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THE CLOSURE OF CONVERGENCE SETS FOR CONTINUED FRACTIONS ARE CONVERGENCE SETS

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We prove that if Ω is a simple convergence set for continued fractions $K(a_n/b_n)$, then the closure $\bar{\Omega}$ of Ω is also such a convergence set. Actually, we prove more: every continued fraction $K(a_n/b_n)$ has a "neighbourhood" $\{\mathscr{D}_n\}_{n=1}^{\infty}$; $\mathscr{D}_n=\{z\in\mathbf{C};\ |z-a_n|\leqq r_n\}\times\{z\in\mathbf{C};\ |z-b_n|\leqq s_n\}$ where $r_n>0$ and $s_n>0$, with the following property: Every continued fraction from $\{\mathscr{D}_n\}$ converges if and only if $K(a_n/b_n)$ converges.

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1. Definitions and notation

We consider *continued fractions*

$$K\frac{a_n}{b_n}=K(a_n/b_n)=\frac{a_1}{b_1}\,\frac{a_2}{+b_2}+\cdots=\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots}};\ a_n\in\mathbf{C}\setminus\{0\},b_n\in\mathbf{C}. \tag{1.1}$$

We say that $K(a_n/b_n)$ *converges/diverges* if its sequence of *classical approximants* $S_n(0)$ converges/diverges in $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$, where S_n is the linear fractional transformation

$$S_n(w)=\frac{A_n+A_{n-1}w}{B_n+B_{n-1}w}=\frac{a_1}{b_1}\,\frac{a_2}{+b_2}+\cdots\frac{a_n}{+b_n+w}, \tag{1.2}$$

and $\{A_n\}$ and $\{B_n\}$ are solutions of the linear recurrence relation

$$X_n=b_nX_{n-1}+a_nX_{n-2}\quad\text{for } n=1,2,3,\ldots, \tag{1.3}$$

with initial values $A_{-1}=1$, $A_0=0$, $B_{-1}=0$ and $B_0=1$. (See for instance [4, *p*. 20].) Since all $a_n\neq0$, it follows that S_n is non-singular. It is useful to introduce the corresponding quantities $\{A_n^{(k)}\}$ and $\{B_n^{(k)}\}$ for the *kth tail* of $K(a_n/b_n)$, which is the continued fraction

$$\frac{a_{k+1}}{b_{k+1}}\,\frac{a_{k+2}}{+b_{k+2}}\,\frac{a_{k+3}}{+b_{k+3}}+\cdots\quad\text{for } k\in\mathbf{N}_0=\mathbf{N}\cup\{0\}. \tag{1.4}$$

With this notation we have $A_n = A_n^{(0)} = a_1 B_{n-1}^{(1)}$ and $B_n = B_n^{(0)}$.

A sequence $\{t_n\}_{n=0}^{\infty}$ of elements from $\hat{\mathbf{C}}$ is a *tail sequence* for $K(a_n/b_n)$ if

$$t_{n-1} = a_n/(b_n + t_n) \quad \text{for } n = 1, 2, 3, \dots. \tag{1.5}$$

Then $t_0 = S_n(t_n)$ for all n, and thus $t_n = S_n^{-1}(t_0)$. Hence, every $t_0 \in \hat{\mathbf{C}}$ gives a tail sequence $\{t_n\}$ for $K(a_n/b_n)$, and if $\{t_n\}$ and $\{t'_n\}$ are two tail sequences with $t_0 \neq t'_0$, then $t_n \neq t'_n$ for all n. Therefore there always exists a tail sequence $\{t_n\}$ for $K(a_n/b_n)$ with all $t_n \neq \infty$. Note that if follows by (1.5) that if all $t_n \neq \infty$, then all $t_n \neq 0$ and $(b_n + t_n) \neq 0$.

We shall consider continued fractions $K(\tilde{a}_n/\tilde{b}_n)$ close to $K(a_n/b_n)$. We shall use $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{A}_n^{(k)}$, $\tilde{B}_n^{(k)}$ and $\tilde{t}_n$ to denote the corresponding quantities for $K(\tilde{a}_n/\tilde{b}_n)$. We adopt the usual convention that an empty product is equal to 1 and an empty sum is equal to 0.

2. Main results

Convergence criteria for continued fractions $K(a_n/b_n)$ are often stated in terms of *simple convergence sets* Ω. That is, $\Omega \subset \mathbf{C} \times \mathbf{C}$, and every continued fraction $K(a_n/b_n)$ from Ω (i.e. all $(a_n, b_n) \in \Omega$) converges. For instance, the Worpitzky disk $\Omega = \{a \in \mathbf{C}: |a| \leq 1/4\} \times \{1\}$ is a convergence set for continued fractions $K(a_n/1)$, and the Sleszýnski–Pringsheim criterion says that $\Omega = \{(a,b) \in \mathbf{C} \times \mathbf{C}: |b| \geq |a| + 1\}$ is a convergence set for continued fractions $K(a_n/b_n)$. In both these examples the convergence set Ω was a closed set. The question we address in this paper is whether this is always so. Or to be more precise: whether we always can take the closure $\bar{\Omega}$ of Ω in $\mathbf{C} \times \mathbf{C}$ as a convergence set, if Ω is a convergence set. The answer turns out to be yes.

Theorem 2.1. *If Ω is a simple convergence set for continued fractions $K(a_n/b_n)$, then so is its closure $\bar{\Omega}$ in $\mathbf{C} \times \mathbf{C}$.*

The proof of Theorem 2.1 is based on the following result which has its own value:

Theorem 2.2. *Let $K(a_n/b_n)$ be a given continued fraction. Then there exist sequences $\{r_n\}$ and $\{s_n\}$ of positive numbers such that each continued fraction $K(\tilde{a}_n/\tilde{b}_n)$ satisfying*

$$|\tilde{a}_n - a_n| \leq r_n \quad \text{and} \quad |\tilde{b}_n - b_n| \leq s_n \quad \text{for } n = 1, 2, 3, \dots \tag{2.1}$$

converges if and only if $K(a_n/b_n)$ converges.

This is the result announced in the abstract. The sequences $\{r_n\}$ and $\{s_n\}$ define a neighbourhood in which every continued fraction has the same convergence behaviour as $K(a_n/b_n)$. It continues the idea of nearness of two continued fractions which was described in [2]. The emphasis in [2] was on describing how large these r_n and s_n could be chosen without disturbing the conclusion of Theorem 2.2, and the results were restricted to certain classes of continued fractions $K(a_n/b_n)$. Theorem 2.2 shows the *existence* of such sequences $\{r_n\}$ and $\{s_n\}$, without restrictions on $K(a_n/b_n)$.

In recent years the concept of *separate convergence* has received some attention:

$K(a_n/b_n)$ *converges separately* if the limits $\lim_{n\to\infty}\zeta_n A_n$ and $\lim_{n\to\infty}\zeta_n B_n$ exist in $\mathbf{C}$ for some "simple" sequence $\{\zeta_n\}$. (See for instance [7].) We shall prove:

Theorem 2.3. *Let $\{t_n\}$ be a tail sequence for $K(a_n/b_n)$ with all $t_n\neq\infty$, and let $\zeta_n=\prod_{m=1}^{n}(b_m+t_m)^{-1}$ for all $n\in\mathbf{N}$. Further let $M>0$. Then there exist sequences $\{r_n\}$ and $\{s_n\}$ of positive numbers such that every continued fraction $K(\tilde a_n/\tilde b_n)$ satisfying (2.1) has the following properties:*
A. The sequences $\{(\tilde A_n+\tilde A_{n-1}t_n)\zeta_n\}$ and $\{(\tilde B_n+\tilde B_{n-1}t_n)\zeta_n\}$ converge to finite values A and B as $n\to\infty$, where $|A-\tilde a_1/(b_1+t_1)|\leqq M$ and $|B-1|\leqq M$.
B. $\tilde S_n(t_n)$ converges to a finite value.
C. The sequences $\{\tilde A_n/\prod_{m=0}^{n}(-t_m)\}$ and $\{\tilde B_n/\prod_{m=0}^{n}(-t_m)\}$ converge as $n\to\infty$ if and only if $K(a_n/b_n)$ converges in $\hat{\mathbf{C}}$.

3. Proofs

We shall use the following formulas and lemmas (notation as in Section 1 and 2):

$$\tilde B_n=B_n+\sum_{k=1}^{n}((\tilde b_k-b_k)B_{n-k}^{(k)}+(\tilde a_{k+1}-a_{k+1})B_{n-k-1}^{(k+1)})\tilde B_{k-1}. \tag{3.1}$$

This formula can be proved by manipulating the recurrence relation (1.3) for B_n and the corresponding recurrence relation for $\tilde B_n$. (See [5].) Both this formula and the following ones require that the tail sequence $\{t_n\}$ of $K(a_n/b_n)$ has only finite elements.

$$B_n=\sum_{k=0}^{n}\left(\prod_{m=1}^{k}(b_m+t_m)\prod_{m=k+1}^{n}(-t_m)\right). \tag{3.2}$$

This one can be proved by induction on n, using the recurrence formula (1.3). (See [3].)

$$B_n^{(k)}=\left(B_{k+n}-B_{k-1}\prod_{m=k}^{k+n}(-t_m)\right)\prod_{m=1}^{k}(b_m+t_m)^{-1}. \tag{3.3}$$

This is a consequence of (3.2). (See [6].) Combining (1.5), (3.1) and (3.3) gives:

$$\tilde B_n=B_n\left\{1+\sum_{k=1}^{n}\left[\frac{\tilde b_k-b_k}{\prod_{m=1}^{k}(b_m+t_m)}+\frac{\tilde a_{k+1}-a_{k+1}}{\prod_{m=1}^{k+1}(b_m+t_m)}\right]\tilde B_{k-1}\right\}$$
$$-\left(\prod_{m=0}^{n}(-t_m)\right)\sum_{k=1}^{n}\left[\frac{\tilde b_k-b_k}{\prod_{m=1}^{k}(-a_m)}B_{k-1}+\frac{\tilde a_{k+1}-a_{k+1}}{\prod_{m=1}^{k+1}(-a_m)}B_k\right]\tilde B_{k-1}. \tag{3.4}$$

Lemma 3.1. *Let* $A>0$, $c_k \geqq 0, d_k \geqq 0$ *and*

$$c_n \leqq A + \sum_{k=1}^{n-1} d_k c_k \quad \text{for } n=1,2,\ldots,N.$$

Then

$$c_n \leqq A \prod_{k=1}^{n-1} (1+d_k) \leqq A \exp\left(\sum_{k=1}^{n-1} d_k\right) \quad \text{for } n=1,2,\ldots,N.$$

This result, which essentially can be found in [**1**, p. 455], is easily proved by induction. The last inequality follows since $\exp(d) \geqq 1+d$ for $d \geqq 0$.

Lemma 3.2. *Given* $K(a_n/b_n)$ *with tail sequence* $\{t_n\}$ *such that all* $t_n \neq \infty$. *Then* $K(a_n/b_n)$ *converges in* $\hat{\mathbb{C}}$ *if and only if*

$$\sum_{k=0}^{n} \prod_{m=1}^{k} \frac{b_m+t_m}{-t_m} \tag{3.5}$$

converges in $\hat{\mathbb{C}}$ *as* $n \to \infty$.

This follows simply from dividing

$$A_n - B_n t_0 = \prod_{m=0}^{n} (-t_m) \quad \text{(proved by induction)}$$

by B_n as given by (3.2). (See [**8**].) By induction it also follows that

$$A_n B_{n-1} - B_n A_{n-1} = -\prod_{m=1}^{n} (-a_m) \quad \text{and} \quad B_n + B_{n-1} t_n = \prod_{m=1}^{n} (b_m+t_m). \tag{3.6}$$

Lemma 3.3. *Let*

$$D_n = \max_{1 \leqq k \leqq n} |B_k|, \quad P_n = \max_{1 \leqq k \leqq n} \prod_{m=0}^{k} |t_m| \quad \text{for } n=1,2,3,\ldots,$$

and

$$\gamma_k = \max\left\{1 \Big/ \prod_{m=1}^{k} |b_m+t_m|, \ |B_{k-1}| \Big/ \prod_{m=1}^{k} |a_m|\right\} \quad \text{for } k=1,2,3,\ldots.$$

If (2.1) holds, then

$$|\tilde{B}_n| \leqq D_n \exp\left((D_n+P_n)\sum_{k=1}^{n+1}\gamma_k(r_k+s_k)\right). \tag{3.7}$$

Proof. From (3.4) we find that

$$|\tilde{B}_m| \leqq D_n\left\{1+\sum_{k=1}^{m}(s_k\gamma_k+r_{k+1}\gamma_{k+1})|\tilde{B}_{k-1}|\right\}+P_n\sum_{k=1}^{m}(s_k\gamma_k+r_{k+1}\gamma_{k+1})|\tilde{B}_{k-1}|$$

$$=D_n+\sum_{k=1}^{m}(D_n+P_n)(s_k\gamma_k+r_{k+1}\gamma_{k+1})|\tilde{B}_{k-1}| \quad \text{for } m=1,2,\dots,n.$$

Hence, by Lemma 3.1 we get

$$|\tilde{B}_n| \leqq D_n \exp\left((D_n+P_n)\sum_{k=1}^{n}(s_k\gamma_k+r_{k+1}\gamma_{k+1})\right)$$

which is less than or equal to the bound in (3.7). □

Proof of Theorem 2.3.

A. Let us first consider the $\tilde{B}_n$-expression. From (3.4) we find that

$$\tilde{B}_n+\tilde{B}_{n-1}t_n=(B_n+B_{n-1}t_n)\left\{1+\sum_{k=1}^{n-1}((\tilde{b}_k-b_k)\zeta_k+(\tilde{a}_{k+1}-a_{k+1})\zeta_{k+1})\tilde{B}_{k-1}\right\}$$

$$+B_n((\tilde{b}_n-b_n)\zeta_n+(\tilde{a}_{n+1}-a_{n+1})\zeta_{n+1})\tilde{B}_{n-1}$$

$$-\left(\prod_{m=0}^{n}(-t_m)\right)\left[\frac{\tilde{b}_n-b_n}{\prod_{m=1}^{n}(-a_m)}B_{n-1}+\frac{\tilde{a}_{n+1}-a_{n+1}}{\prod_{m=1}^{n+1}(-a_m)}B_n\right]\tilde{B}_{n-1},$$

where $(\prod_{m=0}^{j}(-t_m))/(\prod_{m=1}^{j+1}(-a_m))=\zeta_{j+1}$ by (1.5). Hence, division by $(B_n+B_{n-1}t_n)$ $(=\zeta_n^{-1}$ by the second expression in (3.6)) leads to

$$(\tilde{B}_n+\tilde{B}_{n-1}t_n)\zeta_n-1$$

$$=\sum_{k=1}^{n-1}((\tilde{b}_k-b_k)\zeta_k+(\tilde{a}_{k+1}-a_{k+1})\zeta_{k+1})\tilde{B}_{k-1}+(\tilde{b}_n-b_n)\zeta_n\tilde{B}_{n-1}. \tag{3.8}$$

Let first $\{R_k\}$ and $\{S_k\}$ be sequences of positive numbers such that

$$\sum_{k=1}^{\infty} \gamma_k(R_k+S_k) \leqq 1,$$

where γ_k is as given in Lemma 3.3, and let all $r_k \leqq R_k$ and $s_k \leqq S_k$. Then, by Lemma 3.3,

$$|\tilde{B}_n| \leqq D_n \exp(D_n+P_n).$$

Let further $\{R'_k\}$ and $\{S'_k\}$ be positive numbers such that

$$\sum_{k=1}^{\infty} (S'_k\gamma_k + R'_{k+1}\gamma_{k+1})D_{k-1}\exp(D_{k-1}+P_{k-1}) < 1,$$

and let

$$r_k \leqq \min\{R_k, MR'_k\}, \quad s_k \leqq \min\{S_k, MS'_k\} \tag{3.9}$$

for each k. Then the series in (3.8) converges absolutely to a value B', $|B'|<M$, as $n\to\infty$, and the last term in (3.8) vanishes as $n\to\infty$. Finally, $B=B'+1$.

To prove the convergence of the $\tilde{A}_n$-expression, we observe that

$$\tilde{A}_n\zeta_n = \frac{\tilde{a}_1}{b_1+t_1}\tilde{B}^{(1)}_{n-1}\zeta^{(1)}_{n-1} \quad \text{where } \zeta^{(1)}_{n-1} = \prod_{m=2}^{n} (b_m+t_m)^{-1}.$$

The arguments above applied to the first tail of $K(a_n/b_n)$ prove the existence of $\{r_k\}$ and $\{s_k\}$ such that if (2.1) holds, then $\lim \tilde{B}^{(1)}_{n-1}/\prod_{m=2}^{n}(b_m+t_m) = B^{(1)}$, where $|B^{(1)}-1| \leqq M_1$ for any $M_1>0$. Hence the result follows.

B. This follows immediately from the results in part A, since $\tilde{S}_n(t_n) = (\tilde{A}_n+\tilde{A}_{n-1}t_n)\zeta_n/((\tilde{B}_n+\tilde{B}_{n-1}t_n)\zeta_n)$, and the limit B of the denominator expression is $\neq 0$ if $M<1$.

C. Again we first consider the $\tilde{B}_n$-expression. From (3.2) we find that $B_n/\prod_{m=1}^{n}(-t_m)$ can be written as (3.5). Hence, by Lemma 3.2, $K(a_n/b_n)$ converges if and only if $B_n/\prod_{m=0}^{n}(-t_m)$ converges in $\hat{\mathbb{C}}$ as $n\to\infty$. From (3.4) we find that

$$\frac{\tilde{B}_n}{\prod_{m=0}^{n}(-t_m)} = \frac{B_n}{\prod_{m=0}^{n}(-t_m)}\left\{1+\sum_{k=1}^{n}((\tilde{b}_k-b_k)\zeta_k+(\tilde{a}_{k+1}-a_{k+1})\zeta_{k+1})\tilde{B}_{k-1}\right\}$$

$$-\sum_{k=1}^{n}\left[\frac{\tilde{b}_k-b_k}{\prod_{m=1}^{k}(-a_m)}B_{k-1}+\frac{\tilde{a}_{k+1}-a_{k+1}}{\prod_{m=1}^{k+1}(-a_m)}B_k\right]\tilde{B}_{k-1}. \tag{3.10}$$

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We recognize the first series in (3.10) from (3.8). Hence it converges to a finite value B if (2.1) holds with the choice (3.9) for r_k and s_k. In particular, with $M<1$ in (3.9), we know that $|B-1|<1$, which means that B is non-zero. The second series in (3.10) also has terms bounded by $(s_k\gamma_k+r_{k+1}\gamma_{k+1})|\tilde{B}_{k-1}|$. Hence it converges absolutely, and the result follows.

The proof for the $\tilde{A}_n$-expression follows similarly, since

$$\tilde{A}_n\Big/\prod_{m=0}^{n}(-t_m)=(-\tilde{a}_1/t_0)\left(\tilde{B}_{n-1}^{(1)}\Big/\prod_{m=1}^{n}(-t_m)\right). \qquad \square$$

Proof of Theorem 2.2. Let $\{r_n\}$ and $\{s_n\}$ be chosen such that (3.9) holds with an $M<1$. Then the assertions of Theorem 2.3 hold. Now,

$$\tilde{S}_n(t_n)-\tilde{S}_{n-1}(0)=\frac{\tilde{A}_n\tilde{B}_{n-1}-\tilde{B}_n\tilde{A}_{n-1}}{\tilde{B}_{n-1}(\tilde{B}_n+\tilde{B}_{n-1}t_n)}=\frac{-\prod_{m=1}^{n}\left(1+\dfrac{\tilde{a}_m-a_m}{a_m}\right)}{\dfrac{\tilde{B}_{n-1}}{\prod_{m=0}^{n-1}(-t_m)}\cdot\dfrac{\tilde{B}_n+\tilde{B}_{n-1}t_n}{\prod_{m=1}^{n}(b_m+t_m)}},$$

where the first factor in the denominator converges in $\hat{\mathbb{C}}$ if and only if $K(a_n/b_n)$ converges, and the second factor converges to a finite value$\neq 0$. Since $\tilde{S}_n(t_n)$ also converges to a finite value, the result follows if the numerator converges to a finite value$\neq 0$. This holds if we, in addition to (3.9), also make sure that $\sum r_m/|a_m|<\infty$ when we choose $\{r_n\}$. $\square$

Proof of Theorem 2.1. Assume that $K(a_n/b_n)$ is a divergent continued fraction from $\bar{\Omega}$. Then there exist sequences $\{r_n\}$ and $\{s_n\}$ of positive numbers such that every continued fraction $K(\tilde{a}_n/\tilde{b}_n)$ with $|\tilde{a}_n-a_n|\leqq r_n$ and $|\tilde{b}_n-b_n|\leqq s_n$ diverges. This is impossible since every such neighbourhood contains elements from Ω, and Ω is a convergence set. Hence, all continued fractions from $\bar{\Omega}$ converge. $\square$

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