpha(G) = \sup_{H\in\Gamma_\alpha}\{r(G/H)\}, \tag{1}$$

$$s_\alpha(G) = \inf_{\beta\in\alpha}\{r(p^\beta G[p])\}. \tag{2}$$

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THEOREM 1. *Let G be a reduced p-primary Abelian group. Then $r_\alpha(G) \leqq s_\alpha(G)$.*

Proof. For $H \in \Gamma_\alpha$, the following hold (see [**4**]):
(a) $p^\beta G \cap H = p^\beta H$ for all $\beta \in \alpha$,
(b) $\langle p^\beta G, H\rangle = G$ for all $\beta \in \alpha$.
Thus $r(G/H) = r(\langle p^\beta G, H\rangle/H) = r(p^\beta G/p^\beta H)$ for all $\beta \in \alpha$. Now (a) implies that $p^\beta H$ is pure in $p^\beta G$. Thus, if $\{\bar{x}_\zeta\}_{\zeta\in A}$ is a basis of $(p^\beta G/p^\beta H)[p]$, we can choose $x_\zeta \in p^\beta G[p]$ so that $x_\zeta + p^\beta H = \bar{x}_\zeta$. Then $\{x_\zeta\}_{\zeta\in A}$ is linearly independent and so

$$r(p^\beta G/p^\beta H) = r((p^\beta G/p^\beta H)[p]) \leqq r(p^\beta G[p])$$

for all $\beta \in \alpha$. Therefore $r(G/H) \leqq s_\alpha(G)$ for each $H \in \Gamma_\alpha$. Hence $r_\alpha(G) \leqq s_\alpha(G)$.

THEOREM 2. *Let G be a reduced p-primary Abelian group of length α, where $\alpha = \bigcup_{i\in\omega} \alpha_i$ ($\alpha_i \in \alpha_{i+1} \in \alpha$ for all $i \in \omega$). Then $r_\alpha(G) \geqq s_\alpha(G)$.*

Proof. Let G_i be a chain of $p^{\alpha_i}G$-high subgroups of G; that is, $G_i \subseteq G_{i+1}$ for all $i \in \omega$ and G_i is maximal with respect to $G_i \cap p^{\alpha_i}G = 0$. Define $P_0 = G_0[p]$, and for $i > 0$ choose P_i such that $G_i[p] = G_{i-1}[p] \oplus P_i$. Note that for all $\beta \in \alpha$,

$$G[p] \subseteq \left\langle \sum_{i\in\omega} P_i,\ (p^\beta G)[p]\right\rangle;$$

i.e., $\sum_{i\in\omega} P_i$ is a dense subsocle of $G[p]$ in the relative p^α-topology.

Note that $\inf_{\beta\in\alpha}|p^\beta G[p]| = \aleph \geqq \aleph_0$. Either $\lim_{k\to\infty}|\sum_{i=k}^\infty P_i| = \aleph$ or $\lim_{k\to\infty}|\sum_{i=k}^\infty P_i| < \aleph$.

Case I. $\lim_{k\to\infty}|\sum_{i=k}^\infty P_i| < \aleph$. Since $|\sum P_i| = \sum |P_i|$ and since the cardinals are well-ordered, there exists an $i_0 \in \omega$ such that

$$\left|\sum_{i=i_0}^\infty P_i\right| = \lim_{k\to\infty}\left|\sum_{i=k}^\infty P_i\right|.$$

Let K be a neat subgroup of G such that $K[p] = \sum_{i=0}^\infty P_i$. (We need only choose K containing $\sum_{i=0}^\infty P_i$ and maximal with respect to the property of being disjoint from a complementary summand of $\sum_{i=0}^\infty P_i$ in $G[p]$.) Since $K[p]$ is dense in $G[p]$ with respect to the relative p^α-topology, we have, by [**4**, Theorem 2.9], that K is a p^α-pure subgroup of G. Note that G/K is divisible since it is easy to show that $\langle K, p^\beta G\rangle = G$ for all $\beta \in \alpha$. From [**4**, p. 196], we have that K is isotype in G. Thus

$$G/K = \langle p^{i_0}G, K\rangle/K \cong p^{i_0}G/(p^{i_0}G \cap K) = p^{i_0}G/p^{i_0}K.$$

Now $|p^{i_0}K[p]| = |\sum_{i=i_0}^\infty P_i| < \aleph$. Choose L such that

$$(p^{i_0}G)[p] = L \oplus p^{i_0}K[p].$$

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Since $|p^{i_0}G[p]| \geqq \aleph$, $|L| \geqq \aleph$. Then if $\{x_\zeta\}_{\zeta\in A}$ is a basis of L, $\{x_\zeta + p^{i_0}K\}_{\zeta\in A}$ is linearly independent and hence $|G/K| = |p^{i_0}G/p^{i_0}K| \geqq \aleph$, as required.

Case II. $\lim_{k\to\infty}|\sum_{i=k}^{\infty} P_i| = \aleph$. It may happen that $|\sum_{i=0}^{\infty} P_i| = \aleph$ but $|P_i| < \aleph$ for all $i \in \omega$. Thus we proceed to pick out a subsocle S of $\sum_{i\in\omega} P_i$ to obtain

$$\left|\sum_{i\in\omega} P_i/S\right| = \aleph \qquad \text{and} \qquad \langle S, p^\beta G\rangle \supseteq \sum_{i\in\omega} P_i \qquad \text{for all } \beta \in \alpha.$$

Letting K be neat such that $K[p] = S$ will give a p^α-pure subgroup with G/K divisible and of cardinality $\aleph$.

Let $\{R_j\}_{j\in\omega}$ be a subsequence of $\{P_i\}_{i\in\omega}$ with the property that $|R_j| \leqq |R_{j+1}|$ for all $j \in \omega$ and $\sum_{j\in\omega} |R_j| = \aleph$. Note that if $|R_j|$ is now finite for all $j \in \omega$, then $\alpha = \beta + \omega$ and we will choose the K in the following to be a neat subgroup supported by a socle consisting of the direct sum of the socle of a $p^\beta G$-high subgroup of G and the socle of a lower basic subgroup of $p^\beta G$. Thus we may assume that $|R_j|$ is infinite for all $j \in \omega$.

Define Q_n^r, r, $n \in \omega$ as follows. Let $Q_0^0 = R_0$ and $Q_n^r = 0$ whenever $r > n$. Inductively let $R_n = Q_n^0 \oplus \ldots \oplus Q_n^n$, where $|Q_n^j| = |Q_{n-1}^j|$ for $0 \leqq j < n$, and $Q_n^n = 0$ if $|R_n| = |R_{n-1}|$. This can be done by defining, for each $j \in \omega$, λ_j to be the least ordinal whose cardinal is dim R_j (as a vector space over the integers mod p), choosing a basis $\{y_\lambda\}_{\lambda\in\lambda_n}$ for R_n, and defining

$$Q_n^i = \langle\{y_\lambda|\ (\lambda_{i-1} = \lambda \text{ or } \lambda_{i-1} \in \lambda) \text{ and } \lambda \in \lambda_i\}\rangle, \text{ where } \lambda_{-1} = 0.$$

Let $\Lambda = \{\dim R_i|\ i \in \omega\}$. For each $\mu \in \Lambda$ let k_μ be the least element of ω such that $\sum_{i=0}^{k_\mu} |R_i| = \mu$. Then R_{k_μ} is the first member of the sequence with dimension μ. Thus $\dim(Q_{k_\mu}{}^{k_\mu}) = \mu$. Let $Q_\mu = \sum_{n=k_\mu}^{\infty} Q_n{}^{k_\mu}$. Let $\{x_n{}^\beta\}_{\beta\in\mu}$ be a basis of $Q_{k_\mu+n}{}^{k_\mu}$, $n \in \omega$. Let $Q_\mu{}^\beta = \sum_{n\in\omega} \langle x_n{}^\beta\rangle$ (note that $\sum_{\beta\in\mu} Q_\mu{}^\beta = Q_\mu$). Let $S_\mu{}^\beta \subseteq Q_\mu{}^\beta$ be generated by all elements of the form $\sum_{i=a}^{2a} x_i{}^\beta$, $a \in \omega$. We show below that $Q_\mu{}^\beta \subseteq \langle S_\mu{}^\beta, p^\gamma G\rangle$ for all $\gamma \in \alpha$, and $|Q_\mu{}^\beta/S_\mu{}^\beta| = \aleph_0$. Hence if $S_\mu = \sum_{\beta\in\mu} S_\mu{}^\beta$, then

$$\left|\frac{Q_\mu}{S_\mu}\right| = \left|\frac{\sum_{\beta\in\mu} Q_\mu{}^\beta}{\sum_{\beta\in\mu} S_\mu{}^\beta}\right| = \left|\sum_{\beta\in\mu} \frac{Q_\mu{}^\beta}{S_\mu{}^\beta}\right| = \aleph_0 \cdot \mu = \mu.$$

Let $Q = \langle\{S_\mu\}_{\mu\in\Lambda}, \{P_i|\ i \in \omega \text{ and } P_i \neq R_j \text{ for all } j \in \omega\}\rangle$.

Thus by construction we have

$$\left|\frac{\sum_{i=0}^{\infty} P_i}{Q}\right| = \left|\frac{\sum_{\mu\in\Lambda} Q_\mu}{\sum_{\mu\in\Lambda} S_\mu}\right| = \left|\sum_{\mu\in\Lambda} \frac{Q_\mu}{S_\mu}\right| = \sum_{\mu\in\Lambda} \mu = \aleph.$$

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Let K be a neat subgroup of G with $K[p] = Q$. If $\gamma \in \alpha$, then

$$\begin{aligned}\langle Q, (p^\gamma G)[p]\rangle &= \left\langle \sum_{P_i \neq R_j} P_i, \sum_{\mu \in \Lambda} S_\mu, (p^\gamma G)[p]\right\rangle \\ &= \left\langle \sum_{P_i \neq R_j} P_i, \sum_{\mu \in \Lambda} \sum_{\beta \in \mu} S_\mu^\beta, (p^\gamma G)[p]\right\rangle \\ &= \left\langle \sum_{P_i \neq R_j} P_i, \sum_{\mu \in \Lambda} \sum_{\beta \in \mu} Q_\mu^\beta, (p^\gamma G)[p]\right\rangle \\ &= \left\langle \sum_{P_i \neq R_j} P_i, \sum_{\mu \in \Lambda} Q_\mu, (p^\gamma G)[p]\right\rangle \\ &= \left\langle \sum_{i \in \omega} P_i, (p^\gamma G)[p]\right\rangle \\ &= G[p].\end{aligned}$$

Since G/K is divisible, we then have that K is p^α-pure in G. Again by the construction we have $|G/K| = \aleph$, as desired.

Finally we will show that $Q_\mu^\beta \subseteq \langle S_\mu^\beta, p^\gamma G\rangle$ for $\gamma \in \alpha$. Let γ be given and find $m \in \omega$ such that $\gamma \in \alpha_m$. Then, given $x_r^\beta \in Q_\mu^\beta$, we have

$$x_r^\beta - \sum_{i=2^{m+1}(r+1)-1}^{2^{m+1}(r+2)-2} x_i^\beta = \sum_{i=0}^{m} \sum_{j=2^i(r+1)-1}^{2^{i+1}(r+1)-2} x_j^\beta - \sum_{i=0}^{m} \sum_{j=2^i(r+2)-1}^{2^{i+1}(r+2)-2} x_j^\beta.$$

Each member of the sum on the right side is an element of S_μ^β. The left side is $x_r^\beta - z$, where $z \in p^{\alpha_m} G \subseteq p^\gamma G$. It follows that

$$Q_\mu^\beta = \sum_{r \in \omega} \langle x_r^\beta\rangle \subseteq \langle S_\mu^\beta, p^\gamma G\rangle.$$

Note that $|Q_\mu^\beta / S_\mu^\beta| = \aleph_0$ as follows. Suppose that n is odd. If x_n^β is in S_μ^β we can write

$$x_n^\beta = \sum_{i=1}^{m} c_i \sum_{j=a_i}^{2a_i} x_j^\beta,$$

where $0 < c_i < p$ and $i < j \Rightarrow a_i < a_j$. Then $x_{2a_m}^\beta$ appears only in the last term and $n \neq 2a_m$. Thus $c_m x_{2a_m}^\beta = 0 \Rightarrow p | c_m$, a contradiction. Hence $x_n^\beta \in S_\mu^\beta$ for n odd. We claim that $\{x_{2n+1}^\beta + S_\mu^\beta\}_{n \in \omega}$ is linearly independent. If $x_{2n+1}^\beta - x_{2k+1}^\beta \in S_\mu^\beta$, then, supposing $n \geqq k$, we have

$$x_{2n+1}^\beta = x_{2k+1}^\beta + \sum_{i=1}^{m} c_i \sum_{j=a_i}^{2a_i} x_j^\beta$$

with $0 \leqq c_i < p$, and $i < j \Rightarrow a_i < a_j$. Once again we see that $p | c_m$, and thus $x_{2n+1}^\beta = x_{2k+1}^\beta$. Hence $|Q_\mu^\beta / S_\mu^\beta| = \aleph_0$. This completes the proof.

THEOREM 3. *Let G be a reduced p-primary Abelian group and let α be an accessible limit ordinal. Then $r_\alpha(G) = s_\alpha(G)$.*

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Proof. This follows from Theorems 1 and 2 and the fact that if H is a $p^{\alpha}G$-high subgroup of G then $r_{\alpha}(G) = r_{\alpha}(H) + r(p^{\alpha}G)$ and

$$s_{\alpha}(G) = s_{\alpha}(H) + r(p^{\alpha}G).$$

One application of Theorem 3 is as follows. Let G be a reduced p-group of length α, α an accessible limit ordinal. Let H be a p-group and B a basic subgroup of H. Then a necessary and sufficient condition that there exist a group K such that $K/p^{\alpha}K \cong G$ and $p^{\alpha}K \cong H$ is that $r(B) \leqq s_{\alpha}(G)$. (Note that $s_{\alpha}(G)$ can be replaced by $r_{\alpha}(G)$ with no restriction on the limit ordinal α. See [**2**, Proposition 1.7].)

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University of California,
Davis, California;
University of Alberta,
Edmonton, Alberta

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